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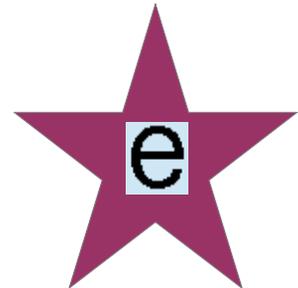
MATHIZATION

Edition 1, as of May 2, 2023

**ARITHMETIC
ALGEBRA
TRIGONOMETRY
ADVANCED TOPICS**
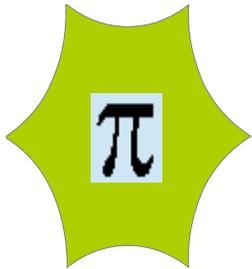
J.P. Albertson

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This ebook is about learning some basic and essential mathematics from the author's perspective and considerations. Mathematics is the study of measurements, numerical relationships, quantifying, solving, and where arithmetic is the general study of counting, which can be called basic or fundamental mathematics. rand of practicality for the average person who just needs some basic knowledge on a particular subject and then goes and does something useful with it. Each other math book is usually a trade-off between the maximum number of topics. This book may be just right for those seeking a way to finally learn math, especially for its unique overview of topics and informal approach, rather than leaving the casual reader wondering if they should have taken some kind of advanced calculus or perhaps being born with exceptional mathematical abilities and thought processes. There is already many other math books available, and to write something new would mean writing something that is hopefully different in some ways, and so as to be considered for that. This book will hopefully make you feel comfortable about mathematics to the point where you actually want to keep reading this and other learning material and applying what you know when needed. Most books, inducing this book, are also lacking in some way and are therefore never completely finished, and at some point, they will be presented to the readers at its current edit or version so as some hopeful and timely use can be made of what is expressed there. This book does not discuss any particular branch of mathematics in great depth, due to the space that would be required, and the particular or simpler style of this book. Still, this book does show you the essentials and power of mathematics. To acquire that power, the reader must give a reasonable effort. Be patient since mathematics books are usually not known for being "fast" or "lite" reading material, and we sometimes need to learn how to learn new things. The decimal number symbols and the language of mathematics is necessary, standardized and used world-wide.

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hundred pages, but obviously, it is not specifically related to the topic of math and science, since it is rather an art and health topic. The combined writings in this math ebook are also meant to help support and enhance each other in some way.

As for trying to learn and-or understand things, it is surely much better to learn slowly and easily, than not at all. Consider math as a helpful tool to learn, build and apply other knowledge, rather than just as a temporary goal to complete some kind of study or course. This book can be viewed before, during, and after any course as a helpful supplement to learning. One thing to always remember is that it is not how much that you know of a seemingly infinite or endless topic, but how you can apply what little you do know so as to make some type of practical and beneficial use of it for yourself and-or others. To learn more about the infinite subject of mathematics, just apply what you already know as the first step to begin leaning new concepts. This process could be called the "fundamental math learning concept" where each new concept is built upon previous concepts. It's like a logical chain, steps or progression of concepts from one to the next following one.   Some concepts can be fascinating and an achievement to learn, but there is nothing within this book that is only for the magically gifted to attempt. Concepts are rather given in successive steps so as anyone could reach and learn them. Even the most highly advanced math topics are built upon and utilize the basics of math such as: **addition, subtraction, multiplication, and division** to a great extent, and therefore knowing just a little bit of math can go a "long way" in terms of practicality or usefulness.

This book also includes some example plain text, readable and editable, **computer source code programs** that are mostly based on the math material found within this book. The computer programming language used is the popular **C programming language**. Even if you may never program a computer, you are still encouraged to take a glance at them when your able for the sake of familiarization. A computer can quickly calculate many mathematical values that would otherwise be very difficult and-or time consuming to do. Computer programming is an interesting, joyful, artful, and rewarding hobby that could even become beneficial to others when making even relatively simple computer programs that can easily calculate values that others such as scientists, hobbyists and businesses could benefit from. Always remember that there is "strength in numbers", "numbers matter", and that nobody is magically born knowing how to do math, but they can learn the knowledge step by step at their own pace and needs.

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Thankyou in advance.

This book is dedicated to my wife Elizabeth, and to our relatives, helpers and friends. Special mention to Mr. Norman Borofski , and also to Mr. Dana Switzer for some academic inspiration in the Pennsylvania area.

This Edition is 1.0 as of May 2, 2023. All previous versions are still very useful, but there are some new writings and edits available in this newer version or edit of this book. For the latest version number or date, etc., search the popular **https://www.youtube.com** video website for: **Mathization ebook**

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INTRODUCTION

This book is intended for readers who are just beginning with arithmetic, algebra, and trigonometry, or for those who always wanted or needed another grasp on these subjects. Knowing how to work with numbers can enlighten you and give you confidence in problem solving, reading comprehension and discussions. This book is composed of four general mathematical topics and a generous appendix, and each is given an entire section of this book. This book is intended for a rapid understanding of these topics by basically using a less formal or less critical approach to mathematics. Due to the style of this book, it can also be used as a basic reference , a helpful supplement or complement to another book or course you are currently taking, an aid for a teacher or those being educated at home, or as a step to a more formal or advanced mathematics course. If a topic or example seems of no certain need or interest, or perhaps too complicated at the moment, please skip over it to another useful topic or example and consider that which was skipped over to be reviewed at a later time. If a topic needs to be referenced at another time, please write down the topic and the page number(s) and-or formula(s).   Some of the concepts are not explicitly noted in the table of contents, and are rather given "on the fly", that is, they are given as you read and come upon them.

With some available liberties of creating an e-book, such as having an unfixed length or number of pages, style, content, an effort to be different so as to be more desired, etc., this book contains many pages needed to convey many thoughts that can be roughly grouped into the following categories: main text and knowledge, examples, figures or graphics (many in a simple homemade form), computer programs, and numeric data tables. Perhaps 100 pages worth of "blank space" is either due to the leftover space of unfilled pages and-or some pages intentionally left blank to help assist with the editing of this ebook. The many pages help give the reader much material to have available and learn from just having one book, especially if your access to other materials is limited and-or having limited funds. There is a saying that can be adapted here, as that there is sometimes a certain quality to be found in having a quantity or volume of material to access and to then consider. If just two pages a day are written, a person could create about a 600 page book during one year. Having a quantity of quality is a rather a better goal than quantity alone, and of which this book has tried to attain.

Many topics that are frequently reserved only for algebra studies are shown in the first section: ARITHMETIC. Here, instead of using algebraic variables to show a concept, actual numeric values are utilized instead. By doing this, understanding basic algebra becomes easier since many of its concepts were already effectively shown using actual numbers instead of variables. Likewise, the transition from BASIC ALGEBRA to the BASIC TRIGONOMETRY section was made to be as smooth as possible. The concepts of trigonometry should also not be overlooked since they seem to arise everywhere in mathematics, and in the least expected places elsewhere in life.

A special section of some ADVANCED TOPICS is included at the end of this book. It is intended to mainly show you how to make some very important calculations when a scientific calculator (a preprogrammed, multifunctional calculator) is unavailable. An inexpensive regular "home" calculator can also be a great assistance throughout this book, especially for performing otherwise tedious and time consuming computations, such as finding the square-root of a number, and for checking results easily and quickly so as you can move on to other things without loosing your train of thought due to lengthy calculations.   There are many calculator programs for computers, and apps (applications, programs) for phones such as RealCalc. The basic versions of these programs are often free to download, and you often will have a generous option of purchasing a fuller version and so as to fairly support the creators of it. If you already have a scientific calculator, or not, you will effectively learn throughout this book what the common buttons, commands or functions on most scientific calculators actually do.   It should be noted that this section can still be considered as a continuation of the topics covered in this book. Don't forget the APPENDIX section, for it also contains much very useful information and derivations related to the other book sections, and is itself a good mathematical read.

At the beginning of the ARITHMETIC section, their is a discussion about the decimal number system. Here, you might find out that there is more to "plain old numbers" than you might think. After that topic, discussions about expressions and units (of reference or measurement) are given. To get through these discussions, you should (hopefully) already of had some initial thought, education, or understanding of the basic mathematical operations of addition, subtraction, multiplication and division.   Those operations are given a more general discussion after these more fundamental concepts are first used to build upon.

SECTION 1: ARITHMETIC

THE DECIMAL NUMBER SYSTEM

The decimal number system is the common number system we are all taught to use as the accepted standard numeric or number system in the modern world. For many people, many of the fundamental concepts of numbers are either unknown, weak, and-or taken for granted. Because of a lack of a firm understanding of the basic concepts and fundamentals of numbers in general, this is where some problems may start in learning math and-or using math in a more practical manner with confidence.

We are first taught to count things, items or objects which may, or may not even be similar. To count (as a verb) things or items is a repetitive (to repeat, "do again, and again") process of steps of a measurement so as to numerically quantify (to express as a numerical representation) the total (all, entire) amount, "number of" or quantity of things or items being considered.   Here, to measure is the act or process to find (often by comparing to some standard or reference) and give a certain type of "size" or "size of it all (things or items)", and that this measurement (the result of measuring) is called the quantity or "the count (as a noun, and a single number result)", sum or total of the items or things. The measurement result, or simply, "the count", is expressed or represented numerically as a single, symbolic and representative number. In a way, a number is much like a special type of word, and a unique word which has a unique meaning. With the decimal numeric system, a number can be a single symbol (0 through 9) called a single digit number or numeral, or a grouping of several of these symbols so as to represent larger counts, values or results of a calculation such as combining or summing two counts together so as to have another representative count (measurement) of things and a number to represent that count.

We measure a quantity or amount of items or things by considering (identifying or reckoning) each individual item or thing and then (numerically) summing each item as an increment of 1 to the total or previous count of items that were already considered. We will consider 1 thing or item can be numerically considered and represented as equal (=) to a count of 1 numerically. Summing (summation) is to take this numeric value or representation of 1 and mathematically or numerically successively combine it to or increment (increase, here by 1) the current count or number value so as to make an updated, larger and more accurate numerical (number) representation.   Before the final or true total or sum is found of all the things, the active (in the process), or "running" sum is called an "intermediate", or incomplete or partial sum of the entire quantity being considered. We see that to count is a process to successively (ie., summing, addition), and numerically quantify (quantize, quantization, account for, calculate) so as to represent any quantity (several, many) or amount of items or things so as to produce a single numerical (number) representation of the total (including all, complete, sum, result) amount, "(final) count" or quantity of them. Each item, whether the same or not, can be assigned and represented as a numeric value of 1. For example, a count of several, here two items can be expressed as: "one and one" in written or spoken form, or (1 + 1) in numeric representation, and the result or sum is written or expressed as: 1 + 1 = 2, and is spoken as: "one plus one, equals two".

In Europe (or the "old world"), the decimal (based on 10) number system has only been in practical use since the 1300's AD, hence about 700 years ago. With their invention of the decimal (base 10, based on a count of 10), number system and positional notation in about the 5th century AD, about 1500 years ago, the people of the country of India ("Hindu's") made many initial advances in mathematical calculations. This superior (positional) counting and recording system is the main model and influence of the Arab (Arabian, middle-east area) numbering system with its more refined and standardized number symbols, and the decimal system eventually became known as the Hindu-Arabic (decimal and positional) numbering system. This system eventually replaced most other numerical systems in use throughout the world, such as the Roman Numeral system (with symbols: I=1, X=10, V=5, etc.) which could represent any number (after a possible calculation, hence not always explicitly, for example: IX=9, VII=7, XV=15, XVI=16) but was generally good for only very simple calculations. The Roman Numeral system is lacking and insufficient for these modern times, and is rarely used. For the decimal system, the name "decimal" is given due to the "deci"or "ten" (10) counting symbols it utilizes. Many people believe this system was brought about from the fact that we have ten fingers. Before writing as we know it was invented, ancient mankind in ancient Africa was already aware of the use and need for mathematics and was already using objects such as fingers, stones or notches (a primitive digital-like, binary system with something=set=1 and-or

nothing=not-set=0) for a long lasting artificial memory or record, and a primitive form of writing to (symbolically) represent, store (record), and communicate things such as a quantity of something. For example, notches on a stick or stone, or perhaps dots on a wall were used for representing quantities or amounts of things like the number (quantity) of apples, people, and-or days. Each finger or notch symbolically (with a symbol) represented a quantity of one. By using two hands, a single and unique quantity of one through ten could be represented by using that many fingers. Thereafter, such as in India, other simple or compact written (for expression and record) symbols were invented so as to easily represent these and larger quantities. Three items (| | |) and-or fingers could be expressed as 3. Four items and-or fingers (| | | |) could be represented and-or expressed as the (more compact) symbol 4. The initial seeds of math were planted long ago in Africa and the usefulness of it were being used ever since, and it was only a matter of time till mankind created machines like calculators and computers to quickly calculate values.

Once writing was developed in ancient times before the decimal system, various written symbols (ex. such as an "X" for ten) could now represent and-or store (in writing as a written record or memory for later use) the 10 possible "finger" or "digit" representations of a quantity.   When the decimal system was developed, the symbols were simplified as these incrementing numerical values and symbols of: 1, 2, 3, 4, 5, 6, 7, 8, and 9. A quantity of "no quantity at all" or "none" had to be considered and was assigned and represented as the a numeric symbol of: 0. We call this the "zero" symbol, and it was created by the Mayan people in the "New World", the Americas. It was quickly adopted by the rest of the world as a standard decimal symbol, often replacing just a blank or empty space previously used for 0. Note that today, a value of ten (or larger) is not considered as one of the basic decimal (ten) counting symbols or numbers in the decimal number system. The ten symbols and values, typically called digits or the digits of a decimal number, used in the decimal system, where each next symbol represents a count or value of one (1) greater than the preceding value are:
0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.   0 will also sometimes be utilized to create numbers that represent quantities equal to or larger than ten (10) which is the next value after an increment (of 1) is applied to 9. 10 ("ten") is therefore now using two decimal symbols and places to represent one unique numerical value or number.   A single symbol or even a large group of these numeric symbols placed together, so as to represent a unique count (number of items) and its corresponding numeric value, is sometimes called a (unique) decimal, or (decimal represented, or representational) number. Each particular number, displayed or written, mathematically represents a particular quantity (often just spoken as simply a "number" due to the common numeric or mathematical symbolic representation of quantities) of something. Numbers used for common counting are simply called as the "counting" (quantifying or measuring) or "natural" (individual units (ex. an apple) that often occur in nature) numbers. When the symbol 0 ("zero", none or nothing) is also considered along with the counting numbers, numbers of this type are classified as the "whole" numbers. For example, if you had 5 whole or individual things and gave them all away, you now have 0 whole or individual things remaining or left. Here, 0 (as a "logical" or "thought" number) is the valid numerical symbol and representation for the quantity left or remaining.

A good method to understand and use these ten (10) basic numbers is to both physically and-or mentally visualize, think, memorize, compare, and draw each number as a group of similar objects (perhaps coins, fingers ("digits" is the equivalent old Roman word), a certain graphic shape, or simple pencil dots , and here below, arranged in arbitrary groups of 4 so as to have a nice orderly appearance and arrangement for faster recognition, counting, and so as to also reduce the possibility of error while summing:

zero	one	two	three	four	five	six	seven	eight	nine	
0	1	2	3	4	5	6	7	8	9	: English words for the first ten numbers. These are generally a form of the North European, Germanic area words.
	*	* *	* *	* *	* *	* *	* *	* *	* *	
			*	* *	* *	* *	* *	* *	* *	
					*	* *	* *	* *	* *	
							*	* *	* *	
									*	

The above dot numerical representation system can also be thought of and-or counted as:

1 2
3 4

5 6
7 8

9

A common question then arises: How then are larger quantities or values greater than 9 numerically or symbolically represented using the decimal numeric system? The answer is to use what is called positional notation. To use positional notation, the symbols are properly (standardized) arranged and set next to each other in new locations or positions of the indicated number, These positions are also called "digit" positions. A digit of a number is composed of a one-symbol number (0 to 9) of the decimal system, and its (digit) position within the overall number. The quantity or value that this arrangement of digits of the number represents (being equivalent or equal to) is not simply the sum of its digits:

Ex.   35 does not represent, and is not equal to the addition of 3 an 5,   or = 3 plus 5,   or = (3 + 5) = 8

First consider that you have a count of 8 and add 1 to it. We know that the symbol to represent this new count is the nine symbol (9):

$$\begin{array}{r} 8 \\ +\ \underline{1} \\ 9 \end{array}$$: * * * * * * * * + * =   * * * * * * * * * = 8 + 1 = 9 , "eight plus one is nine"
                                                                                     "eight and one equals nine"
                                "eight combined with one is nine"   ,   "one added to eight is nine"

If we add 1 to this value of 9, we will either need a new symbol (which doesn't exist in the decimal number system) or a combination of the existing symbols to represent this new value. What symbols that we already have should we use? The apparent choice is to let 9 "roll-over" (as if all the (incremented) symbols were all on a rotating and repetitive whee) or "reset" back to the lowest symbol of 0 (so this position can possibly be reused for representing higher quantities and for further additions) and increase the next leftward digit (considered 0 if not indicated or shown) or decimal number and-or position, by a count of 1:

$$\begin{array}{r} 08 \\ +\ \underline{1} \\ 09 \\ +\ \underline{1} \\ 10 \end{array}$$ or $$\begin{array}{r} 1 \\ 09 \\ +\ \underline{01} \\ 10 \end{array}$$: Leading 0's are alright to express some viewpoint about something
                                        since it will not add any value to the total

                                : * * * * * * * * * + *   =   * * * * * * * * * *   = 9 + 1 = 10

The number 10, spoken as "ten", or sometimes "one, zero", represents the quantity of "nine and (plus) one", (9+1). It can also be said that for the decimal (based on 10) number system, each 1 or count in the next leftward position represents (9+1) = 10 ("ten") of the previous position - of which even for itself represents 10 of any previous existing rightward position in the number. This concept is called positional notation. Notation is an accepted or standard way to indicate or express something.

The specific value of representing 10, or "ten times", more of the previous position is called the positional weight, the digits weight, or simply the weight of that position. Note that weight has nothing to do with the specific digit value (0 to 9) or count in that single position or location. The product (multiplication) of the positional count (the digit shown at that position in the number) and its corresponding positional weight yields (produces) a positional product. The total value or quantity being represented by a series of digits of a number is simply the sum of their positional products, and which produces a positional sum which represents the entire number being considered. Don't forget to add in the first digit's positional product which has a positional weight of one (or (single) "units") only. For example, in the number 275, starting with 2, which is in the position or column having a positional weight of 100 ("one-hundred", or (a) hundred), the positional products are: 2 times 100, which is 200, and 7 times 10, which is 70, and 5 times 1, which is 5. The positional sum is

therefore: 200, plus 70, plus 5, is or equals (=) 275, and this can be spoken of a in terms of its digits from left to right: "two, seven, five", or more often as: "two-hundred, and seventy, and five", or even more simply as: "two-hundred and seventy five".

Digits leftward in a number are called more significant digits since they represent a larger part of the value and positional sum. This is so since their positional weights are larger. The most leftward digit that has a value greater than zero is called the more or most significant digit (MSD). Likewise, digits more rightward are called less or least significant since they represent the least or smallest part of the positional sum. The most rightward digit having a value greater than zero is called the least significant digit (LSD). For approximations of a number, the least significant digits often become unnecessary and are not used or indicated since they only represent a small (as an insignificant small percentage or fractional part of the whole value) value and part of the total value or number. For example, if you have or counted 559 or 561 things, you could say something like "I have about, or approximately, five-hundred and sixty (=560) things". Here, a value of 1 was omitted from the sum, quantity or number, and this value of 1 is a very small value as compared to 560.

To be proficient in addition, a student must first know how to add (and subtract) any two of the basic (0 through 9) decimal numbers. Since addition is normally performed column by column (starting from the least significant digits), and the highest value for each digit is 9, the highest possible column sum is 9 plus 9 which equals 18, plus 1 possible count as a carry or "carry over" from the previous column making a total of 19. The series of whole numbers from 0 to 19 where the next number is a count of 1 higher is: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19

A good method to illustrate the concept of the addition of two numbers is to display or draw an object or group for each number, and then count or mathematically quantify and express them by successively recognizing (ie., "reckoning") the quantity that each number represents, and then increasing (repeated incrementing) the intermediate sum being found so as to find the total quantity or sum of the two numbers and-or the quantity of all of the objects those numbers represent.

Ex.   1   +   2   =   3   =   3
         *   +   * *   *   * *   * * *

As an aid to adding, or rather than use your fingers, you can discretely use dots on paper or in your mind as previously indicated for the basic decimal numbers. Usually a 4-dot system that has a square layout is a practical method.

                    *   *
                    *   *

For higher values, you can simply reuse the same 4-dot system locations, or start another one. For example:

5 would be something like this when expressed as items or dots:

     *   *   *
     *   *

Let's say you need to add 4 and 3. You can first put 4 (the initial running total) in your mind, and then tap or draw out 3 dots (of the 4-dot system) while adding one (1) to the running or incrementing (increasing) sum or total in your mind for each new dot considered. For this example we would not even need all 4 dots of the 4-dot system, but only 3:

              3

         *   *   : You can see that a "visualization memorization" method of knowing
         *   how many dots to make, and-or how many you are viewing needs to
                               available when needed.

The steps in your mind would be something like: You first have 4 in your mind, and you need to make 3 more dots to add to that value. Then make a dot and add 1, making a (updated, modified, intermediate or partial) total of 5. Then make

another dot and add 1 more making a total of 6.   Then make another dot (the final third dot) and add 1 more in your mind, making a total of 7.   The result can be said as: "four plus three, equals seven" or perhaps written as: "4 plus 3, equals 7".   This "dot system" might seem very primitive and almost silly to some, but I think many people, even advanced mathematicians, still utilize it to some extent. It's an alternative to counting with the help of your fingers, but that matter is up to you, so as to use whatever works best for you. Another tip is that you can switch the two values being summed and the result will still be the same. This method often helps to perform an addition. Consider adding 1 and 9. It is easier to start with 9 in your mind than then simply add 1 to it so as to have 10, rather than starting with 1 and then having to add or increment 9 to it.

The result of adding any number with 0 is that same number since nothing was effectively added to that number and the number should remain the same:

Ex. 7 + 0 = 7   Ex. 0 + 4.1 = 4.1

The basic "addition table" is a "look-up" table. With a table, you can view, seek and find corresponding values so as to find the answers or results already solved for. The addition table or chart is a good aid for learning the addition of small numbers.   It can even be extended as needed to included larger numbers and their sums.

ADDITION TABLE

0	1	2	3	4	5	6	7	8	9	. . .
1	2	3	4	5	6	7	8	9	10	
2	3	4	5	6	7	8	9	10	11	
3	4	5	6	7	8	9	10	11	12	
4	5	6	7	8	9	10	11	12	13	
5	6	7	8	9	10	11	12	13	14	
6	7	8	9	10	11	12	13	14	15	
7	8	9	10	11	12	13	14	15	16	
8	9	10	11	12	13	14	15	16	17	
9	10	11	12	13	14	15	16	17	18	
.							.			
.								.		
								.		

To use the table, the sum of any two numbers is at the intersection (meeting, crossing or joining) of the horizontal (left and right direction) row, and vertical (up and down direction) column. For example, if you were to add 3 and 2 you would look for a row or column of 3, and then find the corresponding (here, this means the only other, single and unique part of the row and column pair) intersection of the 2 row or column, and the result (sum) seen at their intersection will be 5.   Just the same, you could of also started by taking the 2 row or column, and then finding the intersection of the 3 row or column.   The result will be the same value of 5.   To be proficient in addition and subtraction, all the possible sums (for the basic (0 through 9) decimal number values) above should be memorized or available without too much difficulty.   If you can't memorize all of it, then the "dot method" to help add two values, as described above, can be utilized as an aid to counting and-or summing to higher values.

If you are given a number from the table, you can then find two numbers (a corresponding pair) that were, or can be, summed up together to make that given number, hence the table above can also be used in a backwards or reverse type of manner for subtraction purposes. That is, given a sum (the total) and you have or take away either the row or column value part, you can arrive at the other corresponding value part required to make that given sum.   Given one number (here the row or column number), you can find the other or "remaining part" corresponding row or column number that was part of that given sum or number.   If you take a number in the table and subtract (or "reverse count", "reverse increment" which is called decrement) either the row or column value, you can then see the other corresponding row or column value (technically called the "difference" when subtraction is being considered) needed to make that number total (or sum). For example, take value 10, and you want to subtract or remove 6 from it. Look up 10 in the addition table, and then look for either the row or column of 6. The other corresponding row or column (required to make that specific pair

with an intersection at 10) will be 4. You could say that subtraction is the "backwards" or "reverse" of addition since you are first given the result and are to find the other (corresponding) number of a pair (two) of numbers that will sum to that value or quantity.

Some examples of how to say multi-digit numbers, and the word "and" is often used for the word "plus" or "added to or with":

10 = "ten" ,
11 = "eleven" , 12 = "twelve"   , 13 = "thirteen" , 14 = "fourteen" , 15 = "fifteen" , 16 = "sixteen" , 17 = "seventeen" ,
18 = "eighteen"   , 19.57 = "nineteen point fifty-seven" or "nineteen point (and) fifty-seven hundredths" ,
20 = "twenty" , 21 = "twenty-one"   , 30 = "thirty" , 32 = "thirty-two" , 40 = "forty" , 43 = "forty-three" ,
50 = "fifty"   , 54 = "fifty-four"   , 60 = "sixty" , 66 = "sixty-six" , 70 = "seventy" , 77 = "seventy-seven" ,
80 = "eighty" , 89 = "eighty-nine" , 90 = "ninety" ,   98 = "ninety-eight" ,
100 = "one-hundred"   , 105 = "one-hundred and five" , 123 = "one-hundred and twenty-three"
1,000 = "one-thousand" , 1123 = "one-thousand , (and or plus) one-hundred and twenty-three"
10,437 = "ten-thousand, (**"and" or "plus"**) four-hundred and thirty-seven"
100,987 = "one-hundred-thousand, (and) nine-hundred and eighty-seven"
1,531,251 = "one-million, (and) five-hundred and thirty-one thousand, (and) two-hundred and fifty-one"
1,907,000,000 = "one-billion, (and) nine-hundred and seven million"

WHAT DOES MEASURE MEAN?

To measure is to find the size of something or the quantity of several things or objects. Today, numbers are the most common way of (numerically and symbolically) representing a quantity or the result of a measurement. Other than that, it is possible to use words such as: "small", "medium", and "big" as measurements, but these can sometimes be ambiguous (unclear, uncertain) to someone else. To measure is done by the act of comparing, or the comparison of one thing to another (often similar) thing that is used as a reference for the comparison. The word of "meter" is similar to the word of "measurement", and is also the name given to the basic unit of length in the metric (measurement, reference) system. A measurement value or number is the numeric result, count or sum of comparing. If the thing was a stone, or stick of a certain dimensions or qualities (ie. width, length, weight), that thing could be used as the basis (reference) of comparison or measurement for all other similar things. A common scaled ruler, having one to several unit sized markings (indications, steps) is basically a replica (a similar representation or copy) of something, such as another standardized or agreed upon reference ruler. The ruler or (measurement) reference gauge is then used as the "rule" or "standard (standard reference)" to consider and use with all similar types of measurements such as lengths, weights, etc. The simplest (non-numeric) kinds of comparison is to state that the things being compared are not the same or are equal, or something is simply more or less than something else without giving a numeric quantity of some specific size or the difference in sizes and-or amounts. To measure quantitatively, numbers and some form of a standardized measurement unit which is a specific instance and-or identification of the reference and comparison value used for a measurement. More about units or "units of measurement" will also be discussed further ahead in this book. For measuring the weight of stones, the units could be some small reference stone, and this "reference-stone" would then be the units (of reference for the measurement) of weight and for measuring the weight of any other thing, including other stone weights and so as to also have a weight reference or unit that is twice or other multiples as heavy. Even the numbers in the decimal number (counting, quantifying) system that we use for everyday counting are in reference to something, and that something is number 1, the numeric unit. Each count greater, higher, or the "next higher or larger number", is based on this reference of 1. Given a number, to find the next higher number, add or combine this 1 to that given number. To measure something quantitatively is to express how many (ie., the number of) units (the standardized thing of reference, comparison to, and measurement of) that something has, or can be compared to something considered as the reference. [FIG 1]

Letting this length be the standard (reference) unit of measurement for all other measurements.

This will be the "rule" of measurement. It is a "unit ruler" or ruler that is only one unit long.

Joining several together will help us measure larger or longer lengths more quickly and prevent error:

The points indicate the start and end of each reference unit. Since they are the same, the ruler consists of many, or a multiple, of that unit.

Indicating how many in an increasing or incremental (integer) order:
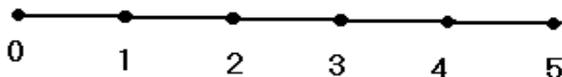
0   1   2   3   4   5   : a ruler

EXPRESSIONS

A mathematical expression is essentially another form or representation of a number, a process (such as a single formula [ie., a "mathematical recipe"]), some mathematical manipulations (such as to use the square root of a number), or steps (explicitly indicated or understood) so as to find a single number or result that represents the given expression for it. That is, the expression or representation, however it is written or contains, and the actual number that it represents, have exactly the same value, hence they are equal (equivalent, =) and either can actually take the place or be (mathematically) substituted for of the other. For example, 5, and the expression of (3 + 2) are equal, and either could take the place or represent the other. Even a representation of just a number can consist of mathematical operations (processes) such as addition (+, "plus" is the common symbol or instruction for addition), subtraction (-, "minus"), multiplication (x, "times") , division (÷ or /, "divided by"), and operands (the numbers or values to be acted upon by the operations), which when taken as one single group, it represents a single value (the given number). Some of this was previously indicated with the basic concepts of positional products and sums, hence, even a number can be "viewed", "thought of", or considered as being an (special) expression and vice-versa since they are equivalent.

> operand   operation   operand   : A basic technical format of a basic expression. The (mathematical) operation is to be applied to, or in reference to the operands (numbers). Some expressions might only have or need one operand.

The 4 most common mathematical operations and their corresponding mathematical symbol(s) are:

Operation	Symbol(s)	Example	Some Common Wording
addition	+	1 + 2 = 3	"one plus two, equals three"
subtraction	-	3 - 1 = 2	"three minus or take away one, equals two"
multiplication	x , ( ) , *,	5 x 2 = 10	"five times two, equals ten"
division	/ , -----	10 / 5 = 2	"ten divided by five, equals two" or
			"five goes into ten, two times, or twice"
			"5 can be subtracted from 10, a total of 2 times".

Ex. 30 + 5   This is an expression containing one operation and two operands. The operation is the addition operation, indicated by the "add", "plus" or "combine" symbol of (+), and the operands are 30 and 5. The symbol usually chosen to indicate an addition (operation, "add") of two values is called the addition or "plus" symbol, and was made popular by **Michael Stiple** (~ 1554) and **Robert Recorde** (~ 1557), although the concept of addition, etc., was already known for thousands of years, and the (+) symbol was beginning to be used in the late 1400's as meaning "and". Recorde is also credited to the subtraction (-) operation symbol, and the and equals (=) or equality symbol.

In a reverse type of manner, it is very possible for an entire expression to be considered as a single operand.

Ex. Considering the last example and having any expression that represents a value of 30 as an operand:

```
(20 + 10)  +  5  =  35        : (The entirety or grouping of: (20 + 10) can be considered as a single operand (to be
   30     +  5  =  35           first "simplified", or solved, to a value of 30) to be added to the operand of 5.
          35      =  35
```

```
(31 - 1)  +  5  =  35
   30     +  5  =  35
          35    =  35
```

We see in both cases that the expression within the parenthesis is equal to (an operand of) 30 when it is "simplified" (by performing or processing the mathematical operations) or represented as a single value. More is said about this below.

TO REDUCE OR SIMPLIFY AN EXPRESSION

To reduce an expression or to "simplify" an expression is to perform the indicated operations within the expression, or arranging it in another manner to effectively produce less or no further operations on the remaining operands. This process is also known as "simplification". As shown above, the simplified expression is 35 after performing the additions. In a natural conversation, you would normally say something like "I have thirty-five of something", rather than say something like "I have thirty, and then I have five more of that something" - which is sort of an indirect or roundabout way of saying a total and single value of 35.   So then, we often see the need to simplify things such as mathematical expressions by solving them or making them uncomplicated, simpler, and nicer (easier to work with).

Ex.   Simplify the expression: 2 + 3 + 4

The simplified form of this expression is 9. That is, the entire value of : (2 + 3 + 4) = 9.

As previously indicated, even basic numbers can be expressed (ie. in a mathematical expression) as a positional sum (of the numbers' digits, or digits of that number). The positional sum is an expression containing the sum of each digit times the positional (decimal system) weight corresponding to that digit or (column) position within the given number.

Ex.   Show 35 as a positional sum of its digits.

Here, we are to write an expression for 35. Use the positional notation concepts summarized below:

Positional Weight   =   Is 10 times the previous (next rightward) position's weight.
Positional Product   =   The value in that position times its corresponding positional weight.
                               You can consider this expression or its value as an operand in the final sum.
Positional Sum   =   Sum of all positional products. This expression represents and equals
                                the total value represented by the number.

Taking 5 as the first digit in question, the weight for this position is understood as 1. This is known as the (basic) "units" (of measurement) or "ones" position. Note that even this weight too is 10 times the previous (next rightward) digits weight which is 0.1. That is, 10 times 0.1 is 1. The second digit will also have a weight of 10 times the previous positions weight. Hence , its weight is 10 times 1, which is 10. This is known as the "tens of one" or simply the "tens" position. Breaking each digit into its' positional product (separated by the addition symbol needed for the positional sum):

```
           35
3 x (10)   +   5 x (1)        : here, showing the expression for the sum of the positional products, or:
   30      +      5           : here, after multiplying for some simplification, showing a positional sum
                                 (of the positional products) expression
```

Note, we now know the second digit does not represent 3, but it actually represents 30, or 30 units, and in strict mathematical or numerical terms, this would be 30 ones.

Ex.   Show the number 3215 as a positional sum.

Taking 5 as the first digit, the weight of the first digit is one (1). The weight of the second digit is ten times that of the previous (first digit's) weight or 1x10=10. The third digits positional weight is 10 times that of the second digits positional weight or 10x10=100. This is known as the "tens of ten", "hundreds of one" or simply "hundreds" position. Likewise, the fourth digits positional weight is 10 times that of the third digits positional weight or 100x10=1000. This is known as the "tens of hundred", "thousands of one" or simply the "thousands" position.   Expressing this information as a sum of positional products of 3215, we now have:

3215

3   2   1   5

(3 x 1000) + (2 x 100) + (1 x 10) + (5 x 1)   : an expression for the positional products. After some simplification:
  3000   +   200   +   10   +   5   : expression for the positional sum of 3215

POSITIONAL WEIGHTS IN THE DECIMAL NUMBER SYSTEM

The discussion above about expressing a number as an expression has lead us to this. The names for the commonly used weights in the decimal system are summarized below. The commas used when writing the numbers, being a visual aid, are optional. Here, the value of one (1) is used only to indicate the position in question since the value in any position can be between and including 0 through 9.

Whole Portion Weights (In Increasing Weights)		Fractional (values smaller than one) Portion Weights (In Decreasing Weights)		
ones or units	1			
tens	10	tenths	0.1	(or 1/10)
hundreds	100	hundredths	0.01	(or 1/100 , and so on:)
thousands	1,000	thousandths	0.001	
ten-thousands	10,000	tenth-thousandths	0.000,1	
hundred-thousands	100,000	hundredth-thousandths	0.000,01	
millions	1,000,000	millionths	0.000,001	
ten-millions	10,000,000	tenth-millionths	0.000,000,1	
hundred-millions	100,000,000	hundredth-millionths	0.000,000,01	
billions	1,000,000,000	billionths	0.000,000,001	
etc.		etc.		

Ex. 5.2 = "five point two"
Ex. 0.0015   or with helpful commas as: 0.001,5

This value above would be commonly spoken as: "one, point, five-thousandths". It is also possible for the value to be described in many other ways such as:
"one-thousandth and (plus) five ten-thousandths"
"one-thousandth and five-hundred millionths"

Considering just digits, rather than digits and their positional weights, this can be spoken or thought as::
"zero, (decimal) point, zero, zero, one, five".

The decimal point (written as a period character = .) is a separator that separates the number of whole units from the number or value that numerically represents some part, fraction or portion of only 1 single unit. Before the decimal point was introduced by John Napier, and who studied logarithms, other symbols and methods were used. A portion or fraction (part of) is only a part of 1 whole unit. A fractional part of a number is that part less than 1. The decimal point also means to the effect: to add this part or portion of 1 unit also to the total. A number that contains a decimal point, and a fractional value, is sometimes loosely spoken as being a "decimal number". The parts of a decimal number can be expressed or considered as:

    whole_value   decimal_point fractional_value
    whole_value . fractional_value   : Ex: 100.5 = "one hundred point five"
    whole_value and fractional value
    whole_value plus fractional value
    whole_value + fractional value

    A value can be a complete fractional value where that value is less than 1. That is, the value
    does not even have a whole portion except the value of 0.

    Ex. 0.5   : The whole portion is 0, and the fractional portion is 0.5 = "zero point five"   =   "five tenths (of one)"

    More advanced mathematics (that will be shown) shows that each positional weight can be represented as an

indicated power of 10. In an expression form, this is a base of 10 with an exponent. The exponent increases by one for each digit position leftward and likewise decreases by one for each digit position rightward. The weights in the decimal number system using exponential notation, and the common decimal equivalent form are:

$$. . . ,10,000 , \quad 1000, \quad 100, \quad 10, \quad 1 \quad . \quad 0.1, \quad 0.01, \quad 0.001, \quad 0.000,1, \quad . . .$$
$$. . . , \quad 10\char`^4 , \quad 10\char`^3, \quad 10\char`^2, \quad 10\char`^1, \quad 10\char`^0 \quad . \quad 10\char`^{-1}, \quad 10\char`^{-2}, \quad 10\char`^{-3}, \quad 10\char`^{-4}, \quad . . .$$

Note for example :   10^2 = 10 x 10 = 100   : basically, the exponent means repeated multiplication, here twice (2)
              10^3 = 10 x 10 x 10   = (10 x 10) x 10 = 100 x 10 = 1,000
              10^4 = 10 x 10 x 10 x 10   = (10 x 10 x 10) x 10 = 1,000 x 10 = 10,000

Note also that 10^0 is the weight of the ones position. That is, 10^0 = 1. Notice that for the whole units weights, that each greater weight essentially has "another zero" placed onto the end of the previous weight. In fact, this is a simple method to multiply any value by 10; simply put a zero at the "end" (in the "ones position") of the number (also move the decimal point one position rightward of course). Ex. 7x10 = 7.0x10 = 70.0 = 70, Ex. 3.52x10 = 3.520x10 = 35.20 = 35.2 Notice also that when 10 is the base of a power, that the exponent indicates the number of zeros in the decimal equivalent. For example 10^4 equals one with four zeros after it or 10,000.

Ex. Here is the positional sum of 0.0015

$$0\,(1) \; + \; 0\,(10\char`^{-1}) \; + \; 0(10\char`^{-2}) \; + \; 1(10\char`^{-3}) \; + \; 5\,(10\text{-}4) \qquad \text{or perhaps more commonly:}$$

$$0 \quad + \quad \frac{0}{10} \quad + \quad \frac{0}{100} \quad + \quad \frac{1}{1000} \quad + \quad \frac{5}{10,000} \qquad : \text{here, the first 0, is for the "ones" whole units position}$$

For an extra note, the first three values in the sum are equal to 0 after simplification (ie, here solving or simplifying the fraction; no or 0 parts of anything is always 0. Also, anything times 0 is always 0), and for the sum, this then leaves us with just:

$$\frac{1}{1000} \quad + \quad \frac{5}{10,000} \qquad \text{combining (summing) these fractions to a single (fraction) value -}$$
the method which will be shown further in this book, we have:

$$\frac{15}{10,000} \qquad : \text{"fifteen ten-thousandths"} \quad (= 0.0015 \text{ after performing the division expression })$$

Here is some more explanation of the weights that are less than 1. Let's begin at the weight of 1000. Since any weight is a product (result of multiplication) of and factor of 10 times more than the previous (lesser) weight, previous weights are always a factor of 10 times less, therefore divide this factor of 10 out of a weight to find the previous weight.   Dividing 1000 by 10 results in 100. Dividing 100 by 10 results in 10. Dividing 10 by 10 results in 1 which is the weight of the "ones" column or position. Dividing 1 by 10 and expressing the weight in various forms, we have:

$$10\,\overline{\smash{\big)}\,1} \quad = \quad \frac{1}{10} \quad \text{or} = \quad 0.1 \quad \text{or} = \quad 10\char`^{-1} \qquad : \text{weight of the tenths column}$$
: a "tenth of one"

$$\frac{1}{10} \quad = \quad \text{"} \frac{one}{ten} \text{"} \quad \text{or} \quad \text{simply: "one-tenth" or a "tenth" (of one)}$$

Dividing the weight of the tenths column by 10 we have the weight of the hundredths column:

$$\frac{\frac{1}{10}}{10} \quad = \quad \frac{\frac{1}{10}}{\frac{10}{1}} \quad = \quad \frac{1}{100} \quad = \quad 0.01 \quad \text{or} = 10\char`^{-2} \qquad : \text{weight of the hundredths column}$$
: a "tenth of a tenth"
:Note, 0.1/10 = 0.01 = 1/100

$$\frac{1}{100} = "\underline{\quad one \quad}_{one\text{-}hundred}" \quad \text{or simply: "one-hundredth" or a "hundredth" (of one)}$$

Many other counting systems such as hexadecimal (has a base 12 instead of the common 10 as used in the decimal system, and often used with machine language/computer programming), octal (base 8), and the binary (base 2) number system that is used for understanding computers and other digital electronic circuits, use the same concepts of positional notation as the decimal (base 10) number system. The basic difference among these systems is that they use more or less counting symbols. The ADVANCED TOPIC section of this book contains a short discussion about the binary number system.

UNITS OF MEASURMENT

To add properly, you must add things that are similar. These similar things are mathematically referred to as having "similar" or "like" (ie. same or common) units (of measuring or reference). Consider the numeric part of a value as: "how many", and the units part of the value as: "of what". Units are said as being that to which the numeric quantity or value (expressed and represented mathematically as a number) is in reference or measurement to. For example, it could be an inch, a mile or meter of which a measurement is based upon or in reference to. For example, you can add (mathematically, or even physically combine or group together similar items, such as into a container) a quantity of oranges to another quantity of oranges, and get a larger net (final sum) quantity of oranges. That is, here the like units (or "identifiers" or "type") of the items or objects being added are oranges. When you see the word "units" used in mathematics, it is a short version of the phrase: "units of measurement".

Ex. 5 oranges + 5 oranges = (5 + 5) oranges = 10 oranges

This mathematical writing or expression also indicates that you can simply add the numerical values or quantities of "like things" that have "like" or similar or units, and the resulting sum will have those same units.

You cannot add a quantity of oranges to a quantity of apples or anything else, since their units are unlike or not similar, and get a larger quantity of either apples or oranges. Their will be more fruit, but the quantity of either apples or oranges will still be the same even when placed in the same basket.

Many times you will be working with "plain" or unitless numeric values. You can consider their units as being "units" or "wholes" as in an entirety or completeness of something, or as "ones" for a more numeric sense.

Ex. 5 + 2 = 5 ones + 2 ones = 7 ones
Ex. 20 + 10 = 20 ones + 10 ones = 30 ones

When the quantities or values being added are measurements, their units are correctly called units of measurement. Again, to add properly, their units, or the unit, of measurement must be the same. If they are not the same, such as when adding "apples and oranges" (unlike units), it is possible to convert one value (its numeric part) to an equivalent representation that will have the same units of measurement which is needed for their combining (addition) and to express the result as having just one unit of measurement or reference.

Ex. 5 ft. + 10 ft. = 15 ft.   : ft. = feet, which is a specific length unit, or unit of length
Ex. 3 ft. + 12 in. =   : in. = inches which is another length measurement unit :

This expression above cannot be simplified as is since the units of feet (ft.) and inches (in.) are unlike. One of the units must be changed to that of the other. For example, the quantity of inches can be converted to its equivalent measurement with units of feet, or the quantity of feet can be converted to its equivalent measurement with units of inches.

Since 1 ft. = 12 in.,   or likewise,   12 in. = 1 ft. ,   and substituting this into the expression above:
         3 ft. + 12 in. = 3 ft. + 1 ft = (3 + 1) ft. = 4 ft.

Note, that which is said about having similar or like units for addition, also holds true for subtraction and other mathematical operations.

Occasionally, a values units must be the combination of unlike units that were used during the mathematical operation or definition. The concept of the (necessary) use of several units (as like one new unit) is sometimes called "derived units", and may or may not be identified as a single unit name. For example, speed (or average velocity) equals distance, which has units of measurement of length, divided by a time value with its own units of measurement of time.

speed (length units/time units)   =   Distance (length units)   : ex. of a value that can have multiple or
                                              Time (time units)   **derived units**.

Hence, the units for speed is a length unit divided by a time unit which are unlike units of measurement.

Ex. 50 mi./hr. = 100 mi.   : perhaps you went 100 miles in 2 hours, your average speed is then 50mi/hr
                          2 hr.   Why is the word average used? Because usually, or normally, you sometimes
                                                 will go slower, and perhaps even a bit faster than 50mi/hr during your travel.

The units for speed in this example is (miles/hour), and the speed in this example is said to be "fifty miles per (1) hour".

Sometimes, combinations of units may be assigned a single unit name. For example, in physics (physical science) and electronic studies, a quantity or number of charged particles (Q), typically electrons from atoms, is measured in reference to a specific amount of charged particles called a coulomb (C) of charge. Hence, the units for charge are coulombs (C). Current (I) is defined as the amount of this charge passing a point per second (s) of time (T):

$$I = \frac{Q_c}{T_s}$$

Hence, the units of measurement for current is coulombs/second or (c/s). This quotient of c/s, or 1 coulomb per 1 second, is commonly assigned the new and single (derived) unit name, or identifier, of amperes (A), that is (c/s) = A.

Ex. $\frac{10c}{5s}$ = $\frac{2c}{1s}$ = 2 c/s = 2A

When dividing quantities or values that have like or similar units, the units are effectively "canceled out" (and-or reduced to 1, since anything times 1 does not change that value) as if they were some value or algebraic variable (an alphabetical symbol(s) representing, and as a "placeholder", a numeric value or quantity) by themselves. The result is said to be unitless or a ("plain", literal, "strict" and specific value only) numeric value.

Ex. The ratio (which essentially is, and resolves to a quotient, which is a division result) of 10 ft. to 5 ft. is 2, and not 2 ft.:

$\frac{10 \text{ ft.}}{5 \text{ ft.}}$ = 2   since 5 ft. x 2 = 10 ft.   also, = (5 x 2)ft. = 10ft.

The result is not indicated as 2 ft. since 5 ft. x 2 ft. = 10 feet x feet = 10 ft.^2   : "ten feet squared" or more
                                                                                                                          commonly as "ten square feet"
  Note: $\frac{10 \text{ ft.}}{5}$ = 2 ft. ,   since 2 ft. x 5 = 10 ft.

That is, the units of a result of a multiplication or division operation where only one operand has units, will have those indicated or expressed units, and for the division operation, the units must be in the "top number" or dividend operand.

Even the plain, unit-less numbers that we use are in actually reference to the value of 0. If given the number 5, it is a value that is in reference to 0, and is a count of 5 counting numbers and-or positions greater than 0 on the continuous, unbroken "string" or "line" of numbers called the number line.

BASIC MATHEMATICAL OPERATIONS

ADDITION

When we "add" quantities, we are performing a mathematical operation called addition which is essentially joining, mixing-in, summing, or combining; usually making something greater or larger. The standard symbol for the addition operation is the "plus" symbol (+). But remember, the quantities being added must be "like" or "similar" quantities. That is, they must have the same units or units of measurement to be added, otherwise, there will be an awkward "apples and oranges" situation where they can't be combined as needed to produce one numeric result having just one of those units of either apples or oranges. Formally (some accepted standard), the operands or values of an addition operation are called the augend and addend. These both are infrequently called summands since they compose, or are part of, the sum (result of a summation or addition process) value. The result of an addition operation is called the sum which means the total value of all the combined numbers.

number + another number = sum : This can be read or spoken as: "number plus another number equals (the) sum".
"sum" is sometimes called the "total"

Ex. 5 + 2 = 7 : "five plus two equals seven" or "the sum of 5 and 2 is seven"

augend + addend = sum : formal or fundamental addition (symbolic) expression, expressed horizontally, or:

   augend   : addition expressed vertically   : augend is a word somewhat similar to augmentation,
+ addend   and other similar words, most containing the letters aug,
   sum   and has the basic meaning of increase or change. The words augend and addend are not heavily used today. Sum has its origins in the word summit as in the (noun, thing) peak (highest part) of a mountain, and (verb, action, "summation") to go to the top or peak.

When the units are different, as in feet and miles, a conversion of one of the quantities can be made so that the units are identical for the addition.

Ex. Add 5280 feet and 2 miles.

5280 ft. + 2 mi. : The above placed into or expressed as a mathematical expression

Since the values being added have different or unlike units, a direct addition cannot be performed such as would incorrectly result in either 5282 feet or 5282 miles. By converting (representing) 5280 feet as (equivalent to) 1 mile, we will have values with like units, and the addition can then be properly performed:

5280 ft. + 2 mi.   After converting or expressing 5,280 feet to it's equivalent value in miles (with miles units), we have:
  1 mi. + 2 mi.   : The operands now have like units and can therefore be combined or added.
  (1 + 2) mi.   : An equivalent expression that indicates that the numeric parts (of the similar units) will be added.
   3 mi.   : After adding the numeric part of the values with like units, and keeping and expressing the
              same, common, identical or like units in the total.

Below is an example that shows mathematical expressions for both horizontal (left and right) and vertical (up and down) addition. Vertical addition, because of its practicality, is usually the preferred method to calculate the sum of any two numbers since corresponding digit positions can be aligned into columns having the same weight value. Adding the digit values of each column produces a sum having the same weight as the columns weight. The addition begins in the rightmost column. Showing horizontal addition, or other mathematical operations, is a good way to get comfortable with writing or presenting mathematical expressions or formulas (standardized or commonly accepted (known) expressions).

Ex. Add 35 and 24

Horizontally: 35   +   24
                  (3 x 10) + (5 x 1) + (2 x 10) + (4 x 1)   : expression showing the sum of the positional products
                     30   +   5   +   20   +   4   : after some simplification we have this positional sum
                              59   : since: 30 + 5 = 35 ,   35 + 20 = 55 , and   55 + 4 = 59
Vertically: 35
             + 24
               59

checking (to help verify, locate any errors, and accept the result):

sum = (sum of tens columns positional products)   +   (sum of ones columns positional products)

                  ((3 x 10) + (2 x 10)) + ((5 x 1) + (4 x 1))
                       (3+2) x (10)   +   (5+4) x (1)   Or: (30 + 20) + (5 + 4)
                          5 x 10   +   9 x 1   50   +   9
                            50   +   9   59
                                    59

        A carry or carry-over is produced when the sum of corresponding (same column) digit positions exceeds 9 as in the decimal system. This was also mentioned during the discussion of positional notation.

Ex.   35
         + 9

Here, by first adding the rightmost or "ones" column, we get 14 which is two digits, hence greater than the maximum digit value of 9. Simply "carry" or move the second digit (the 1), which has a positional weight of 10, to the second column which has a (corresponding) positional weight of 10, or 10 more than the previous, and continue to add.

        1   : this 1 could be shown as 10, but the zero is often not written since it does not contribute to/affect the sum.
       35   Or the rarely used   35   : In this method, you need not start
      + 9   non-carry method: + 09   adding at the ones column.
       44   (summing corresponding   14   : (5 x 1) + (9 x 1) = 5 + 9 = 14
                     positional products):   + 30   : (3 x 10) + (0 x 10) = 30 + 0 = 30
                                                      44

Again, each unit (1) of carry is 10 of the previous digits positional weight.

checking:
                          35   +   9
                  ((3 x 10) + (5 x 1)) + (9 x 1)   with some simplification of the expressions:
                  (30   +   5) +   9
                         30 + 5 + 9   : expressed as a sum of the positional products
                          30 + 14
                            44

Note above, in the expression (30 + 14), that this is an intermediate simplified expression showing that the 5 ones and 9 ones were combined (added). Often intermediate expressions like this are not shown, but it can assist you in understanding or remembering the steps you took to simplify an expression or to solve an equation (two different expressions that are equal each other).

Another, more popular, method to check an addition problem is with subtraction. Subtracting either operand from the sum

should yield the other operand of the addition operation. This was mentioned during the discussion about the "addition table". Using the last example:

```
    44      or:      44            : instead of "carries" (of 10) as used with addition, we now have "borrows" used
  - 35            -   9              with subtraction.  Essentially, these are reverse concepts of each other.
     9               35              In common speak, people may say "a borrow of 1", or "a carry of 1", however,
                                     this 1 is actually 10 of the smaller weight, next rightward position that receives it.
```

Extra, an advanced technical note: The "(calculation and result comparison) checking process" for an addition operation can be formally expressed as subtracting either operand from both sides of the addition (operation) expression or formula:

augend + addend = sum   subtracting the addend value from each side:

augend + addend - addend   = sum - addend   since subtracting or removing a value from itself (the
                                                        same value) results in, and is equal to, 0:
                                                        value - value = 0,   therefore,   addend - addend = 0 :

augend + 0 = sum - addend   since adding 0 to any value or sum does not change it, we can express this as simply:

augend = sum - addend   by "switching sides" (for clarity) this can be expressed as the examples indicated:

sum - addend = augend   in more generalized or representative expressions:

From: sum = value1 + value2   therefore,   sum - value1 = value2   and   sum - value2 = value1

SOLVING FOR AN UNKNOWN VALUE OF AN ADDITION PROBLEM

This discussion is about solving addition problems where the sum is not necessarily being found, but one of the numbers (ie. operands; formally the augend or addend) that is part of the given sum. As you will see, the method may seem trivial at first, but has far reaching potential as the basis for solving equations ("equa" is from "equate" and "equals" meaning that which is equivalent or the same value) as in math and-or algebra. To equate is to make or state an equivalence to, usually with the "equals" (=) mathematical symbol, and it can be interpreted as "is", "is this", "is equals or equivalent to this", "is the same as this". Some of this discussion was mentioned in the last topic.

If you add 2 to 3, you get 5:

   3 + 2 = 5   : the (entire) expression of: (3 + 2) is being equated to 5, and vice-versa.
                             Commonly spoken as: "three plus two equals five", or
                                       "three plus two is the same as five"

Hence, if you take or remove that 2 back out of the total sum of 5, by subtraction, you should arrive at the other number that is part of the sum (here, 5), which is 3. Or, if you take 3 out of the 5, you should arrive at the other number which is 2.

   5 - 2 = 3   and   5 - 3 = 2

Therefore, we see a method of solving problems such as the following that is identical to 3 + 2 = 5 that was first given above , except that one operand here is now unknown and is to be found:

   3 + ____ = 5   or as: ____ + 3 = 5   : after simply switching the operands being summed,   and:
   2 + ____ = 5   or as: ____ + 2 = 5

Instead of ____ which means "blank", "unknown", or "find this value", you might also have the word "number", or a letter symbol such as used in algebra for a symbolic (ie. a placeholder or representative) number, such as ?, n , x, or whatever happens to be used to represent and express a value that is to be found.

Ex. Solve this equation :   ____ + 4 = 7   : To solve this equation, we are to find the number which makes
                                               the equation true, hence we are to find the correct "unknown"
                                             value that makes the equation true so as to have a balance
                                             (equivalence) on both sides of the equals symbol. To have
                                             this balance, both sides must always be kept equal.

To find the other specific or corresponding number that is part of the sum, remove or subtract the known or given (operand) number, the 4, which is the known other part (operand) that made that sum, from 7 to see what is the remaining or other part that made up or is part of that sum of 7:

   7 - 4 = 3   , hence,   ____ = 3   : we find that the "unknown" or "unknown value" equals 3

Checking by substituting the value of 3 into the "unknown values" placeholder or position:

  ____ + 4 = 7

   3   + 4 = 7   : this equation is correct since both sides, or expressions, of the equality sign, equal 7:
     7   = 7

In the formal methods of equation solving, which will be describe further ahead in this book, 4 will actually be subtracted from the total values on both sides of the equals signs, and not just from 7, during the process of solving for the unknown

value. This formally keeps both sides of the equation always in "balance" (equivalent or equivalence) when they are both mathematically acted upon or processed together:

| left-side | = | right-side | : a simple description of an equation. Each side can be any valid expression, and that both expressions equal the same value or number. |

$$\underline{\quad} + 4 = 7$$: the (incomplete, unsolved) equation in question

$$\underline{\quad} + 4 - 4 = 7 - 4$$: expressing the subtracting of 4 from both sides of the equation

$$\underline{\quad} + 0 = 3$$ since any value, known or unknown, plus (+) zero is itself or still the same value, this can be expressed as:

$$\underline{\quad} = 3$$: this indicates that the unknown value is equal to 3, and it is the same result as shown above

If 4 was not also subtracted from the left-side, and only subtracted from the right side, we would of created an unacceptable "inbalance" of the equation, and which would lead to an incorrect result than the true or actual result such as when given:

$$\underline{\quad} + 4 = 7$$

$$\underline{\quad} + 4 = 7 - 4$$: an incorrect method to solving this equation for an unknown, because 4 is being subtracted from just one side of the equation, rather than both:

$$\underline{\quad} + 4 = 3$$: more advanced math (where signed numbers are considered) will show that the solution here is: $(3 - 4) = -1$. and -1 is obviously not equal to 3, the true result being found. In short, an unbalance (non-equivalence, error) and incorrect result was created by not applying the same mathematical operation to or upon each, and entire, side or expression. The application and incorrect process used here could be said as being wrong since it was only "one-sided"; applied to just one side of the equation, and not correctly to both sides of the equation.

ADDITION WITH FRACTIONAL PARTS

The decimal point (.) within a number is used to visually separate the number of "whole" (complete) or unit values from that which is a fraction ("part of", or "portion of", you can also think of the word fracture [break/shatter into pieces or smaller parts]) of one single "whole" or unit. That is, (true) fractional parts of 1 are always less than 1. A piece of a whole thing is always less than the whole or entirety of which it came from. If you like, you can consider the decimal point as meaning "add this (fraction part) also", or as a small hidden plus (+) symbol. For example, 5.2 can be considered as 5 + 0.2, or 5.0 + 0.2. This is also the positional sum expression for 5.2. Addition (or subtraction) of values that may contain fractional parts is performed in exactly the same manner as those without.   Just be sure to add corresponding columns, that is, they must have the same weight (as when units must be identical) to added properly.   A quick and simple method to do this is to align the decimal points of the two or more values (and of the resulting sum), and then all the corresponding columns will be in alignment and easier to work (process, add, combine) with.

Note that even if one or both numbers do not have a decimal point, you can still place one there. Adding leading or trailing zeros to a number will not change the value of a number.

$$4 + 2 = 4.0 + 2.0 \quad , \quad 4.3 + 2 = 4.3 + 2.0$$

Ex. Add nine-tenths (0.9) and one-tenth (0.1)

```
     1
   0.9              : nine-tenths,  plus
 + 0.1              : one-tenths,  equals
   1.0              : (nine + one) tenths,  or (9 + 1) tenths  =   ten-tenths or one
```

Ex. Add 1043 and 2.1005

```
   1043.0000        : Any leading zeros in the whole number part, or trailing zeros in the fractional part
 +    2.1005          are insignificant and will not change the numbers or the result  after using them in
   1045.1005          mathematical operations.
```

SUBTRACTION

Subtraction, or to "subtract", is to reduce, remove, deduct, take-away, take-out, or decrease a value, quantity or its representative ("quantized", "quantization") numerical value or number by some amount. The accepted symbol for a subtraction operation is the "minus" (-) symbol. The word minus is based up or "rooted" in the words and meanings of: "tract" (to pull, remove or take away from), "smaller", "subject", "piece" ("sub" word prefix), "less" or "lesser", "minimal", "minimum", "minor" (for "minus"), and "diminish" (to "lessen", "reduce"). The general "formula" for a subtraction operation is:

      minuend - subtrahend = difference   : The difference is sometimes called the remaining or value "left over". In monetary use, such as at a store, it is often called the "change" value.   Change has another more important or technical meaning such as how much did a value, such as the minuend value, change in value so as to be equal to that difference value. The answer here is that the subtrahend value is equal to that change value of the minuend value. The word difference is rooted in the words "different" and "diverse".

Minuend and subtrahend represent the operands of a subtraction operation or problem, and the difference represents the result or output of the subtraction operation. You can think of the subtrahend as that which is being subtracted. Actually, a difference or difference value is the simplest, and perhaps the greatest mathematical measure or representation of the comparison of two values. Given the values of the minuend and subtrahend, we can say that the minuend is difference larger than the subtrahend, and the subtrahend is difference smaller than the minuend. The mathematical measure of the comparison of two similar values by division is called a ratio which yields a ratio value or result, and this will be discussed further ahead in this book, and it basically indicates how many times larger one value is than the other value, or how one value changes when another value changes by some amount (usually by 1, so as to have a fundamental rate or ratio).

Note that by "switching (the left and right) sides" of the equality sign, or equation, that this could also be correctly expressed or written as:

difference = minuend - subtrahend

To advance your math skills, and in particular, for solving for some unknown value in an equation, you should try to feel comfortable with switching both sides of an equation.

A good method to illustrate the concept of subtraction is to display a group of objects that represents the larger number (the minuend) and remove (perhaps circle or cross-out the objects if paper is being used) the number that is to be subtracted, the answer, formally called the difference, will be that which is "left over" or remaining.

Ex. Show 3 minus 2 = 3 take away 2 =   ✳̸ ✳̸ ✳   = 1   , and-or:

        3   -   2   =   1   : "three, take-away two, is one"
      * * *   -   * *   =   *

To be good at subtraction, you first need to know the sum of any two of the basic (0 through 9) decimal values. These values may represent groups of objects. For example if you were to add a group of 3 and a group of 4 together, you could create a new and larger group of 7 objects:

        3 + 4 = 7   : (* * *) + (* * * *) = (* * * * * * *)

If you had a basket containing 7 apples, you could say, for example, that in the basket there is a group of 3 apples, and a group of 4 apples, and a total of 7 apples if you added those two groups together, perhaps to make just 1 group. If you took out, or mathematically subtracted, either group of apples, you would be left with the other group of apples. To

illustrate this point, "grouping symbols" (here, parenthesis) are used around each group of which can be regrouped into two smaller groups, of which a group can be removed or taken away during the subtraction:

$$7 - 3 = 4 \quad : (*\,*\,*\,*\,*\,*\,*) - (*\,*\,*) = (*\,*\,*\,*)(*\,*\,*) - (*\,*\,*) = (*\,*\,*\,*) = *\,*\,*\,* \quad \text{and:}$$
$$7 - 4 = 3 \quad : (*\,*\,*\,*\,*\,*\,*) - (*\,*\,*\,*) = (*\,*\,*)(*\,*\,*\,*) - (*\,*\,*\,*) = (*\,*\,*) = *\,*\,*$$

When both groups (here, groups of apples) are combined together into one new (and larger) group, either physically or mathematically with addition, you will then have the sum of objects of both groups, and the number of objects in the new combined group:

$$4 + 3 = 7 \quad : (*\,*\,*\,*) + (*\,*\,*) = *\,*\,*\,* + *\,*\,* = *\,*\,*\,*\,*\,*\,* = (*\,*\,*\,*\,*\,*\,*)$$

Above, parentheses were used to express each group being considered. Parentheses ( ), and brackets [ ], are common grouping symbols used in math. Grouping symbols are sometimes used to clarify an expression, and-or to treat it as like one value.

Like addition, subtraction problems are often expressed horizontally, but are usually solved vertically "by hand" (manually):

```
     minuend
 -  subtrahend
     difference
```

To check the result (the difference) of a subtraction problem is to sum (add) the subtrahend to the difference. If this result is equal to the minuend, the result is correct. As done in addition, in subtraction, values or quantities with only like units can be subtracted.

In "vertical subtraction", to borrow is to "reverse carry". That is, the borrow of 1 is equivalent to 10 times the positional weight which receives it.

Ex.
```
    45
 -   7
```

Here, without using signed number concepts that will be discussed ahead, 7 cannot be taken from or out of 5 since "5 does not have enough", or 5 is simply less than 7. We can make enough by borrowing 1 (or 1 tens) from or "out of" the second digit (the 4). We are actually taking 10 ones since the positional weight there is 10. The digit value which gave us these 10 ones must then be reduced by one. These 10 ones borrowed are added to the 5 to make a sum of 15. Now, 7 can be taken from 15, leaving 8.

checking: 15 - 7 = 8   since   8 + 7 = 15

```
   3
  4(15)
 -    7
   3 8
```

checking:
```
            1
          38        :  difference
       +   7        : + subtrahend
          45        : =  minuend
```

Ex.   Subtract 7 from 105

```
    105
 -    7
```

Obviously, 5 is less than 7. Since the second digit position has no value other than 0 in it, there is nothing immediately there to borrow from. To place a value there to borrow from, the second digit can borrow from the third digit (the 1 shown). Borrowing 1 from the third digit leaves zero. Since the weight of the third position is 100, it is actually borrowing 100 which is 10 times that of the previous digits weight (the second digits weight) of 10.

```
    0
    1(10)5
 -       7
```

Now that there is a value in the second digit position, the first digit can borrow from it. Borrowing 1 from the 10 leaves 9:

```
    0  9
    1(10)(15)
 -         7
    9   8              : difference  =  minuend - subtrahend  =  difference
```

checking:
```
               1
             98         :  difference
           +  7         :+ subtrahend
             105        :  minuend           : minuend  = difference + subtrahend  =  minuend
```

When several zeros are "in a row", and you need to borrow, start the borrowing at the first leftward non-zero value.

Ex.   Subtract 5 from 3002   Showing this with enumerated (numbered) steps of the calculation process:

```
1)   2                2)   2 9
     3(10)02               3(10)(10)2              : showing some intermediate steps
 -        5            -          5
```

```
3)  2 9  9            checking:    111
    3(10)(10)(12)                 2997
 -           5              +    5
    2 9  9 7                     3002
```

The result of subtracting 0 from any number is that same number since nothing was effectively subtracted or removed:

   Ex. 7 - 0 = 7

If any number is subtracted from 0, the results are somewhat similar to that mentioned above, but since the concepts of signed numbers, particularly negative numbers, have not been discussed yet in this book,, examples will not be presented here.   We all seen or heard of 0 degrees on the thermometer, and even "negative temperatures" where the temperature is actually less than the reference value of 0. For example, -10 ("negative ten") degrees Celsius or Fahrenheit. Anyway,

without these more advanced concepts, it is common sense that nothing can actually be physically (as opposed to logically) subtracted or taken away from nothing or 0.

Extra: The checking process for a subtraction operation can be formally expressed as:

 minuend - subtrahend = difference   ,   after adding the subtrahend to each side of this equivalence (same values):

 minuend - subtrahend + subtrahend   =   difference + subtrahend

If you subtract value2 from value1, and then add that value2 back, the net result is that you have added or subtracted nothing or 0:

value1 =   value1 + value 2 - value2   = value1 + 0 = value1   = value1 - value2 + value2 = value1 - 0 = value1

minuend = difference + subtrahend   or =   difference + subtrahend = minuend   here is a generalization:

value1 + value2 = value3   : a generalized addition formula,   solving for value1 by subtracting value2 from each side:
value3 - value2 = value1   : a generalized subtraction formula, also:
value3 - value1 = value2   solving for value3, here, as the minuend, from either of these two equations:
value3 = value1 + value2   : minuend = difference + subtrahend   or = subtrahend + difference

SOLVING FOR AN UNKNOWN VALUE OF A SUBTRACTION PROBLEM

This discussion is about solving subtraction problems where the difference is not necessarily being found, but one of the numbers, either the minuend or subtrahend operand of a subtraction operation or problem. This discussion is similar to the previous discussion of SOLVING FOR AN UNKNOWN VALUE OF AN ADDITION PROBLEM.

For example, if we take 3 away from 7, the value of 7 is reduced to a value of 4:

7 - 3 = 4

If we do the exact reverse and add (rather than subtract) or combine this value of 3 (that was taken out of the 7) back with or to the value of 4 (the difference, or "remainder" if you will, of which 7 was reduced or changed to), we should effectively arrive at the original value in which 3 was subtracted from.

4 + 3 = 7

Now, with this information in mind, if given:

7 - ___ = 4   : "Seven minus some number is equal to 4.". We know that 4 added to some number, that we are
                              to find, will sum to 7 and that this is the value that was subtracted from 7.   Expressing this
                              mathematically, we get, and are to effectively solve the new (addition) problem or equation of:

4 + ___ = 7   : "Four plus some number is equal to seven."

Actually, in more formal equation solving, which will be discussed further ahead in this book, this new equation shown above is easily arrived at by taking the original equation given: (7 - ___ = 4) and adding ___ (which represents some unknown value that we are solving for) to each side. If you want to, you can also substitute (replace) the ___ symbol with ?, x, n or something to symbolically (ie. a symbol) represent a number to be found. When any value is both given to or added (even if its 0 or nothing), and then taken away, removed, reduced or subtracted (regardless of the order of events), it results in a value of 0, and effectively eliminating that value. We do this to eliminate a value from one side of an equation and effectively move it to the other side of the equation. Again, that which is done to one side of an equation,

must also be done to the other side of the equation so as to keep the equation in mathematical true (equals) balance. Now, simply solve for ____ just like the method shown previously for solving addition problems (hint: subtract 4 from each side):

____ = 7 - 4 = 3

The problem above may have been stated as:

___ - 3 = 4

Usually, we have been told verbally how to solve this when checking a subtraction problem: "add the difference and subtrahend" to find the minuend, hence:

___ = 3 + 4   or: 4 + 3 =   : Which is effectively the result of adding 3 to each side of the above previous equation,
                                              and eliminating the 3 on the left side since 3 - 3 = 0. Simplifying the right side we have:
___ = 7   or: 7 = ___

Now, let's give some more reasoning to this, and it's based on the above discussion. If we reduce or takeaway a number by 3 and get a smaller number (the difference), we can do the reverse and add or combine that which was taken away back to this new or reduced number to get the original number before any subtracting had taken place.

        Before moving on to other topics, it must be mentioned that addition and subtraction are logically, inverse operations of each other. You can think of an inverse operation as an operation that will "undo" or "reverse" the last operation.

Ex. Starting with 10, and then performing an addition operation on it:

     10 + 5 = 15

To solve for the original value of (10) given this value of (15), you must perform the inverse operation of addition on it, which is a subtraction operation:

     15 - 5 = 10   : Subtracting 5 is the inverse process or operation of adding 5.

If you first started with (15) and then subtracted (5):

     15 - 5 = 10

To solve for the original value of (15) given this value of (10), you must perform the inverse operation of subtraction on it, which is an addition operation:

     10 + 5 = 15

This is exactly what we did at the beginning of the example.   Notice that the value of (5) was used in both inverse operations. Performing an inverse operation is also a way to check a mathematical problem. Likewise, most other mathematical operations have a corresponding inverse operation:

addition and subtraction,
multiplication and division,
powers and roots,
logarithms (basically an exponent) and antilogarithms (basically a power),
trigonometric functions and inverse, "arc" or "angle" trigonometric functions

A lot of ideas have just been mentioned above in this current topic which may very well seem a bit complex or overwhelming, and perhaps even unnecessary at first. Just give it a look over once in a while, and you will begin to find more and more reasoning, justification and understanding of it.

MULTIPLICATION

Multiplication, or to multiply, is a "short-hand", or quick method of representing, expressing and performing repeated addition. "Multi" is a word prefix that means: many, such as in the word multiple, hence multiplication basically means many or multiple repeated additions of a value or something. Similarly, multiplication means a combination of the words: "multiple" or "many", and "apply" or "application", hence to apply addition multiple times or repeatedly.

The symbols for multiplication are (x), (**.**), or sometimes parentheses or other grouping symbols surrounding the operands. For example: (5)(3), here 5 is being multiplied by three. (5)(3) = (5) x (3) = 5 x 3. The basic "formula" for multiplication is:

  operand1 x operand2   = result   : the operands of a multiplication problem are called factors, and the
                                                 result is called the product, therefore, this can be expressed as:

    factor1 x factor2   = product   after assigning formal names or identifiers to each factor:

  multiplicand x multiplier = product   : common multiplication expression or formula

The symbol for the operation of multiplication is often called and spoken as the "times" symbol. It means repetition or the number of times to repeat the addition of the given number, here the multiplicand.   The (x) symbol for multiplication is thought to be a plus sign that is turned or turning as if in a repeated process. At about 1700, the dot (.) symbol was sometimes used between two factors being multiplied so as to avoid using the (x) symbol which looks like the variable or "(number) placeholder" (x). This book does not use the dot symbol (**.**) for multiplication very much, but rather uses the parentheses around each factor or operand of the multiplication expression. Computers programs often use the star or asterisk symbol (*) between two factors of a multiplication expression, and this symbol was already sometimes used for multiplication, possibly to look like a modified (x) symbol for multiplication. The multiplicand and multiplier are the operands of a multiplication operation. The product (means something that was "produced") is the result of a multiplication and-or an expression to multiply.

The multiplier is the number of times the multiplicand is to be repeatedly added (summed) to an initial sum or starting value of zero or nothing. The multiplicand and multiplier are also called "(the) factors" of that result or product since these factors will have influence (effect) on the resulting value (the product).

Adding any number just once (to a starting or initial sum of 0) is the same as multiplying that number by 1:

7 + 0 = 7   and,
7 x 1 = 7   : multiplying any number by 1 does not change it's value. 7 is being added to an initial working sum or
                       total of 0, only 1 time, and the result is therefore 7. Note that this is also equivalent to 1 x 7 ,
                       and 1 multiplied or increased 7 times, or added to itself 7 times is : 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7

Adding 5, ten times would result in an addition expression or problem (to solve for the result) looking like this:

5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5   : sum = 50

That is a very long addition problem or expression, and a "running total" of many "intermediate sums" would have to be written down or remembered as you try to simplify or solve it. Since the above expression is of the repeated addition of the same value (5 added to the running (processing) total (sum), or to itself, 10 times), transforming this expression into an equivalent and simpler multiplication expression, we get:

5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 5 x 10 = (5)(10)   : multiplication is repeated addition

Since both expressions are said to be equivalent in value, they are shown as equated (ie. equals) together.

5 x 10   : product = 50,   and this is a much "shorter" expression for 5 added to itself 10 times.

Multiplication with a zero operand: The product of any value multiplied by 0 is 0.

Ex.   4 x 0 = 0   : With respect to addition, four added zero (or none) times is equal to zero.
                            Remember that 4 x 0 is also equal to 0 x 4 = 0 + 0 + 0 + 0 = 0

Ex.   0 x 20 = 0   : With respect to addition, zero added to itself twenty times is zero.
                            20 multiplied to itself zero (or none) times is equal to 0 since no addition to the initial
                            starting sum of 0 has taken place.

To solve common multiplication problems, you must know from memory, or by using a multiplication table, the products of
many factors. Usually, all the products of factors between 1 and 10 are memorized, such as shown in the multiplication
table below:

A MULTIPLICATION TABLE

	1	2	3	4	5	6	7	8	9	10
1 |	1	2	3	4	5	6	7	8	9	10
2 |	2	4	6	8	10	12	14	16	18	20
3 |	3	6	9	12	15	18	21	24	27	30
4 |	4	8	12	16	20	24	28	32	36	40
5 |	5	10	15	20	25	30	35	40	45	50
6 |	6	12	18	24	30	36	42	48	54	60
7 |	7	14	21	28	35	42	49	56	63	70
8 |	8	16	24	32	40	48	56	64	72	80
9 |	9	18	27	36	45	54	63	72	81	90
10 |	10	20	30	40	50	60	70	80	90	100

To use the table, the resulting product of any two factors (indicated at the top row, and at the left column) is the value at
the horizontal and vertical intersection. This table can also be used for division where the value at the intersection
becomes the dividend. Also notice in the table that there are two methods to find the same product:

multiplicand x multiplier = product
multiplier x multiplicand = product   : the product will be the same for both expressions and equations

Either value at the top and left side can be considered as either the multiplicand or multiplier. This can be very easily
verified by representing the multiplicand and multiplier in a physical or graphical form (such as with objects). For example,
3 x 2 can be represented by 3 objects and 2 groups of those three objects and which is the same as adding three to itself
twice: 3 x 2 = 3 + 3. Expressing this physically or graphically, where an object represents 1:

              *   *   *   : (3 objects, first group)
              *   *   *   : (3 objects, second group)

Since multiplication is repeated addition, the product is the sum of all the objects. Since there are 6 objects, the product
or repeated sum is 6.

Without removing or adding any objects, the product of 3 x 2 can be represented by tilting the above objects on to their side, producing 2 objects and 3 groups of those 2 objects, hence 2 x 3 = 2 + 2 + 2 :

```
    *    *          : ( 2 objects, first group )
    *    *          : ( 2 objects, second group )
    *    *          : ( 2 objects, third group )
```

The sum is still 6. Hence: 3 x 2 = 2 x 3 = 6   : Verifies that the multiplicand and multiplier factors can
be exchanged and still produce the same product or
result, just like the numbers (augend and addend) of
an addition problem can be exchanged.

Ex. There are 7 piles with 10 stones in each pile. How many total stones are there in all the piles?

Since we are totaling (summing) the number of stones, we are to add each (identical) group, or pile, of stones a total of 7 (piles) times.   Since this is a repetitive or repeated addition problem, we can represent and solve this more easily as a multiplication problem:

        10 stones   x   7   =   70 stones   : or more advanced as: (10 stones per [for each] pile) x 7 piles = 70 stones

The are all sorts of rhymes and apparent "tricks" to remember, and-or to check, the basic multiplication table and problems. For example, to remember the 9's products, take that specific multiplier (or multiple of 9) and subtract 1 from it. This will be the first or most significant number of the product. The other digit (if any) of the result will be that number which when combined to the first number found sums up to 9.

Ex. 9 x 4

Subtracting 1 from 4 leaves 3. and the number that when added to 3 that creates the sum of 9 is 6 (you can simply use subtraction to find this next digit value: 9 - 3 = 6).

      9 x 4   =   36   : note that for the product, here 36, that its digits sum to 9:   3 + 6 = 9

Ex. 9 x 7

      The first digit will be (7-1) = 6, and the second digit will be 9 - (7-1) = 9 - 6 = 3

      9 x 7 = 63   : note that the sum of the two digits of the product is 9:   6 + 3 = 9

Perhaps the most simple method to use as an aid to basic multiplication is the fact that if you already know:

      multiplicand x multiplier = product

Then you can easily solve for:

      multiplicand x (multiplier + 1) = product   : Increasing the multiplier by one
                                                                  to find the next product.

Ex. If you already know that 6 x 4 = 24 , and you then want to know what 6 x 5 is, then simply add one more 6 to the value of 24, since after all, multiplication is still repeated addition, and that you can use a previous sum or product as an intermediate sum of another multiplication operation.   Expressing this method and example in mathematical statements:

      Find 6 x 5

```
6  x  5  =   6 + 6 + 6 + 6  + 6         : expressing the multiplication as its equivalent addition
6  x  5  =  ( 6 + 6 + 6 + 6 ) + 6        expressing the first 4 additions as multiplication we get :

6  x  5  =  (6  x  4)  +  6             : equivalent expressions, and what this discussion is about
6  x  5  =     24     +  6             : If you knew 6 x 4 = 24, you can then find 6 x 5 with just another addition of 6.
6  x  5  =           30               : If you knew 5 x 6,  you would have the same result of 30.
                                      : If you knew 5 x 5, then 5 x 6 =  (5 x 5) + 5  =  25 + 5 = 30
```

Multiplication, like addition and subtraction, when done without a calculator, is usually performed vertically. To multiply this way, the positional product of each digit of the multiplicand is multiplied to (or by) the positional product of each digit of the multiplier once, producing an intermediate product (or intermediate sum to be added together, since a product is technically a sum of repeated additions) that will be part of the sum.

Ex.   Multiply twenty and seven together.
         (This example also shows how to multiply values (factors) greater than 10 that are not listed in the basic table.)

```
     20   x  7          : expressed horizontally

       20              : expressing this vertically
    x   7
```

```
       20
    x   7
        0        : 0 x 7 = 0 , a positional sum of 0
 +    140        : 20 x 7 = (2 x 10) x 7  =  (2 x 10 x 7)  =   (2 x 7) x 10 =  14 x 10 = 140,  a positional sum of 140
      140        : summing up these positional products (or intermediate sums)
```

```
Ex.     28
       x 17
        56      : 8 x 7 = 56
       140      : 20 x 7
        80      : 8 x 10
 +     200      : 20 x 10
       476
```

A more popular vertical method is to not use the positional products, but to use just the single digits of the multiplicand and multiplier, and utilize any carry's:

```
Ex.     28
       x   7
```

First, the 8 is multiplied by 7. The resulting product is 56. The 6 is placed into the "ones" digit position of the product (which may be an intermediate product that will be used in the final sum). The second digit of this intermediate result, the 5, is a "carry" to its corresponding "tens" column and is added to the product of the next multiplication. That is: 7x2=14, and then 5 is added to 14, making a sum of 19. This result is placed in the next leftward and significant available position as the product of that multiplier digit is constructed:

```
     5
    28
 x __7_
   196        : steps of:  8 x 7 = 56,   7 x 2 = 14,   14 + 5  =  19   (or a 9 with a carry of 1 into the next column)
```

The basic reasoning that the carry of 5 was not added to the 2 before performing the multiplication is that, like above, 8 is being multiplied by 7, and 20 (not 70) is to be multiplied by that 7 also giving:

```
  28 x 7  =  ((2)(10)  x (7)  +  (7 x 8)  =   20  + (8 x 7)           or:      (20 + 8)     : "both twenty and eight, multiplied
                                                                          x _____7       by seven, and their products
                                                                                           added together"
  (20x7)  +  (8x7)      : or  expressed as:     (8 x 7)  +  (20 x 7)
   140   +   56                                  56    +    140
       196                                           196
```

Continuing with the above discussion, multiplying by the positional products (such as the 20 indicated above) is not necessary as long as you place an appropriate amount of "trailing" zeros onto each next intermediate product(s) and continue multiplying with just the single value or digit in that digits position. This is perhaps the easiest and most popular method to multiply. The reason behind this method is that when you multiply with just the value in that digits position, instead of the positional product, you are effectively dividing the correct intermediate product by the positional weight of that digit, and placing in the "trailing zeros" will correct this (effectively multiplying, the intermediate product, by the positional weight that was effectively "divided out" using this process). Excluding the first time, each time another digit of the multiplier is used for multiplying, the intermediate product or sum is shifted leftward one digit position which essentially means placing in another trailing 0 and effectively multiplying the result by the positional weight of the multiplier digit. This multiplication by 10 is often thought of as simply shifting the decimal point one digit rightward. That is, when any value is multiplied (or divided) by 10, or some power of 10 such as $10^2 = 100$, the resulting digits are always the same, and only the decimal point will moved. Leading or trailing zeros may sometimes be needed, and placed before (such as when dividing by 10) or after (such as when multiplying by 10) the number, and only if needed. For example(s):
3 x 10 = 3.0 x 10 = 30 , 25 x 10 = 250   , 2.5 x 10 = 25 ,   2.615 x 10 = 26.15   ,   350 / 10 = 35.0   ,
3 / 10 = 3.0 / 10 = 0.3   ,   3 / 1000 = 3.0 / 1000 = 0.003   and   0.003 x 1000 = 3.0 = 3

Ex.   Multiply 28 by 17

```
    5
   28
 x 17
  196       : 8 x 7=56, carry the 5;  2 x 7=14,  14+5=19
 +280_      : First, place a trailing zero=0;  8 x 1=8 (instead of 8 x 10 = 80),  and   2 x 1=2 (instead of 20 x 10 = 20)
  476
```

also note, 196 = 28 x 7 = (20+8) x 7 = (20 x 7) + (8 x 7) = 140 + 56 = 196
also note, 280 = 28 x 10 = (20 x 10) + (8 x 10) = 200 + 80 = 280

Ex.   Solve 107 x 304

```
      107
   x  304
      428       : 7 x 4  = 28,   carry the 2;   0 x 4 = 0   and then that   0 + 2  = 2 and place it ;  4 x 1 = 4 and place it
     0000       : place a trailing 0.   Here, all three products will also be 0.
   + 32100      : now place another, or 2 trailing zeros here, then continue multiplying
     32528
```

It should also be mentioned that some of the steps presented in the above discussions about multiplication imply a mathematical concept formally known as the distributive property or simply "distribution". It will be described and used further ahead in this book, but here is a simple explanation:

Ex. Given   5 + 5 = 5(1) + 5(1)   ,   We know this repeated addition can be expressed using multiplication as:
              5 x 2   = 5(1) + 5(1)   and this can then be expressed as:
              5 x (1 + 1) = 5 (1) + 5 (1)   hence:
              5 (1 + 1) = 5(1) + 5(1)   : This has the general format of the concept of distribution, where a product of
                                               two factors, where one contains several terms (here as expressed additions or
                                               a sum), is equal to the sum of the products of each term and the multiplying
                                               factor . Here both sides and expressions equal 10.

Rather than simplify the second indicated factor of (1+1) to a value of 2, with the distributive concept, we can use a reverse type of process to that mentioned above, 5 can be multiplied to each number that is a part of the sum that comprises this factor of (1 + 1):

              5 (1 + 1)   "distributing" (essentially multiplying) the factor of 5 to each term (ie. a partial sum of the
                                        second factor) to create a new expression:
           5(1) + 5(1)   : this equivalence was also shown above
              5 + 5

The main point to remember is that an initial factor multiplied to another factor that is composed or expressed as sum of numbers is equal to the sum of all the products of that initial factor and each member of the factor that is composed of a sum of numbers.

Ex.   7 (1 + 1) = 7(1) + 7(1)
            7(2)   =   7   +   7
             14   =   14   : checks

Ex.   7 (1 + 2) = 7(1) + 7(2)
            7(3)   =   7   + 14
             21   =   21   : checks

Ex.   15   =   (10 + 5)   =   3 (10 + 5)   =   3(10) + 3(5) = (10 x 3) + (5 x 3)   =   30 + 15 = 45
     x 3   x   3
        45   15   : 5 x 3
                    +   30   : 10 x 3
                           45

        Here is another helpful discussion about multiplication, especially with multi-digit operands, and it is intended to give more clarification and understanding to using a "left-shift" in the intermediate sums during vertical multiplication solving. A left shit by one digit position is equivalent to multiplying a number by 10. This discussion is somewhat detailed, and you may skip over it till perhaps another time.

Ex. Given 1, if we left-shift this one position to the left we have: 10 , and this is the same as multiplying 1 by 10 and the product is 10. If you were to repeatedly sum 1 a total of 10 times you would have:

     1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 1 x 10 = 10   : Hence, simply left-shift the digit leftward one position
                                                                       and place a trailing 0 into the ones position or column.

Ex. Given 2 and we left-shit this one position, we get 20, and this is the same as multiplying 2 or= 2.0 by 10.

Notice that the result has the same digits used, but they are essentially shifted to the left one position and a trailing 0 is

put at the end of the number (in the ones position or column). This new value is 10 times more than the given value. 20 is ten times more than 2. If you were to repeatedly sum 2, a total of 10 times you would have:

  $2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 2 \times 10 = 20$

Hence, simply left-shift the digit(s) leftward one position and place a trailing 0 into the ones position or column.
Extra: 2 can be expressed as (1+1), and, we would have: $(1 + 1)10 = 10(1+1) = 2(10) = (10)(2) = (1)(10) + (1)(10)$
The right hand expression is an example of what is known as applying the "distributive law" (essentially multiplication) to the left hand expression. The left hand expression is essentially a "factored form" of the right hand expression. The factors are (1 + 1) and 10 which is the common factor in both indicated (sum of) products or terms on the right hand side.

$2 \times 10 = (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1) + (1+1)$ which could be expressed as:
$2 \times 10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$ with some grouping:
$2 \times 10 = (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1)$ which could be expressed as:

$2 \times 10 = (1 \times 10) + (1 \times 10)$: which results in 1 left-shifted once, plus, 1 left shifted once (and not 1 left-shifted
                                                   twice), and this is: 10+10, and these are a group of "partial (part of) sums" of 20.

Now consider this, let symbol N symbolically represent the value of any basic number that you can think of:

N	=	N	or =	N	=	N

        N   N   or =   N   =   N   N x 10 = N (1 x 10) = N ((1)(10)) = N(10)
      x 10   x (10 + 0)   x (10 + 0)   x (1 x 10)   Note, we **cannot** use distribution above,
                   N(10) + N(0)   N(0)   since the second factor: (1 x 10) is not a
                                          +   N(10)   sum of values such as: (1 + 10)

These can also be expressed as:

  $Nx10 = N(10) = N(10 + 0) = N(1 \times 10)$: note, you cannot use distribution on the right expression
                                                    to simplify it, since the second factor (1 x 10) is not a
                                                    sum of terms. $N(1x10) = N((1)(10)) = N(1)(10) = N10 = 10N$
                                                    The result is **NOT**: 1N + 10N = 11N

Considering the indicated product on the right hand side, we essentially have 1 with a left-shift being multiplied to N:
That is: $N(1x10) = (N)((1)(10)) = N(10)$. Anything multiplied by 1 is that same number, such as 10(1) = 10. Hence we have: $N(1)(10) = N(10)$, and when we left-shift all the digits of N, we have the product of it and 10. Symbolically using an appended 0 to indicate a left- shift or multiplication by 10:

  $N(10) = N0$: When N is a plain "integer" number that does not contain any fractional portion past the decimal
                      point, when it is multiplied by 10, N is left-shifted, that is, every digit in the number of N is shifted 1
                      position leftward, and a trailing 0 is placed rightward of N's last digit. If N does contain a fractional
                      decimal number part, then simply move the decimal point one position rightward when multiplying
                      by 10, for example: If N=3.49, N(10) = Nx10 = (3.49)(10) = 34.9
                                           If N=5 , N(10) = 5(10) = 5.0(10) = 50 = 50.0

Ex. $N(11) = N(1 + 10)$

We see that N will be summed 11 times, and this is equivalent to summing N once, and then combining that (partial) sum to the summing of N ten times.

$N(11) = N + N + N + N + N + N + N + N + N + N + N$ which can be expressed as:
$N(1 + 10) = (N) + (N + N + N + N + N + N + N + N + N + N)$
$N(1 + 10) = (1xN) + (10xN)$: this notation is also an example of the "distributive law" that will be discussed

Hence, the result is N, plus N left-shifted once. If N=15, N(11) = N(1 + 10) = N + N0 = 15 + 150 = 165

Ex. To multiply a value by 20.   Since 20 = (2 x 10) , this expression indicates that the summing of a value twenty times (ie, repeated addition, which is multiplication) is equivalent to summing up that value twice to first produce a (partial) sum, and then summing up that new (partial) sum a total of ten times to find the final sum or product.   We now know that we can left-shift a value one digit position to quickly have the product of it and ten:

N x 20   =   N(20)   = N(2 x 10)   = (Nx2)x(10)   =   (Nx2)0   : here, the last expression indicates
                                                                              the left-shift of the value of: (Nx2), and
                                                                              putting, or appending, a trailing 0

Ex. N x 23 = N(23) = N(20 + 3) = N((2x10) + 3) = 2(N)(10) + Nx3 = 2(N0) + Nx3

To multiply a value by 100, simply perform two left shifts on that value and place on, or append, two trailing zeros since:

  N(100) = N(10 x 10) = (N(10))10 = (N0)10 = N00

Ex.   573(100) = 57300

Clearly, the number of left-shifts when multiplying by a power of 10 is equal to the number of zeros in that value. For example, to multiply a value by 1000, simply perform three left shifts on that value and place on three trailing zeros.

        To help multiply (or divide) by or to a large constant (steady, unchanging in value) number that has many digits quickly when not using the assistance of an electronic calculator, it is very helpful to first make a table of values of all the products (or repeated additions) of that constant value and all the single digit decimal values from 0 through 9.

Ex. If 307,629 is to be multiplied by any other value:

Table Of 0 through 9 Multiples Of The Given Value

307,629 x 0 = 0   Or by adding 307629 to each previous product to find the new product
307,629 x 1 = 307,629   307,629 + 307,629   or = 307,629 x 2 = 615,258 :
307,629 x 2 = 615,258   615,258 + 307,629   or = 307,629 x 3 = 922,887 :
307,629 x 3 = 922,887   922,887 + 307,629   or = 307,629 x 4 = 1,230,516 :
307,629 x 4 = 1,230,516
307,629 x 5 = 1,538,145
307,629 x 6 = 1,845,774
307,629 x 7 = 2,153,403
307,629 x 8 = 2,461,032
307,629 x 9 = 2,768,661

```
        307,629
     x    57249
        2768661       : write down the corresponding products in the table, and any trailing 0's needed,
       12305160          then sum these intermediate products so as to have the resulting product
       61525800
     2153403000
 + 15381450000
     17611452621
```

The above is also one method to use when the number of digits of the operand(s) and-or the result is excessive for the device and cannot be directly entered into and-or displayed as a result by the computer or calculator.

MULTIPLICATION WITH DECIMAL FRACTIONS

Multiplication by a number with decimal fractional values, with, or without. a whole or fractional part is performed in the exact same manner as when multiplying just whole or "integer" values that do not contain a fractional part. The fractional part of a decimal (system) number numerically represents a part less than one (1), and this is expressed numerically to the right of the decimal point in a decimal number.

Example of a number containing a whole part and a fractional part: 7.32 = 7 + 0.32 = whole part + fractional part =
= whole part . fractional part   Here, the whole part is 7, and the fractional part (of 1) is 0.32

Simply multiply the values and (initially) ignore any possible decimal points as you process or calculate the result. The decimal point of the result must then be properly placed or positioned. Simply add up the number of digits that were not only indicated, but actually used (during the calculation) rightward of the decimal point in each of the operands, and this is the number of places to move the decimal point leftward in the product. Verification of this procedure is shown below, though it requires some slightly more advanced concepts of fractions.   Commonly, the words "decimal digits" or "decimal places" are taken to mean the digits of the fractional part of a decimal number that are to the right side of the decimal point.

Before continuing, you may first want to consider this about the basic concepts of both the decimal number system and fractions or fractional values. This paragraph section may seem a bit complicated at first to some, and they may skip over this till perhaps a later time. In the decimal system, when we count up to 10, each of the ten steps, each being an equal or equivalent step, and here for this specific example,will result in an increment or increase of 1 to the entire number of which the increment is being applied to. Note that the actual value (applied to the entire number that is being incremented) of each step is technically, the (decimal system's) actual or specific value that the increment represents to the entire decimal number, and this value is the numeric weight of that (digit, or column) position in question or reference when just the decimal numeric system itself is being considered. and each step is one of the ten possible steps (before digit or column position "rollover" back to the start or 0), or "one of ten (steps)" for each digit, column or ("decimal") position, and this can therefore be expressed numerically as: 1 of 10. This concept, and each step or one increment out of ten possible increments (per digit position or column), can be mathematically expressed as an indicated fractional part as: 1 of 10 = 1 per 10 = (1/10) or simply: 1/10, and this does equal 0.1 as expressed using the most basic or fundamental decimal numeric form as just one value or number. Notice that the expression (1/10) does look like a common division expression or problem, and that the value of 0.1 is also equal to the result of the indicated division expression. Since each step is only part of the entirety or whole thing, the actual value of each step is said to be only a fractional part. Considering even the number 1, it too, for example, can be, or considered as being composed of ten equal steps. It need not be exactly ten steps, and another example or number system could have more or less steps, but we are now just considering the basics of the decimal system with its common 10 steps in each digit position or column. Ten increments or (fractional) steps, each being (0.1) numerically, will produce the numeric values or sums, starting at 0.0, of 0.1 through 0.9, and 1.0 = 1.   Remember, that in the decimal system, each count or step in the next greater or leftward position is equal to ten, "ten more", or "ten of", steps, and therefore, each step or increment in the next lesser or rightward position is "ten less", or "a tenth" of each step in the current or reference position. Some more will be explained about the fundamentals of fractions later, and we will now continue with the current multiplication discussion.

```
Ex.    1           1.0        : one decimal digit used  and,
      x 1    =    x 1.0       : one decimal digit used
        1           00
                 + 100
                  1.00        : decimal point has been moved two places leftward
```

Ex.   (1)(0.1)

Note that multiplying anything by one is always equal to that same value. Here, a tenth of one is being multiplied by one:

  (1)(0.1) = (0.1)(1) = 0.1

       0.1   : one decimal place used
   x ___1
       0.1   : after moving the decimal place one position leftward

The "one-tenth" (of one) description is easily seen when this decimal (fractional) value is converted to a standard fractional form with some help from the concepts of positional notation and-or scientific notation (SN) which will be discussed ahead.

0.1 = (1)(0.1) = (1) (_1_) = (1)(10^-1) = _1_   : "a tenth of one", and since any value multiplied by 1 is that value
                          (10)   10

Ex.   (0.1)(0.1)

A good knowledge of the decimal system, which can happen with some routine practice, will quickly give an answer of "a tenth, of a tenth", or "one-tenth, of one-tenth", or simply, "a hundredth" or "a hundredth of".

(0.1)(0.1) = (1)(1) = _1_ = 0.01
                 (10)(10)   100

standard method: 0.1
                       x 0.1
                          01
                      + 000
                        0.01

Ex.   (0.5)(2.0)

(0.5)(2.0) =   (5) (2)   After simplification (using multiplication) of the indicated factors in the top (the numerator)
                 (10) (1)   and bottom (the denominator) values of the fraction or division problem:

             = _10_ = 1
                 10

standard method: 2.0
                       x 0.5
                          100
                      + 000
                        1.00

For another check, multiplying any value by 2 is equivalent to adding that value twice:

```
          1
         0.5
    +    0.5
         1.0
```

Ex. (0.3)(0.5)

$$(0.3)(0.5) = \frac{(3)(5)}{(10)(10)} = \frac{15}{100} = \frac{15}{10^2} = 15(10^{-2}) = 0.15$$

checking:
standard method:
```
           1
          0.3
      x   0.5
           15
      +   000
          0.15
```

Ex. (1.5)(0.2)

(1+ 0.5)(0.2)

First note that any number divided by itself is equal to 1 or 1/1. Below, 10/10 = 1/1 = 1 will be used. Also, if you have "ten-tenths" of something and add "five-tenths" of something, then it's obvious that you have "ten plus five tenths" of that something, hence "fifteen-tenths" as shown below. The tenths are used here as the same or "like" (or common) units, and the numeric parts can then be combined.

$$\frac{(1+5)(2)}{(1\ \ 10)(10)} = \frac{(10+5)(2)}{(10\ \ 10)(10)} = \frac{(15)(2)}{(10)(10)} = \frac{30}{100} = 0.30$$

standard method:
```
           1
          1.5
          0.2
           30
      +   000
          0.30
```

DIVISION

Division is to divide (separate or split into groups) something into two or more smaller parts. The symbols to indicate or express a division operation are shown below. The separate parts may possibly not even be the same value. For example, you could divide a bag containing 6 apples among 2 people. Perhaps you will give one person 4 apples, and give the other person 2 apples. Here, the apples would be completely (but not balanced or evenly (equally)) divided out among the people. Formally, in mathematics, division is to separate that something into smaller parts all having the same size and-or numeric value, hence the result of a division operation is generally thought to be "even" (equally, the same, or in balance).

For the example noted above, to have an even division, each person would receive the exact same value of 3 apples. With 6apples for 2people, we can divide the number of apples by the number of people to see how many apples each person will receive. 6apples / 2 people = 3 apples / 1person = "3 apples per person". (6/2)=3. If there were 3 people, for an even division (or "distribution" of and-or to) of the 6 apples, each would receive 2 apples since 6apples / 3people = 2apples /1people = "2 apples per person". (6/3) = 2. The apples are then said to have been "evenly divided (of-and) out" to the people. We see that the word "even", or "evenly", actually means that each part or portion is in balance to any other parts, that is, each part and-or its numeric value is the same or equal to all the other resulting parts after the division is completed. For example, if something was divided into 3 parts, each part will have the same numeric value, and for example, if 30 was divided into 3 parts, numerically expressed as: 30/3, each part of the three total parts will have the same numeric value of: (30/3) = 10 = "ten". The sum of each of each smaller parts (same) value will equal the original or total value before division was performed on it: (10 + 10 + 10) = (10)(3) = 30, and this multiplication is actually a way to check a division problem, and here it's: (30/3) = 10.

Mathematically, division is the "inverse" of multiplication (repeated addition), hence, dividing is essentially performing repeated subtraction. Essentially then, the "starting minuend" is called the dividend, and the divisor is effectively the subtrahend. A quotient is the result or output of a division operation, and it is how many times the divisor can be subtracted from (or "go into") the dividend.

In relation to division, you will often hear the question that is similar to: "How many times can value1 (the dividend) be divided by value2 (the divisor)?", or in other words: "How many times can value2 (the divisor) go into value1 (the dividend)?" A way to state the division process (that is somewhat like a factoring process) and result (quotient), is that it is how many times that value2 (the divisor) must be increased or multiplied so that it equals value1 (the dividend). This is the same as saying how many times bigger that value2 (the dividend) is than value1 (the divisor).

Consider: let: value1 = factor1   and   value2 = factor2

(factor1)(factor2) = product

$$\frac{product}{factor1} = factor2$$: "product is factor2 times bigger than factor1" , product = (factor1)(factor2)
: "factor1 can go or divide into product a total of factor2 times"

$$\frac{product}{factor2} = factor1$$: "product is factor1 times bigger than factor2" , product = (factor2)(factor1)
"factor2 can go or divide into product a total of factor1 times"
: product = (factor1)(factor2) = (factor2)(factor1)

The symbols for division along with the general "division formulas" are:

dividend ÷ divisor = quotient   : quotient is a word derived from quot and quota which basically means
                                                     the current numerical value or size as in "how many (times)" or "how much"
dividend / divisor = quotient   You may sometimes hear something like "Today's quota is 7 sales", which
                                                     means to the effect: "Today's goal or output of selling is to make 7 sales".

$\dfrac{\text{dividend}}{\text{divisor}}$ = quotient : In this "fractional like" form or expression, the quotient is and represents how many times greater (or sometimes, less) that the dividend is compared to the divisor.

$\overset{\text{quotient}}{\text{divisor} \,|\, \text{dividend}}$: this is often the form to use when manually calculating division by hand

Before the concept of division by using repeated subtraction is shown, perhaps the simplest way to understand the process of division or how to divide, is by observing and comparing. For example, if you had 6 objects (such as coins, or dots on paper), how many times (ie. comparisons) can 1 of those objects correspond (ie. match) to all of those objects? In other words, what is 6 divided by 1, hence how many times can 1 be subtracted or taken away from 6? The question may also be stated as something like: How many times can 1 thing "go into" (or compare to) 6 things?

6 divided by 1, graphically:

```
    *    *    *    *    *    *          : 6          now repeatedly compare to, or matching to 1 (or "1 to 1"):

    *                                   : 1 time   (1 comparison or correspondence, there are now 5 dots left)
         *                              : 2 times  (2 comparisons or correspondences, there are 4 dots left)
              *                         : 3 times
                   *                    : 4 times
                        *               : 5 times
                             *          : 6 times  (there are now 0 dots left, the division process must stop)
```

We see that 6 divided by 1 is 6. It could be said that "1 can go into 6, a total of 6 times". Any value divided by 1 is always equal to that same value. If you are using objects such as coins, you can physically remove the necessary coins, or if you are using objects drawn on paper, you can circle or cross-out the objects after each comparison to indicate that they have already been used or considered.

Ex.   6 divided by 2:

```
    *    *    *    *    *    *          : 6          repeatedly comparing 2 (the divisor) to each 2 new and
                                                       remaining parts to be considered of the 6 total parts:

    *    *                              : 1 time
         *    *                         : 2 times
              *    *                    : 3 times
```

6 divided by 2 is 3.   "2 can go into 6, a total of 3 times" or "2 can be subtracted from 6 a total of 3 times" or perhaps something like: "6 can be divided by a group containing 2, a total of 3 times",
,   or "there are 3 groups of 2, within, or out from, 6"

Ex. 6 divided by 3:

```
    *    *    *    *    *    *          : 6          repeatedly comparing to 3 (comparing 3
                                                       to each new 3 and remaining parts of the 6):

    *    *    *                         : 1 time
              *    *    *               : 2 times
```

6 divided by 3 is 2.   "3 can go into 6, a total of 2 times"

Ex. 6 divided by 6

    *   *   *   *   *   *   : 6   repeatedly comparing to 6 (or the 6 smaller parts
                                      to the 6 total parts):

    *   *   *   *   *   *   : 1 time

6 divided by 6 is 1. Any value divided by itself (the same value) is always equal to 1.
Notice in the examples that the larger the divisor is for a given division problem, the
smaller the result or quotient, hence it could be said that the divisor and quotient are
(mathematically) inversely related, that is, as one value goes larger in value, the other
or corresponding value goes smaller in value, and vice-versa.

6/2 = 3   As the divisor goes larger, the quotient goes smaller. This is an (mathematical) inverse relationship.
6/3 = 2   A direct (mathematical) relationship is where if a value grows, a corresponding value also grows.
6/6 = 1

If you had a quantity of 30 and kept removing or subtracting 5 from it, you will make 6 "even subtractions". The word "even" here essentially means a full or complete division and that there was no part, difference or remainder (r), of the 30, left over after all the subtractions.

1. 30	2. 25	3. 20	4. 15	5. 10	6. 5
- 5	- 5	- 5	- 5	- 5	- 5
25	20	15	10	5	0 = r

Or: (((30 - 5) - 5) - 5 - ...)   (... means "and so on")   : expressing 30 divided by 5 as a
                                                         repeated subtraction expression
Or: 30 - 5 - 5 - 5 - . . .

An easier way to express this subtraction problem is with division. The problem could be written as:

$5\overline{)30}$: "thirty divided by five", or = 30 divided by 5

You can check the result (the quotient = 6) for correctness by repeatedly subtracting it from the dividend (30). If you get the original divisor (by getting 5 even subtractions = 5), the result is correct. Since division is essentially repeated subtraction, you can check a division problem with another division problem where the dividend is divided by the quotient. If this quotient value is equal to the original divisor, the division was correct and the quotient (result) is correct.

Ex. $5\overline{)30}^{\,6}$ is correct since $6\overline{)30}^{\,5}$ This is so since:

                           30 = 30
     5 + 5 + 5 + 5 + 5 + 5 = 6 + 6 + 6 + 6 + 6   : "sum of six, fives" = "sum of five, sixes"
            5 x 6   =   6 x 5   : the repeated addition, expressed as a multiplication

Given just the left hand side of the equation, 5 can be removed (ie. subtracted) from 30, a total of 6 times.
Given just the right hand side of the equation, 6 can be removed from 30, a total of 5 times.

To perform and check the results of division without using the slower and more tedious method of repeated subtraction or performing another division, you can use multiplication (which is repeated addition as seen above). To check if the quotient is correct, multiply the divisor and quotient, and if the result (a product) is equal to the dividend, the division problem was performed correctly:

Since $\dfrac{dividend}{divisor}$ = quotient   We mathematically have after multiplying both sides by the same value (so as to keep both sides of the equation in balance, or the same value), here, the divisor:

$\dfrac{dividend}{divisor}\dfrac{(divisor)}{(\ 1\ )}$ = quotient (divisor)   After some canceling, this can be simplified or reduced to just:

dividend = (quotient)(divisor)   : CHECKING A DIVISION PROBLEM (using multiplication). We see that the quotient and divisor are essentially two (of the many possible) corresponding factors of the dividend. If one of these factors is wrong, their product will not match or check (be equal to) with that dividend.

Since 5 can divide into, "go into", "sum into", "be taken from", or be part of the total sum of 30, a total of 6 times, repeatedly adding 5, to an initial sum or start of 0, six total times can be represented by a simple multiplication problem:

0 + 5 + 5 + 5 + 5 + 5 + 5 = 30   or simply:
5 + 5 + 5 + 5 + 5 + 5 = 30   which can be (mathematically) written or expressed in a simplified form as:

5 x 6 = 30   : Note, both 5 and 6 are factors of their product, or 30. Dividing 30 (as a dividend) by either factor will yield (produce, or result to) the other factor. Multiplying these factors together will yield that dividend.

As noted above, dividing 30 by 5, we get 6, and if we divide 30 by 6, we get the other factor which is 5. So if then given a multiplication problem of the form:

____ x 6 = 30   : or =   6 x ____ = 30   or:

____ x 5 = 30   : or =   5 x ____ = 30

To solve these for the unknown factor, simply divide 30 (the product of factors) by the given factor, so as to find the other (corresponding, correct) factor.

Division with a zero as an operand: 0 divided by any value, results in a quotient that is same value of 0.

Ex. $\dfrac{0}{3}$ = 0   : "zero or no thirds, equals zero thirds, or simply zero". In reference to subtraction, 3 cannot be subtracted from 0 even once, since there is nothing there to subtract from that dividend. Zero, or no parts of 3 parts is 0.

This checks when multiplying the quotient and divisor together, yielding the dividend of 0.

3 x 0 = 0

The quotient of any value divided by 0 is not definable, hence this form of a division operation is often called "undefined". This comes from the reasoning that their is no quotient, that when multiplied by a 0 divisor, that will yield a product equal to the given dividend. Also, all values multiplied by 0 result in a product of 0.

Ex. $\dfrac{10}{0}$ = undefined   since: 0 x **?** = 10   :here, ? does not represent a specific value, but rather indicates an endless question, or one that can't be answered, hence an unsolvable or undefined value, and equation.

Note that when checking the product of a multiplication by or with a 0 factor, that when using 0 as the dividing factor, that other non-zero factor of that product cannot be confirmed:

Ex. 4 x 0 = 0

Using what was mentioned previously: product / factor = other factor of that product:

checking: $\dfrac{0}{4}$ = 0   : checks   and $\dfrac{0}{0}$ = ?   (not 4)   : here, ? means undefined

When using a modern electronic computer or hand-held electronic calculator, division by zero is usually noted as the symbol (E) displayed, and which means "Error" because division by 0 as a dividend or factor does not have any result.

How would you solve for the unknown, here a divisor (or denominator), represented as ____ , of a problem of the form:

$\dfrac{10}{?}$ = 5   : 10 divided by what number is 5 ?
                              (here, 10 is the dividend, and 5 is the quotient)

First, we know that to check a division problem, we multiply the quotient and divisor to get the dividend, (that is, the quotient and divisor are both (corresponding) factors of the dividend) hence:

  5 x ____ = 10

And by the above discussion, dividing a product of two factors by either factor will give the other corresponding or correct factor. Therefore, to find the value of the other factor (here, being equal to the divisor in question), we divide 10 by 5:

  $\dfrac{10}{5}$ = ____

  $\dfrac{10}{5}$ = 2

checking: $\dfrac{10}{2}$ = 5   : checks since 5 x 2 = 10

Let us now estimate the value of a quotient. Consider this problem:

6 $\overline{)\,15}$

From the multiplication table or our memory, we know that the quotient is greater than 2 and less than 3 since 6 x 2 = 12, 6 x 3 = 18, and 15 is numerically located someplace between 12 and 18. 2 will be chosen as a start since 6 can go into 15 at least 2 times (but not 3 times) and there will now be a remainder (the value or difference "left over" or unprocessed amount of the dividend, here 15) of the quotient or "starting minuend" since the division has not been completed yet. Hence, the result or quotient will be 2, plus some fractional value of the divisor, here 6. That is,

    quotient =   number of times the entire divisor can be used in the division, or evenly divide into the dividend
              + any portion (fraction) of a possible one single division by the divisor to account for any remainder

or   quotient = number of even (whole or entire) divisions + any portion (fraction) of one division   ,and for the example:

quotient = (2 + some fractional value of 6)   : and for checking: (this quotient value) x 6 = 15
"Some multiple (a value between 2 and 3) of 6, equals 15". Since it's a value between 2 and 3, its not one of the whole or "counting numbers" (sometimes called "integers" for incrementing or incremental "steps" of numbers). Here, the quotient value will have both a whole part and a fractional part, and this kind of a decimal number is sometimes called a "real number". Consider that values are not always incremented by 1, but could be incremented or changed by smaller values such as a part or fraction of 1, and hence the need for real numbers to both represent these increment values and results (such as a sum).

Here, the whole part (the value of 2) of the quotient, can be interpreted as being 2 whole or entire sixes, hence "twice" or occurring 2 times in that dividend, and therefore, at least 2 even divisions are possible. Since there is a remainder "left over" to account for the entire dividend, added (or "plus") to this quotient value is some numeric (fractional, less than 1) value that represents this fractional part of a partial division, and not a complete division. This division is applied to the remainder or what is left of the dividend. The numeric representation, using decimal positional notation, of a fractional part of anything or any value is always less than the entirety or whole thing which can be represented numerically as 100% ("one-hundred percent") or just 1.00 = 1, and hence any fractional value or part will be less than (< is the "less than" mathematical symbol) 1.0

Here is how much of the dividend (here for this example, it's 15) is "left-over" or remaining after all the "whole divisions" by 6:

```
15  -  (6 x 2)                    or:          15     : here, optionally using a repeated subtraction method
15  -   12                                   -  6
    3            : remainder of               9
                   the dividend            -  6
                                             3     : 3 is the remainder of the dividend of 15, and 3 is only a part of one
                                                     whole or entire value of  6. It could also be said that since only 3 is
                                                     remaining, that only 3 of the parts, of the 6 parts of the divisor, can
                                                     evenly divide or "go into" this value of 3.  Hence, there will not be a
                                                     (1) full or complete division upon the remainder.
```

This remainder (here, it's 3) of the dividend (here, it's 15) is clearly less than 6 and it is therefore numerically only some part or fractional value of 6. Even though 3 is not less than 1, it is still only some fractional (part) value of 6 since 6 is being considered as a single (1) "whole quantity" of reference now and 3 is only a part (portion, or a fraction) of it. The fractional part of the quotient, which represents the numeric value of an incomplete, fractional or partial of one complete possible division, can be said as (and equivalent to) "3 parts of the 6" are left over or remain. The whole part of the quotient is 2.

The quotient can be said to be:

"Two plus three parts of six"   =   2 + 3   : a quotient is expressed in terms of the divisor, that is, how many
                                              6   times the divisor can go into that quotient.

That is,   15   =   2 + 3   : note also that 3 is equal to half of 6
            6   6   The quotient could be expressed as "Two and a half" of that divisor value,
                                        hence two and (plus) half of a (one) complete division by the divisor.
    .
The quotient is always considered as being in reference to the specific divisor that divided the dividend. The fractional part of the quotient is, and numerically represents 3/6 = 1/2 = 0.5, and is not, for example: 3/5 or 3/7, or 3/15 or any other value. That is, the remainder of one complete division is expressed as a fractional part (or remainder left) of the divisor used for the division, and is not expressed as a fractional part that the remainder of the dividend is to, or of, the entire dividend, (ie. here, not as 3/15 = 0.2).

The quotient can be expressed as equal to:

Since: $\dfrac{dividend}{divisor}$ = quotient   and   dividend = (divisor)(quotient)

$\dfrac{15}{6}$ = 2 + $\dfrac{3}{6}$ 15 = (6) (2 + $\dfrac{3}{6}$)   or= $\dfrac{(6)}{(1)}\dfrac{(2}{(1}$ + $\dfrac{3)}{6)}$

quotient = 2 whole divisions (or subtractions) + numerically expressed or indicated part of a one division by the divisor
since a complete division is no longer possible. In short, this value will be and represents a fraction of 1 division.

quotient = 2 whole divisions (or subtractions) + 3/6 = 1/2 = 0.5 = "half" of a 1 complete division by the denominator

The representation (halfs, thirds, fourths, tenths, thousandth, etc.) of the dividend is often not so apparent given a strict numeric decimal value that has a fractional value, and then again, just how do you numerically represent half or any other portion anyway, especially at this stage in the book?   The simple trick to overcoming this problem is to increase the remainder, and any further remainders, so that the divisor can divide into it at least once, rather than some fractional portion. At each stage of this "continued division", one extra (lesser weight) digit will be placed in the quotient. This is the basis of "long division" which will be discussed next.   Observe the actual quotient of the division problem above, expressed in various forms or expressions:

2 + 6$\overline{)3}$ = 2 + 6$\overline{)3.0}$, or = 2 + $\dfrac{3}{6}$, or = 2 + $\dfrac{1}{2}$, or = 2 + 0.5 = 2.5

     Now, another example and more analysis will be shown below so as to give a fuller understanding of the basics of division.

Ex. 31 divided by 5

5$\overline{)31}$ Since 5 x 6 is 30, and 5 x 7 is 35, we will start with 6 times that the divisor of 5 can go into, or be subtracted completely from this quotient of 31.

$\quad\dfrac{6}{5\,\overline{)31}}$: 5 x 6 = 30
$\quad$- 30
$\qquad$1   : 1 is the remaining (or remainder) portion of the dividend which cannot be evenly divided by 5. The division cannot continue normally, and is therefore an incomplete division remains. This 1, by itself, is NOT part of the quotient (the result of the division). A quotient is a value that is how many times the divisor can go into, or divide into, the dividend. Hence, a quotient is how many times the dividend can be divided by the divisor.   Having said this, the remainder of the quotient is NOT be added to, or expressed as to be added to, the quotient. Here, for this example, the quotient is NOT: 6 + 1, 6.1, or "6 with a remainder of 1 (one more division?)", since 1 is part of the dividend, and is not part of the quotient. Hence the quotient is also NOT: 6 + 1/31 , or 6 + 1/6.

Sometimes the quotient might be expressed as: Quotient and Remainder of the dividend left over
Q and R   , and for the above example:
Q=6 and R=1
R is technically, not part of the Quotient

Here, 6 is the number of ("multiple") times the divisor can "go into (wholly, completely, entirely or evenly)" the dividend. Since there is a remaining part (the 1) of the dividend left, and 5 cannot go into 1, this "last", incomplete or "partial"

division is simply expressed or indicated and added to the quotient (as a fraction or part of a (one) complete division). The quotient is therefore:

6 + 5⟌1   =   6 and 1_   =   6 + 1/5   : Note that the added in, division of 1 by 5, to the quotient,
                    5   is essentially adding in a fractional part of the divisor value.
                                                       One-fifth of a whole (one) division by the divisor is one-fifth of the 5.
                                                       One-fifth of five = ((1/5) of 5) = 1 which does equal the remainder
                                                       of the dividend left over due to an incomplete division.

And for an even deeper understanding:

5⟌ 31   The dividend can be expressed in many ways such as:

5⟌ 30 + 1   which can be expressed as:

30 + 1   :a division expression, and it is also considered a fraction (here, of 5) which is equivalent to:
    5   (Knowledge, understanding about fractions may be needed here, and is discussed further ahead.)

30 + 1_   simplifying the first division in this sum, we have the same quotient as shown previously:
5   5

6 + 1/5   = 6 + 0.2 = 6.2 . Checking: (5)(6.2) = 31

       It's easier to learn manual (by "hand") division by example, so as to solve (find, arrive at) the result (the quotient), rather than by "wordy" and seemingly complex explanations. An example is shown below. This example is sometimes referred to as "long division". Even though the dividend, divisor, or quotient values may contain both an integer (basically a whole part) and fractional part (a value less than 1), the process of performing the operation with "long division" is usually done completely with (single digit) integers (basically whole numbers or the "counting numbers"). The fractional part of the dividend is of course still considered and utilized during the division, but it will be first included as part of the dividend expressed as only a whole number. If the divisor has some number of "decimal places", the decimal point in both the divisor and dividend is moved the same number of decimal places rightward, essentially ridding the fractional part of the divisor and possibly the dividend. Doing this to the divisor, and then the dividend, the decimal point in the quotient will also be adjusted correctly to its proper position. The actual division of the dividend by the divisor begins at the most significant (leading) digits of the dividend and continues to the least significant (trailing) non-zero digit. If the divisor cannot divide at least once into the remaining portion of the dividend yet to be processed, the result or remaining part of the division process is called the remainder (of one division) and is expressed as being part (a fraction of 1 to be added into the quotient, and not a full increment of 1) of the quotient.

About moving the decimal point in both the divisor and dividend, and the same number of places:

Since: quotient = dividend
                         divisor

Mathematically, the same quotient will result if you multiply both the divisor and dividend, or both numerator and denominator of a fraction - creating an "equivalent (same quotient value) fraction", by just the same value, for example, 10 or some other power of 10, such as 100, 1000, etc.   When you move a decimal point rightward in a value, each time you move it rightward, you are essentially multiplying each new value by 10, and-or multiplying the original value by another (incremental, integer) higher power of 10. First consider that any value divided by itself is equal to 1.   Ex. (10/10) = 1, and that multiplying any value by 1 is that same value:

$$\text{quotient} = \frac{\text{dividend}}{\text{divisor}} = \frac{(\text{dividend}\,(10))}{(\text{divisor}\,(10))} = \frac{\text{dividend}}{\text{divisor}}\,\frac{(10)}{(10)} = \frac{(\text{dividend})}{(\text{divisor})} \times 1 = \frac{\text{dividend}}{\text{divisor}}$$

As an introductory example of the actual working method of dividing "by hand", first consider any value divided by 1. The result, the quotient, is surely and always that same value as the dividend:

   1 | 123   : one-hundred and twenty three divided by one

```
      123
  1 | 123            : the divider of 1 can go into the first digit of the dividend once, put a 1 in the quotient
  -  1              : 1 x 1 = 1, then subtract this, and then "bring down" a remaining digit of the dividend, and
     02              : continue to divide, here, 2 divided by 1, is 2, put a 2 in the quotient
  -   2              : 2 x 1 = 2, subtract this, and then bring down a remaining digit of the quotient
     03              : continue to divide, here, 3 divided by 1 is equal to 3, so put a 3 in the quotient
  -    3             : 3 x 1 = 3, subtract this, and the remainder of the dividend is 0, therefore, the division is complete,
       0               and it was an even or perfect division with no remainder of the dividend left over, and therefore,
                       no partial or incomplete division will be expressed as part (summed to) of the quotient.
```

Ex.   Divide 172 by 5

First, 5 cannot divide into the first leading, significant digit (the 1), so then check if it can divide into the first two leading digits (17):

```
      3 .                     34.               +
  5 | 172.0      or    5 | 172.0                         5 | 20.0
     15                     170     : (34x5=170)
     22                      20
```

17 - 15 is 2. If this difference was greater than or equal to 5 (the divisor), an error has obviously been performed. The error would be that the corresponding digit in the quotient is too small, yielding a product that is too small, and then a difference that is too large and being greater than the divisor itself. Since 2 is less than 5, we can continue the division by first "bringing down" the next digit of the dividend. Here, 22 is an intermediate difference or remainder (of the dividend). To verify this, we note that the 3 in the second digit (from the decimal point) of the quotient actually represents 30, and 30 x 5=150, and 172 - 150 = 22. Since 22 can be divided by 5, it is not the final remainder of the dividend, and that must always be less than the divisor. The (integer or whole number) division continues:

```
      34.
  5 | 172.
  -  15
     22
  -  20        : 5 x 4 = 20
      2  = r    : r = remainder of the dividend left  (here, r or R = 2,of the dividend of 172)
```

Since any integer multiple of 5 cannot divide "evenly" (at least once) into 2, then 2 is the remainder (r) left of the 172. 2 is obviously less than 5.   The remainder is always less than the divisor.

When checking a division problem where there is a remainder (of the dividend) left over, the remainder is added to the product of the quotient and divisor, so as the total or sum will equal that dividend. For the last example:

```
    2
   34          :   quotient
x   5          : x divisor
  170
+   2          : + remainder
  172          : = dividend       : checks
```

The quotient in the last example may also be written or noted as:

```
       2
  34 and  5    or =   34 and 2/5   or =   34 + 2/5   or  more simply and commonly as:   34 2/5 = 34.(2/5) = 34.4
```

We see that a quotient is sometimes explicitly expressed as:

quotient = (even, whole or entire divisions) + (remainder/divisor)   which equals:
quotient = (even, whole or entire divisions) + (partial or fraction of 1 division)   : as an expression, or as:
quotient = (even, whole or entire divisions) **.** (partial or fraction of 1 division)   : as one (numeric, decimal) value

Using this notation, checking to see if the product of the quotient and divisor equals the dividend requires knowledge of more advanced mathematics (that will be shown ahead), specifically the use of adding or combining fractions. Note that the remainder is expressed in the notation above as part or fraction of the divisor (not as a part of the dividend or quotient). The value of this fraction though is equal to the value of the fractional part of the quotient when it is expressed as just a single standard decimal number.

The division does not have to stop at the remainder of the dividend left. If you "bring down trailing zeros" and continue to divide, you will automatically get the decimal equivalent of the remainder as part of (and added directly to) the quotient.

```
      34.4        : Note, 0.4 is the fractional part of the quotient, and this part equals 2/5 as indicated above.
 5 │ 172.0          Also, 5 x 0.4  = 2  = whole value remainder of the dividend, and 2/5 = 0.4
    15               Again, at this point, 2 is the remainder "left over" of the dividend, and is NOT the remainder of the
    22               quotient.   2/5 (or 0.4) can be considered as the numeric expression for the remainder or final part of
    20                a one (incomplete) division, and which is to then be added to as (a fractional) part of the quotient.
    20        : bringing down a "trailing" 0
    20
     0        : here,  0 indicates the division is complete when there are no more digits of the quotient left.
               172 has been fully divided by 5.
```

```
checking  :      22
                34.4
            x      5
               172.0
```

Ex. Divide 1 by 0.2

```
  1.0  units        or  =   0.2 │ 1.0    : here,  strict numeric values with unexpressed units can be understood
  0.2  units                                and-or treated as having units of "ones" or "wholes".
```

Here, we can multiply both the numerator and denominator by 10 to clear or rid any of them of a fractional portion, or simply by moving the decimal point in both the divisor and dividend one position rightward: Note, that 10/10 = 1, and multiplying anything by 1 is still that same value numerically, and is without issue or worry of obtaining an incorrect result:

$$\frac{1.0 \text{ units } (10)}{0.2 \text{ units } (10)} = \frac{10 \text{ units}}{2 \text{ units}} = 5 \qquad \text{or commonly as:} = \quad 2 \overline{)\ 10.0\ }^{\ 5.0}$$

As seen above, when dividing one quantity by another quantity whose units are the same, the result is a simple or "unit-less", plain and "strict", numeric value only. The problem could also have been expressed as:

$$\frac{1.0 \text{ units}}{2 \text{ tenths of a unit}}$$

Since there are 10 tenths of a unit in each 1 unit, after multiplying the numerator by 10 for the (converted to) equivalent number of tenths of a unit in that value given (here it's 1 only) , we will have like or similar units to properly work with:

$$\frac{10 \text{ tenths of a unit}}{2 \text{ tenths of a unit}} = \frac{10 \text{ tenths}}{2 \text{ tenths}} = 5 \quad , \quad \text{checking:} \quad (5)(2 \text{ tenths}) = (5 \times 2) \text{ tenths} = 10 \text{ tenths}$$

This new (derived) units of: "tenths of a unit", or simply "tenths" is verified below:

$$1 \text{ unit} = \frac{10 \text{ units}}{10} = \frac{10 (1 \text{ units})}{10} = \frac{10 (\text{ units})}{(10)}$$

The value of : $\frac{(\text{units})}{10} = \frac{(1)}{(10)} \text{ units}$, is commonly called: "one-tenth of a unit" , a "tenth of a unit", or simply "tenths". (1/10) unit = 0.1 unit

1 unit = 10 tenths of a unit   or simply :
1 unit = 10 tenths

This unit of tenths could have been called something else, but they are kept as is to reflect that they are derived from a whole (1) unit, and that the decimal system is based on ten = 10.

For the above example, it is perhaps easier to also visualize the division when the dividing is by a whole or integer number of units (like 2 tenths) rather than some fractional (like 0.2 units) or mixed (whole + fractional) number of units: [FIG 2]

(1/ 5) graphically:

Letting the circle represent 10 tenths instead of 1.0 units.

each section is 2-tenths instead of 0.2 units
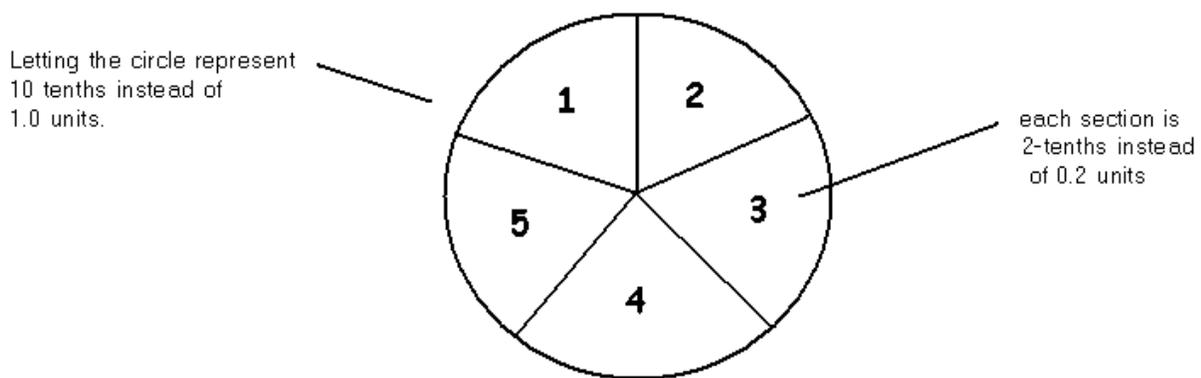
1 is the numeric representation of the whole or entire thing in question, and is graphically represented as one complete or whole circle. Note here that 2-tenths is equal to or composed of 1 tenth and 1 tenth. Each section of the circle divided into 5 parts or sections, has a numeric value or representation as (1/5). (1/5)=0.2 and this can be considered as:

$$\text{quotient} = \frac{\text{dividend}}{\text{divisor}} = \frac{1}{5} = 0.2$$

Consider these general graphics or visualization representations below of the division of any specific value greater than or equal to 1. The entire circle area represents the specific value used as the dividend, the number of segments of the circle represents the divisor value, and each equivalent segment size or (area) portion of the circle represents the quotient value: [FIG 3]

value
3

value
2

value
1
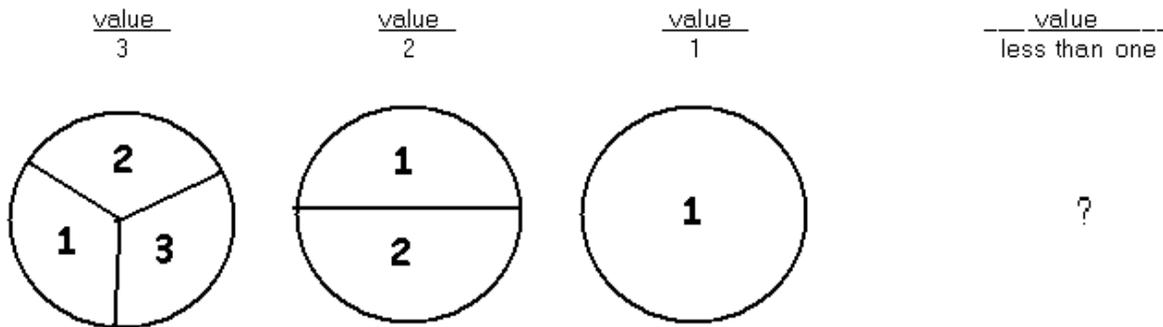
value
less than one

For the first 3 values divided, the quotient (represented as a portion or segment area of the circle, and is equal to the size of each portion or segment of that circle) is clearly less than the value in the dividend or numerator, but for the circle or value on the right, the quotient or result is actually a bigger circle or value, that is, the quotient is actually larger than the dividend value. This is a reason why we changed the divisor numbers to be greater than or equal to 1 for the divisions in the above examples, such as when the dividend and divisor are first multiplied by 10 before the division starts.

A very helpful method to perform "long-division" (a manual division computation, by pen and hand) is to take the worthwhile effort to create a table of all products of the divisor and integers from 1 to 9 before performing the division calculation by hand. Some of these products may not be needed due to the specific values used in the division expression or problem, but they will be there (already calculated) if needed. This will eliminate a lot of possible guesswork, and therefore it can save some time, and may even prevent some errors.

Ex. Divide 5.2 by 741,279

Below, rather than use multiplication to create a useful table for division, repeated addition is utilized since it (addition) is easier in itself and requires one less repetitive step to create the table. When the division cannot be complete (as like it with an "even division" that would be complete at some point, and not have an "endless" quotient composted of unending digits), there is a good chance all the multipliers (0 to 9) will be needed and used in the quotient. Sometimes, one or several specific digits repeat over and over in a quotient:

```
           741279   (1)        :  741,279 x 1
        +   741279
           1482558  (2)
        +   741279
           2223837  (3)
        +   741279
           2965116  (4)
        +   741279
           3706395  (5)        : 741,279 x 5
        +   741279
           4447674  (6)
        +   741279
           5188953  (7)
        +   741279
           5930232  (8)
        +   741279
           6671511  (9)        : 741,279 x 9
```

Using this created table of the single digit integer products (or multiples) of the divisor:

```
                    0.0000070149
       741279 |  5.2000000000
          -   5188953       : 7   , optionally indicating each new digit of the quotient during this process.
              110470            : each difference here should always be less than the divisor,  then "bringing down"
          -         0      : 0     a zero will essentially multiply this difference by 10 so the division can continue
              1104700
          -   741279      : 1
              3634210
          -   2965116     : 4
              6690940
          -   6671511     : 9
                 19429
```

For practicality, here, the quotient is "rounded" (basically means limited to, and approximated) to only 9 decimal places is: 0.000,007,015   The underlined 5 means that this value was rounded at that decimal place and precision.

Division can be used to check the result (a product) of a multiplication operation.   Given the product of two factors, divide the product by one of the given factors, and if the result (a quotient) is equal to the other factor, the product is correct.

Ex. 5 x 4 = 20   now checking with: 20/4 = 5   and, or   20/5 = 4   : we get the other factor or operand

It is interesting to note that electronic computers can perform multiplication by repeated addition. Subtraction is performed using a special form of addition known as complement (value(s)) addition. Division can be performed by repeated subtraction. Hence, it is possible for a computer to perform all the basic mathematical operations by using addition. Computers can be designed to contain just the electronic circuitry to perform one main mathematical operation such as addition, and therefore eliminate or reduce some circuitry and complexity needed for the other mathematical operations, and making the computer less expensive.

USING MULTIPLICATION AND DIVISION TO MEASURE SMALL THINGS

Here is an example of using division to find that which is normally not practical. Something small, such as a grain of sand or rice, cannot be weighed using many weight measuring ("scales") devices. A method will now be shown as to how to weigh these small objects. This method utilizes the fact that if you magnify (increase) something either physically or mathematically by a multiplication, that you can find that original something by a corresponding "demagnification" (a decrease, factorized) using division.

Given 1 grain of rice, if you multiply the number of grains of rice by 500, you essentially have also multiplied its' weight by the same value. Now, this total weight of 500 grains of rice can be measured by using a commonly available fine scale. If these 500 grains of rice weigh 1 ounce, you can use division to find 1 grain of rice and the weight of 1 common, "average", typical, or usual grain of the type of rice being considered.

1 grain x 500 = 500 grains

$\dfrac{500 \text{ grains}}{500}$ = 1 grain

weight of 1 grain x 500 = weight of 500 grains

weight of 1 grain = $\dfrac{\text{weight of 500 grains}}{500}$ =

$\dfrac{0.333 \text{ oz.}}{500 \text{ grains}}$ = $\dfrac{0.000667 \text{ oz}}{(1) \text{ grain}}$

mathematically ("solving for 1 grain") :
(dividing the product by one factor, to find the other factor)

Considering the weight of each grain of rice:

Mathematically solving for the weight of 1 grain:

Now substituting the numeric values given or measured:
(weight of 500 grains = 0.333 oz, [depends on the type of rice])

: Weight of 1 grain =~ seven ten-thousandths of an (1) ounce".
: After dividing both numerator and denominator by
  500 we get less than a thousandth of an ounce per (one) grain.
  Later in this book, it will be shown that the entire
  numerical equation shown is called a "proportion" (for a
  given problem or system) and can be re-used, for example,
  so as to now find the weights of any number of grains of rice.
  Though the known reference weight (of either 500 grains, or
  1 grain of rice) and the weight you will be trying to find will be
  different; not in balance, mathematically, both sides will still
  be in balance, and that balance is that each side has the
  same exact quotient value. Here, each side is equal to
  (0.000667 oz/grain). 0.000667oz = about 0.0189 grams
  = about 0.019g = 19 milligrams.

Also, given:

500 grains of rice = 0.333 ounces

If we divided both sides of this equation by 0.333 we can find the number of grains of rice per ounce of weight:

$\dfrac{0.333 \text{ ounces}}{0.333}$ = $\dfrac{500 \text{ grains of rice}}{0.333}$

1 ounce = (about, ~) 1500 grains of hard, dry, uncooked rice

: For the type actually measured, it was a white rice
  variety, that was almost 1/4 inch long, and about
  3/32 of an inch wide at the thickest part in the
  middle. It appeared to be a "medium grain" sized
  of rice. The rice was also hard, dry, and uncooked.

As a check of calculation correctness, and a possible improved value, the above process should be repeated with another set of near randomly selected rice grains so as to have an average value will be as correct as possible for the weight of a

rice grain, for most future circumstances.   Scales to measure low weights, perhaps a hundredth of a gram, are available, and it is possible to construct your own. If you multiply the number of grains per ounce by 16, you can calculate an estimate of the number of grains per pound (16 ounces). Doing this, we find that there is about 24000 grains of rice in a pound of rice.

To weigh something such as a grain of sand, the basics of the above method can also be utilized. First, you would have to count many more grains of sand just to have a weight of usable value for a scale. This is a very impractical method for most people.   To overcome this, you can first find out how many grains are in a certain volume (a measure of 3 dimensional space) of sand, say one cubic centimeter or inch. A cube, as seen in [FIG 20A], is a solid with all it's 3 dimensions, length, width, and height all being the same value of length, and for our purposes here, they will have the same length of 1 inch. A cube shape is like that of a square area, an (bounded or specific) area of plane, hence a planar area which has then been moved upwards a height equal to the length of a side of that square shape.   This creates a volume or (3 dimensional) spacial area called a cube. A cube is the basic or standard unit of measurement for volumes or spacial areas.   For this example, if you find that there are 50 grains of sand lined-up per ("in a", "in one", "for", or "equivalent to") half and inch = 0.5in, then there are (50 x 2) = 100 grains of sand lined-up per 1 inch. There are then (100 x 100) = 10,000 grains of sand per square inch, of course, this assumes that the each grain of sand is roughly the same (average) size in the length and width dimensions, if they were not, you would have to first count and then estimate or average how many are actually in the length and-or width dimensions. Continuing, there are (100 x 100 x 100) = 1,000,000 grains of sand per cubic inch, again, assuming each grain of sand is roughly the same height.

weight of 1 grain   x   1,000,000   =   weight of 1,000,000 grains   mathematically:

weight of 1 grain of sand   =   _weight of 1,000,000 grains_   :after dividing (each side of the equation)
                                              1,000,000   by 1,000,000

If this cubic inch of sand weighs 3 ounces, that is:

weight of 1,000,000 grains of sand = 3 oz: again, dividing each side by 1,000,000:

weight of 1 grain of sand   =   ___3 oz.___   =   0.000,003 oz.   : 3 millionths of an ounce
                                    1,000,000

A note on how similar atoms of a given element are typically counted. Since there are so many atoms in a given mass of atoms, and which therefore cannot even be seen so as to be counted by a person, the number of atoms in a given mass are counted by the weight of that given mass. This is because, for each gram of mass of a certain given element, there is a known corresponding weight and amount of atoms.

SOMETHING ABOUT INFINITY

Infinity (ie. "non-finite", "non-ending", "endlessly", "without end") does not, and cannot, have a specific value, and is rooted in a concept of something that is, and can be or conceived (but perhaps not necessarily easily provable) to be as possibly and probably unending. Outer space is not a process, but rather a thing, and is said to be infinite in size, "unending" and can therefore be without any specific measurement or value of a size or distance. A process can be repeated without end, hence that process is said to be an infinite process or simply infinite. If there will always be a "next second" of time, or at least in thought or theory, time can therefore be said as being something that is infinite.   Something, even when counting or using numbers, can be infinitely large (without bounds) or infinitely small (almost 0, or almost nothing).   The digits of Pi (3.14**...**), that is mathematically part of all circles, are said to be unending and therefore infinite. Some conceivably infinite processes may have just recently began, such as some machine that just started counting higher from an initial value of 0, and hence an infinite process may have not always been in existence, and hence it was not always infinite in some way, but perhaps infinite in only one kind of way, direction or ability now.

Ex. Since 1 can always be added to a number to create or find the next larger number, this is an infinite process, and here, the sum is also infinite in value:

    0 + 1 + 2 + 3 + 4 + 5 + **. . .**   : an infinite process of summing, and here the sum is not any specific value, but is an
                    unknown value that could be described as infinitely high and-or unending value

Below are some more notes and ideas about the topic of infinity to perhaps consider and inspire some further thoughts, and perhaps may flare the temper of a few strict mathematical purists. If you are new to math, you may simply skip over this topic without any problem, and return when you are able.

When the divisor of a division problem gets smaller and smaller (said as "approaching 0", "becoming near to 0", "nearing 0", "approaching a limit of 0", but never actually being equal to 0 because that would mean the divisor and a real valid division problem no longer exists, and also that division by 0 is undefined or an error) you could conceive that the quotient can possibly have an aspect of infinity (its mathematical symbol is: ∞; which somewhat looks like the decimal number 8 symbol tilted on its side), and rather than speak in terms of theory, its' (largest possible, or conceivable) numeric decimal system integer value can be practically conceived or imagined as a never ending series of 9's or: 9999 **. . .**   , even though infinity is a unending process and not a specific numeric value. The **. . .** ("ellipses" = "and so forth") symbol essentially indicates to repeat the process unendingly. Sometimes, when a result in an unending process is sufficient or acceptable, the process can be stopped.

There is also no limit to how small something can be. Things and-or numbers such as this are said to be "endlessly or infinitely" small. The smallest numeric value can be conceivably expressed as a process of an endless number of zero's between the decimal point and the number 1 such as: 0.00000 **. . .** 0001 , or using other expressions such as 0.1^n, or 1^-n, as integer (n) approaches infinity. Consider when the divisor of a division problem gets larger and larger, and-or the dividend gets smaller and smaller, the quotient will become (infinitely or endlessly) smaller and smaller.

It is widely accepted now that 0.9999... = 1 since 1/3 + 1/3 + 1/3 = 0.3333**...** + 0.3333**...** + 0.333**...** = 0.999**...** = 3/3 = 1. If it can be (maybe controversially) conceived that 0.3333**...** can be represented as 0.3333**...**3, then it should also be conceived that 0.0000**....**1 can be thought of as that infinitely small number, something or part (here represented with common decimal, base 10, notation) that when summed to 0.9999**...** will mathematically express and cause that value to "finally roll over" and sum to 1. Note also that whereas 0.9999**...** = 1, we now likewise conceive that 0.0000**....**1 = 0 since it will approach and effectively be a value of 0. Some will argue that 0.333...3 and 0.000...1 have or express a "last value or digit", and hence, the notation does not represent or express an infinite value, but by using the **...** symbol, the infinity concept still exists, and therefore, the "last digit" argument does not hold since there isn't a last digit due to the process.

For some extra ideas on the above concepts of infinity, there are two types of infinity, and it's best to think of them as a non-ending process without any specific numerical value. The largest (decimal system) number or the number that could represent the infinitely big could be written as: (9999...9) The number that could represent the infinitely small could be written as: (0.0000...1) Why use the 1 digit at the end? Because in our number system we begin counting with 1, and

therefore, the conceivable number, expression or notation that represents the infinitely small should include this first digit that is at least greater than 0 or nothing, and using 1 indicates that there is "something" and numerically represented. The two closest numbers next to 1, one being smaller, and the other one being bigger can be conceived or imagined as: (0.9999...9) and (1.000...1). Their conceived difference is (0.0000...2) which is larger than the conceivably smallest possible decimal (infinite) number (0.0000...1) greater than 0. Some would argue that (0.9999...9) is the same as 1, but that is like saying (1.0000...1) is the same as 1, and then that is like saying (0.9999...9) is then equal to (1.0000...1) which, as just mentioned above, has a difference between them of (0.0000...2), and not 0.0 as when they are actually equal. Still, you could argue that (0.0000...2) is an infinite process and that it will also approach a value of 0. It is fair to equate 0.9999....9 to 1.0 since in theory, there is to be no empty space between two adjacent points and no other number between two adjacent numbers, and even though 0.999....9 is expressed as (an infinite) process and not a specific number, and which "zeros in" or converges to 1.

THE ORDER OF OPERATIONS

The order of operations is a commonly accepted and basic plan for simplifying and solving mathematical equations. It is used so that there is a formal-like consistency and less chance of misunderstandings and error. The plan states which operations of an expression should be performed first or have precedence (more importance or priority). Due to this, it is also a consideration and guide for expressing equations so as they may be correctly solved by someone who is observing the order of operations plan. The 4-step plan is given below in steps of precedence.

1. Simplify powers, roots, and logarithms.

Ex. 5^2 = 25   : a power example, , here: "(a base of) 5, "raised" to the second (2) power" or "5 square"

Ex. $\sqrt{25}$ = 5   : a root example, , here: "the second (2) or square root of 25"

Ex. $\log_5 25$ = 2   : a logarithm example, here: "the log of 25, using a base value of 5, is 2" , 5^2 = 25

The concepts of powers, roots, and logarithms will be covered further ahead in this book. A logarithm expression is simply an expression for the mathematical operation to find or solve for the exponent of the base value of an indicated power value.

2. Simplify expressions that are in grouping symbols.

Grouping symbols surrounding an expression indicate that the expression is to be possibly simplified first before any other mathematical operation can be performed on the resulting (and usually simpler) expression or value. Grouping symbols are often used to force precedence and-or to bound and clarify an expression so as to prevent a misunderstanding and error. If there are other grouping symbols contained within or between grouping symbols, sometimes this situation or expression is said as having "nested" or "inner" grouping symbols, the expressions within them are simplified first and from the innermost to the outermost (ie. inner to outer). That is, each corresponding pair of grouping symbols will be successively cleared or canceled (canceled-out or removed) if possible. Hence, this step may also be called: **Cancel or clear grouping symbols.**

Some examples of grouping symbols are:

( )   : parentheses and other similar shapes such as brackets: **[ ]**

---- : division symbol or fraction symbol
Here, the dividend or numerator represents one value, and the divisor or denominator represents one value. That is, either or both might be expressions that can be simplified first, and then the division operation can be performed so as to produce a quotient.

3. Perform multiplication and division.

This is done (for conformity to avoid any confusion or ambiguity) from the leftward side to the rightward side of the expression. Usually, performing division has slightly more precedence than multiplication since it contains the division grouping symbol. Performing the division will effectively clear that grouping symbol, producing a quotient value which can then be possibly used as an operand for another operation.

4. Perform additions and subtractions.

Like multiplication and division, these are also done from leftward to rightward in the expression.

Note that for the second step, simplify expressions in grouping symbols or clear grouping symbols, that steps 1,3, or 4 may be needed first, during, or "along with" for this process.

It is possible to express the concept of the order of operations using just a few words such as: "perform the most advanced operations first". Another plan to simplify expressions is what I call the CDC plan which is closely related to the plan given above, and will be presented further ahead in this book. Below are some basic working examples of using the order of operations.

Ex.   5 +1 x 7

Without following the order of operations method, and simplifying or solving from left to right, 5 would first be added to 1, and then this sum would then be multiplied by 7. This is how some "chain-logic" (evaluated left to right, as written, entered or processed) "home-use" or "4-function" calculators, and even some outdated or simple computers, would evaluate the expression, but for conformity, we are encouraged to use the order of operations from now on that considers the (order of) precedence or importance of the mathematical operations, rather than their (left to right expressed) order.

Without following the order of operations:

    5 + 1 x 7   When explicitly expressed and evaluated as: (5 + 1) x 7 using "chain-logic" :
      6 x 7
        42

Following the order of operations:

The 1 would first be multiplied by 7, and then this product will be added to 5. The result is very different than the last result even though both original expressions are the same. This method is the preferred method used by "algebraic logic" or "scientific" calculators and computers to evaluate an entered expression.

    5 + 1 x 7   is evaluated as, using the order of operations: = 5 + (1 x 7)
      5 + 7
        12

To achieve this same result using a chain-logic calculator, evaluate the higher-ordered or most advanced operations of the given expression first.   If you want, you can also rewrite the expression with some grouping symbols to clarify how it is to be evaluated (processed, simplified or solved):

    5 + 1 x 7   by using some grouping symbols:
    5 + (1 x 7)   or perhaps by switching the two operands of the addition operation:
    (1 x 7) + 5
      7   + 5
        12

When an expression inside grouping symbols is simplified to a single value, it is possible and desirable to remove (clear or rid) the grouping symbols that surround it as long as it does not change the intended order or operations and therefore, the result.

Ex.   5 + (2 + 3)

      5 + (5)   : after using step 4 and performed the addition operation expressed within the grouping symbols
      5 + 5   : step 2, basically cleared/rid the grouping symbol that are not needed now
                     If you want, you can consider   5 + (5) = 5 + 1(5)   =   5 + 5 , so as to remove the grouping symbol
        10   : step 4, addition

Ex.   5 + 3(3(7x4)) + 2

    5 + 3(3(28)) + 2   : step 3, after simplifying the innermost grouping symbols, here (7x4) = 28
    5 + 3(84) +2   : step 3 and 2, multiplication (to clear grouping symbols), here 3(28) = 84
    5 + 252 + 2   : step 3 and 2, using multiplication which then clears the grouping symbol
        259   : step 4, using addition, here, the order of the summing of the values, does not matter

Ex. Simplify 5 + 3 x 10

Since multiplication has more precedence than addition, this expression is usually written or expressed, and simplified as:
5 + (3x10)   : for some basic home use calculators you might have to use these steps in the order of: (3 x 10) + 5

Here, the grouping symbols clearly indicate or "force" a precedence of performing the operations. That is, here 5 is to be added to the entire value enclosed in the grouping symbols. This value within the parentheses is an expression which must be simplified first whether you are using a calculator or not:

5 + (30)   : after performing step 3
5 + 30   : after performing step 2
  35   : after performing step 4

Ex.   5 x $\underline{10}$ + 3
          2

or   ($\underline{5}$ x $\underline{10}$) + 3 : due to the precedence of multiplication, a grouping symbol is shown here for visual clarification
     (1   2)

    (5 x 5) + 3   : after performing step 3, (division, here, 10 divided by 2)
      25   + 3   : after performing step 3, (multiplication), then step 2
        28   : after performing step 4, addition

The more you practice with the simplification or "simplifying" expressions, and the order of operations, the more "natural" it will feel and become to you.

COMMUTATIVE AND ASSOCIATIVE LAWS

Multiplication and addition are said to be commutative (based on the word "(to) commute" which means "moveable", "changeable", "transportable"). That is, if you exchange or "switch" the operands (values the operators are to be applied to) of the operation, the result will still be the same. This may be done to help clarify, express, or evaluate an expression.

Ex. 5 + 2 = 2 + 5   : * * * * * + * *   =   * * + * * * * *   : somewhat like a mirror image.
       7   =   7   * * * * * * *   : both have the same sum.

Ex. 4 x 9 = 9 x 4   : maybe you don't know 4 x 9 offhand, but perhaps you have already memorized 9 x 4
       36   =   36

Multiplication and addition are also said to be associative (to associate, join, pair, connect, align or group with). That is, you can associate any two or more operands together by grouping (usually with parenthesis; ( ) , or backets; [ ]) and the result will be the same.

Ex. 5 + 2 + 3   =   (5 + 2) + 3   =   5 + (2 + 3)
       10   =   7   + 3   =   5 + 5
       10   =   10   =   10

Ex. 2 x 4 x 3   =   [2x4] x 3   =   2 x [4x3]   =   (3 x 2)(4)
       24   =   8 x 3   =   2 x 12   (6)(4) = 6 x 4
       24   =   24   =   24   24

FRACTIONS

A part is less-than the whole of which it comes from. The whole thing is "100%", all, complete, entirety or "one" (oneness and numerically 100%=1, or the undivided something). When only a part or parts of a whole are to be expressed mathematically, fraction or fractional notation (simply called "fractions") is used to numerically represent this part or fractional value of the whole. A fraction of something is a piece, portion, segment or part of that whole something that was essentially broken or divided into parts or pieces. The part(s) in question is said to be a fraction, or fractional part of the whole. A fraction or part is numerically represented as a value that is likewise less than the whole (or entire) thing or amount expressed numerically. If we let 1=1.0=100% numerically represent the entire thing, then a value less than 1 will represent a lesser or smaller part(s), portion, or fraction of that whole thing.

Symbolically: (fractional part(s) of the whole part) < (whole part)   : A fractional part(s) is less than (<) the
                                                                                                        whole part of which it was from.

Numerically: (value(s) or sum of fractional or part values is less than 1) <= 1.0

        Ex: 0.5 < 1.0   : Here, a part numerically represented as 0.5 is less than the whole (1)
        Ex: 0.5 + 0.5 = 1   : Here, the sum of two parts numerical representations sum to 1.
        Ex: 0.3 < 1.0
        Ex: 0.3 + 0.5 = 0.8 < 1

We can often recognize that a given part is only a fractional part of something. For example, 5 is only a part of 8. 5 is the part or portion of the entire, all or whole value of 8. 5 is different and less than 8, and this can be mathematically verified by subtracting the two values to show that there is a difference in values: 8-5 = 3. 3 is the difference or amount of separation between those two values of 5 and 8.   For example, 200 is only part of 203. 200 is a fraction of 203, and their difference is 3.   Rather than express a numerical relationship of two values as a difference, a better way to quantify and express their numerical relationship is to show how many times bigger or smaller one value is to the other (reference) value. The resulting value will show the decimal equivalent value that the part is with respect to the whole which can be expressed numerically as 1.0 (= 100% = all, entire, whole). 5 of, from, or out of 8 and expressed as a fractional expression is: 5/8 , and this value = 0.625 after division is performed , 200 out of 203 = 200/203 = 0.985   Even though the difference was 3 for each example, this difference value of 3 does not numerically represent how much the part or portion of the whole is numerically. 3 doesn't quantify the portion, but rather only the difference.   The values of 0.625 and 0.985 just shown do. The value of 0.985 is much more than 0.625, and is therefore a larger fractional part. 200 is therefore a larger part or portion of 203, than as 5 is to 8. Since 0.985 is close in value to 1, it indicates that the part or portion in question is almost the entire value in question. 8 or 8 parts = 8/8 , or 203 of 203 parts = 203/203, would result that the fractional value in question can be represented with and has a value of 1, and this indicates that the fractional value in question is equal to the entire, whole part or value.

We have seen that quantifying and expressing the numerical relationship of a part to the whole by using the difference value does not numerically represent how much that the fractional portion is of, represents, or how much it is of the entirety or whole part, but it rather just expresses or indicates how much different, and not the similar, the two values are. As just seen in the two examples above, the difference may be the same, and yet the portions of the whole are much different. A difference value does not quantify or numerically express a portion value. A quotient value can quantify or numerically express a portion. The mathematical operation will be same as a division problem, but the concept and terminology is different since we are not actually trying to divide any parts up into even smaller parts, but rather expressing just the numeric relationship of one part in relation to, or with respect to, another part or the whole part.

The basic mathematical (standard) "formula" or expression for a fractional value is a numerator value divided by a denominator value.

 numerator   : The word numerator is from the word numerate, meaning to count or assign a number to or of.
  denominator   : The word denominator is from the word denominate. Nominate means to assign a name to,
                                          and here it's a number name or identifier. To denominate is to completely assign a name,

hence denominator means the complete number to or of.   The numerator number used is in reference to the denominator number used.

You can roughly think of this as: $\dfrac{\text{parts}}{\text{total parts}}$: "parts to total parts"   or "parts of the total number of parts"

We see that this is essentially the same as a division expression and problem, except for the names used. Comparing the two types of expressions, the numerator is similar to the dividend, and the denominator is similar to the divisor. When something is divided, such as into (fractional) parts, it is somewhat obvious that a division type of expression will be used to express this.

If a whole (1) thing or quantity is evenly split or divided (a division problem) to 2 equal parts, we can represent this numerically as this division expression:

$\dfrac{1}{2}$: this also expresses the numerical representation or value of each similar (fractional) part. This numeric expression is therefore called a fraction. After dividing, the quotient is called the fractional (in decimal number form) value. Each identical fractional part is equal in value, here, (1/2) = 0.5

The numerical value of, or given to, each separate (and identical) part is the value of this fraction (but only when a whole or 1 is divided). Each part is said to be "one-half" of the whole. Numerically, each part has a value of (1/2), or the equivalent decimal value of 0.5, when the 1 is divided by the 2. 0.5 is clearly less than the whole of 1. [FIG 4]

part = $\dfrac{1}{2}$

part = $\dfrac{1}{2}$

the whole, all or entirety = 1.0 numerically and graphically
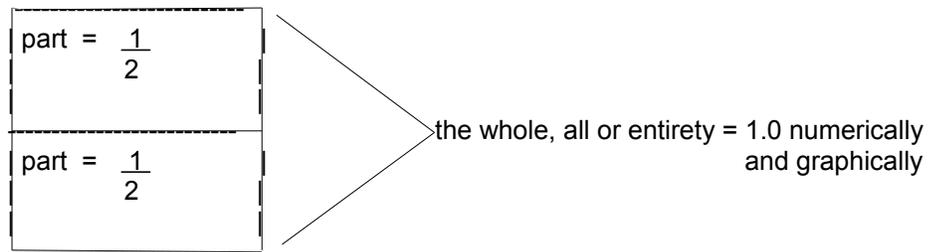
The sum (mathematical, and-or physical combining) of all the fractional parts will always equal that which was divided into those fractional parts. For the example just shown:

part + part   = whole, all or entirety that was divided up into two parts
(1/2) + (1/2) =   1/2 + 1/2   =   0.5 + 0.5 = 1.0

Here is how to express the whole (1) of something divided into 4 equal sized parts: $\dfrac{1}{4}$

Each part will also have this numeric value of (1/4) or= 0.25   Considering just two of these four parts taken, put or summed together as a group, then what fractional part of the whole does this represent? Clearly, since two parts is twice as much as one part, the fractional value must be twice as much:

one-fourth and one-fourth equals two-fourths   or: 2 x one-fourth = $2 \times \left(\dfrac{1}{4}\right) = \dfrac{2(1)}{4} = \dfrac{2}{4}$: "two-fourths"

$\dfrac{1}{4}$ + $\dfrac{1}{4}$ = $\dfrac{2}{4}$ = 0.50

0.25 + 0.25   =   0.50   : or= 0.25(2) = 0.50 since repeated addition can be represented as multiplication

Though it is possible, do not interpret the value of 2 in (2/4) as meaning 2 wholes being divided up into 4 parts, since for this problem we are dealing with only one whole, or entirety of a thing, numerically expressed as 1, that was divided up. So therefore, a numerator can be defined as the number of (all similar) parts in question or being considered of the whole, and a denominator is said to be the total number of parts that the whole was evenly divided up to. You can visually read the mathematical notation of the fraction symbol as meaning "of", "out of" or "from".

Ex. $\dfrac{1}{2}$: "one, out of two" or "one part, out of two parts" or simply as "one, out of two" or "one, of two" , or
"one-half" = "one-half of the whole" = (1/2) of (1) = (1/2) of (1/1) = (1/2) (1/1) = (1/1) (1/2) = (1) 1/2 = 1/2

Ex. $\dfrac{2}{6}$: "2, out of 6" or "2 parts, of something divided or separated into 6 total parts" ,
This can be considered as equal to: (1/6) + (1/6) = 2 (1/6) = (2/1) (1/6) = 2 / 6 and-or = 1/3 = 0.333...

If given a fraction and you perform the indicated division, the resulting quotient is the decimal equivalent of that fraction.

Ex. $\dfrac{1}{2}$ = $\dfrac{0.5}{2\overline{)1.0}}$ = 0.5   Below is another ex. where the numerator is half the denominator,
 - 1 0 and that the total fractional part(s) in question is also numerically
 0 0.5 = 50% of the whole or denominator:

Ex. $\dfrac{3}{6}$ = $\dfrac{0.5}{6\overline{)3.0}}$ = 0.5   : checking, 6 x 0.5 = 3
 30
 0

Ex. $\dfrac{4.5}{6}$ = $\dfrac{0.75}{6\overline{)4.50}}$ = 0.75   : Compared to the last example, this larger quotient indicates that the
 42 portion or fraction is larger. As the numerator comes closer and
 30 closer in value to the denominator, with little difference, the fractional value
 30 will approach (become nearer and nearer in value to) that of the whole or
 0 value of 1=100%. The closer this value is to 1, the fractional part in
 question becomes greater and nears the whole part. For this example, the
 part(s) in question could be said to be 0.75, or 75% of the whole (or entirety,
 entire) part. The portion of the whole part is 0.75.

Ex. $\dfrac{5.4}{6}$ = 0.9   : 0.9 is close in value to 1.0, and it indicates that 5.4 is a large
 part or portion of the whole (100% = 1) or entirety of 6 parts.

Ex. Of the following fractions, which fraction is larger, that is, which represents a larger value
 and hence a larger portion of the whole, all, or entire thing?

 $\dfrac{2}{3}$ or $\dfrac{3}{5}$ after performing division to find their decimal equivalents we can
 often compare them more easily:

 0.66666... and 0.60

Therefore, the part, portion or fraction of 2/3 is (slightly) larger than 3/5.   0.666666 > 0.60 and (2/3) > (3/5)

If the denominators of the fractions are the same (or mathematically made to be the same as done when adding fractions), you can then easily find the largest fraction simply by observing which numerator is the largest. Actually, one quick method of adding fractions is to fist convert each of them to their decimal equivalents value, and then perform the addition. Though the addition of fractions has not been covered yet, here is the above example solved

with the "addition of fractions with the same (or "like") denominator method" so as to find which fraction is larger:

$\dfrac{2}{3}$ and $\dfrac{3}{5}$ Converting each fraction to an equivalent (valued) fraction (numerically, it will still have the same value) where the denominators of both fractions are identical or the same:

$\dfrac{2(5)}{15}$ and $\dfrac{3(3)}{15}$: both denominators changed to 15, hence the fractions can be visually compared easily

$\dfrac{10}{15}$ and $\dfrac{9}{15}$: Note 2/3 = 10/15 = 0.66666**...**   and   3/5 = 9/15 = 0.6
These are the same values as shown above, and this is because these new fractions created from or using each are (mathematically, in value) equivalent fractions. It is also incorrect to think that a fraction can only have one equivalent fraction associated with it since there is actually an infinite number of equivalent fractions possible for any fraction or portion of the whole (1).

Now by just observing the two numerators in the fractions above, since 10 is obviously greater than 9, then 2/3 is the larger fraction.

Given any two fractions and knowing the process of converting each of those fractions to an equivalent valued fraction, and both with like denominators, observe that you can quickly see which is larger by what is called "cross multiplying" the numerators and denominators of the two fractions. For the give example, since (2)(5) = 10 is larger than (3)(3) = 9, and both of these values will have the same denominator (here 15 which is the product of both denominators), the fraction of 2/3 is therefore larger than the fraction of 3/5. Another method, perhaps not so obvious, to find which fraction is larger, is to subtract one fraction from the other. You can first convert both fractions to their decimal equivalent, or to fractions that have identical denominators so they can then be properly subtracted. Assuming both fractions are greater than 0, if the resulting value (the difference) is greater than 0, then that minuend fraction is larger than the subtrahend fraction.

This book tries to avoid the (weak, but practical) concept of "cross multiplying" because, it is often taught, accepted and-or taken for granted, without first knowing the mathematical verification as to why it can be used. The "just do this and you will have it" attitude, should in many cases, be unacceptable when you want to have a firm grasp or understanding of math and what you are doing with it, and so as to then understand more advanced concepts that are built upon the previous concepts.

The whole part, all, or the entirety of something can be thought of as, or represented by, a complete circle which can represent and be noted as equivalent to and expressed as any numeric value.   Given a certain or specific fractional value of any entire value (entirety, all, whole), each part, portion, or fraction is numerically the same regardless of the specific whole or entire value being considered. For example, given a fraction of (1/3), or "one-third" of any value, (1/3) = 0.33333... =33.3% of that whole=1=100% value or circle. The actual numeric value corresponding to that indicated fractional value of (1/3) depends on what the whole value actually is. A portion or fraction of something is always less than the whole or complete value, however, the actual numeric value of that portion may be less than or greater than 1. For example, consider this:

Whole, all, entirety or complete value	Indicated fractional value.	Actual numeric value of the indicated (here, (1/3)) fractional value of the whole.
1.0	(1/3) of 1	= 1/3 = 0.33333... = whole part divided by 3 = (whole part)/3
1.5	(1/3) of 1.5	= 1.5/3 = 0.5 : an example showing that the "whole part or entirety, may not always be an integer or counting number value, and may be greater than 1

| 2 | (1/3) of 2 | = | 2/3 = 0.66666... | : note that this is the same as having (1/3) and (1/3), or (1/3) + (1/3) which is (2/3) = ""two-thirds" when summed: (1/3) + (1/3) = (0.33333...) + (0.33333...) = 0.66666... = (2/3) |

3   (1/3) of 3   =   3/3 = 1.0

4   (1/3) of 4   =   4/3 = 1.3333...   : here we see that the actual "fractional value" or portion of 4 is actually larger than 1

5   (1/3) of 5   =   5/3 = 1.66666...

6   (1/3) of 6   =   6/3 = 2.0 = 2 parts of 6 , which can be expressed as: 2/6 = 1/3
                                                            2 parts of 6 equals 1 part of 3 = 0.33333...

MIXED NUMBERS

A "mixed number" is an expression of a numerical value that (explicitly) indicates both the number of complete or whole parts (hence a multiple of a whole, completes or entirety), and a fraction value that expresses the parts or portion of just one whole value.

mixed number = whole part + fractional part

Mixed numbers are often found in food recipes, such as for example, to indicate "two and a half cups of flour" or= "two cups plus a half of a cup", and the numeric representation of this using a mixed number method is:

2 1/2 cups of flour,   or perhaps just as: 2 1/2 cups flour   : 2 is the whole number of filled or complete cups or cup
                                                                                            units, and 1/2 is a fraction of 1 cup full.

If the decimal value of a fraction is less than one, the fraction is said to be a proper fraction. When the numerator is larger in value than the denominator of a fraction, the fraction is said to be an improper fraction since it represents a value greater than one, (1), since at the fundamental level, fractional numerical values are usually meant as, or understood to be less than the whole, or 100% of something = 1 numerically. A fraction is (usually) meant to be only a (smaller) part or portion of something.   A mixed number has both a whole part and a fractional part, hence it has a mix or several values.

First consider values such as 0.1 and 0.95. Any value that numerically represents a part that is less than the whole (entirety, all, 100% = 1.0) is only a fractional value, part or fraction. 10 by itself is not, or does not represent, a fractional part, however when you (only) have 10 of 100 entire things, which can be expressed numerically as 10/100, that value of 10 is now only a fraction or part of the whole (considered as 1=100%) or entire 100 things. Expressing this in the most basic decimal form, we see that 10/100 = 0.10, which clearly is less than 1, and hence it is only a fractional part of the whole. It can be said that "10 is (only) a fractional part of 100".

Ex. $\dfrac{9}{4}$ = 2   with a remainder (of the dividend) of 1   : 9/4 is an improper fraction

Expressing the above quotient, of the indicated division, as a mixed number (containing both a whole value and a fractional value of the divisor), this is written as:

2 $\dfrac{1}{4}$ which actually means: 2 + $\dfrac{1}{4}$:"two and one fourths". This is a mixed number. 2 is the whole part,
                                                                    and 1/4 is the fractional part. The addition of both parts, is not
                                                                    always explicitly indicated and is "understood" as being so.

That is, 2 is not being multiplied, but added to $\dfrac{1}{4}$: If 2 was being multiplied to 1/4, the multiplication would have to be
                                                              explicitly indicated such as: 2(1/4), or = (2)(1/4), or = 2 x (1/4)

Also note that when division is performed on the fractional part of a mixed number, we get the decimal equivalent of the mixed number. The value will also equal the improper fraction:

$\dfrac{9}{4}$ =   2 $\dfrac{1}{4}$ =   2 + $\dfrac{1}{4}$ =   2 + 0.25   =   2.25   : 2.25 is like a "mixed number" that contains a whole part
                                                                              and a fractional part.

Now that improper fractions can be converted to mixed numbers, mixed numbers can also be converted back to improper fractions. To do this, multiply the whole number part of the mixed number by the denominator of the fractional part of the mixed number, and then add the resulting product to the numerator of the fractional part of the mixed number. The resulting sum is a numerator of a fraction whose denominator is the same as used for the fractional part of the mixed number.

Ex. $3\frac{2}{4}$ = $\frac{(3 \times 4) + 2}{4}$ = $\frac{12 + 2}{4}$ = $\frac{14}{4}$

      The conversion process shown above is actually a "short-hand" method of adding fractions correctly of which could verify or check the method and example shown above. For the last example, the fractions to be added, and their sum, would be:

$$\frac{3}{1} + \frac{2}{4} = \frac{14}{4}$$

Hence, a mixed number represents an expression that is the sum of the number of whole (or entire) parts, and fractional parts. Note that 3/1=3. Any value can be divided by or "placed over" or divided by one, and the value will not be changed. But as mentioned previously, the fractions cannot be summed (added or "combined") until their denominators are equivalent. When the denominators of the fractions are the same, the fractions are said to have like or common denominators, and this is much like that which is necessary when adding things such as measurements that need to have the same units (of measurement or reference) so as to produce a result having those same units. Fractions with the same denominators can be said to be "like fractions". More on the addition of fractions will be shown later.

As another check to the above example, first consider that any value divided by itself it 1: (3/3) = 1

$$3 + \frac{2}{4} = (1 + 1 + 1) + \frac{2}{4} = \left(\frac{4}{4} + \frac{4}{4} + \frac{4}{4}\right) + \frac{2}{4} = \frac{12}{4} + \frac{2}{4} = \frac{12 + 2}{4} = \frac{14}{4} = 3.5$$

Note also, and previously discussed, that: $3 + \frac{2}{4} = \frac{(3 \times 4) + 2}{4} = \frac{14}{4} = 3.5$

EQUIVALENT FRACTIONS

      If you divide a given circle into two equal parts that are the same size or portion, say an upper and lower half, each part is said to be one-half, or numerically as (1/2), of the entire 1 circle. Each part and fraction that numerically represents these parts are equal and therefore they are called equivalent fractions. Here, each part is numerically: (1/2). [FIG 5]
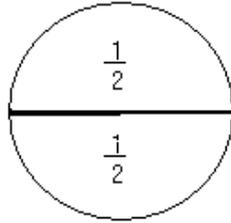
The 1 in the numerator position of the fraction represents one part in question or being considered, and the 2 in the denominator position indicates that the whole (here a circle) was divided into 2 equal parts. If you divide a whole (1) circle into 4 equal parts, each part is said to be one-fourth of that whole (1) circle. Numerically, each part is: (1/4). [FIG 6]
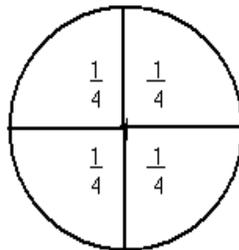
Clearly, by observation, two parts (one-fourth and one-fourth, or= two-fourths) of the circle that was divided into 4 equal parts is equivalent to the portion of one part (here, one-half) of the circle that was divided into 2 equal parts. Numerically, that is:

$$\frac{2}{4} \quad = \quad \frac{1}{2}$$
                         : both fractions are equal in value, here it's 0.5 .
                         : "2 is to 4, as is   1 is to 2"

Each fraction has a value of 0.5 which means its portion is 50% of the (entire 100%) circle. Each part is half (= 1/2) of the circle, and even if the circles are of different sizes, because the portions of the circle(s) being referred to are mathematically the same fractional value of 1 circle, regardless of its actual size. A portion or percentage value is a basic generalization that is in reference to any whole or entirety regardless of its actual specific sizes or values involved. Equivalent fractions represent the same portion or fraction of something.

2/4 comes from the fact that there is 2 parts in question of the circle that was divided into 4 equal parts. Actually,
1/4 + 1/4   or (1/4) x 2   =   2/4 = 0.5 of 1 = (0.5)1 = 0.5   Also: (1/4) + (1/4) = 0.25 + 0.25 = 0.5

The fractions of 2/4 and 1/2 are said to be equivalent fractions since they represent the same quantity or value (here it's half or 0.5 of the circle, hence the same portion). The fraction of 2/4 has larger values for both the numerator and denominator, and it is said to be of "higher terms", that is, it has a higher numerator and denominator. Likewise, the fraction of 1/2 is said to be of "lower terms" since its numerator and denominator are lower in value than the other fraction. Still, the numerical values of both fractions represent an equal part of the circle. Actually, to make any equivalent fraction,

simply multiply or divide both the numerator and denominator of a fraction by the same value. For example, a quick and simple way to find the next higher term equivalent fraction of 2/4 can be found by multiplying the numerator and denominator by the same value, such as for example, 2:

$$\frac{2}{4} = \frac{2 \times 2}{4 \times 2} = \frac{2(2)}{4(2)} = \frac{4}{8}$$

hence, $\frac{4}{8} = \frac{2}{4} = \frac{1}{2}$: = 0.5 , equivalent fractions, each represents 0.5 of the whole (1)

Given these numeric values, it can be said that 4 parts of a circle divided into 8 equal parts represents the same portion of a circle as 2 parts of a circle that was divided into 4 equal parts.

The fraction of 1/2 for the group of all possible equivalent fractions shown is said to be the fraction with the lowest terms. Also for this fraction of 1/2, since both the numerator and denominator cannot be evenly divided by any whole number greater than or equal to one, it is in lowest possible terms of all other equivalent fractions.

As a simple verification that they are equivalent fractions, consider that multiplying any value (including fractional values also) by 1 will not change that value. If any fraction is multiplied by say (2/2)=1, that is, the numerator and denominator are both multiplied by the same value, it is equivalent to multiplying by 1, and we know that multiplying any value by 1 does not change that value.

since $\frac{2}{2} = 1 = \frac{1}{1}$,   multiplying this to any value, the value will not change:

$\frac{2}{4} = \frac{2(2)}{4(2)} = \frac{4}{8}$: the fractions look different (in the numerators and denominators), but they
                                              are numerically equivalent in value, here it's 0.5

Reversing this process, if we divide the same value from both the numerator (num.) and denominator (den.), the resulting fraction is still an equivalent fraction. For example:

$$\frac{2}{4} = \frac{2/2}{4/2} = \frac{\frac{2}{2}}{\frac{4}{2}} = \frac{1}{2}$$: after simplifying the num. and den. fractions

Again, a way to show or verify that these are equivalent (equal) fractions (of the whole) is to convert (represent) each to its decimal form by performing the indicated division.

$\frac{4}{8} = 0.5$, $\frac{2}{4} = 0.5$, $\frac{1}{2} = 0.5$:The fractions must be equivalent since
                                                            each represents a value of (0.5).

Ex. Show that 2.00564 / 7 and 114.32148 / 399 are equivalent fractions or portions.

By using a calculator to perform the indicated division so as to find their quotient values:

$\frac{2.00564}{7} = 0.28652$ and $\frac{114.32148}{399} = 0.28652$: they are shown to be equivalent fractions numerically
                                                                    Each fraction represents a portion or total (sum of
                                                                    portions) that is about 28.7% of the whole (100%) or
                                                                    entirety.

There is another, less used method to create equivalent fractions described in the BASIC ALGEBRA section of this book.

Sometimes it could be helpful to express a given fraction as an equivalent part or fraction of 1, or as an equivalent fraction of 1 part out of a total number of parts. Doing this can give a basic grasp of the fraction when expressed as a single (1) portion of all the portions, or as a portion of 1.

Ex. 0.5 to 7 = $\dfrac{0.5}{7}$: 0.5 parts out of 7 parts   = 0.07143 when expressed as a single decimal value

Equivalent fractions are usually solved by finding the multiplier needed to apply to both the num. and den., but since a value of 1 is used as a value in the equivalent traction being created, it become easier to complete that equivalent fraction.

0.5 is to 7, as 1 is to what? $\dfrac{0.5}{7} = \dfrac{1}{x}$: here, x is used in place of the word: "what"

Notice that if you divide the numerator here, by itself, that the result is 1. To keep the fraction the same value, this must also be done to the denominator:

$\dfrac{0.5}{7} = \dfrac{(0.5/0.5)}{(7/0.5)} = \dfrac{1}{14}$: 0.5 parts out of 7 parts can be numerically expressed as 1 part out of 14, or 1 part of 14

OR:

0.5 is to 7, as what is to 1? $\dfrac{0.5}{7} = \dfrac{x}{1}$

In a similar manner to the above, if we divide the denominator by itself, we will have a value of 1, and to keep the fraction as having the same value, we must do the same to the numerator:

$\dfrac{0.5}{7} = \dfrac{(0.5/7)}{(7/7)} = \dfrac{0.07143}{1}$: 0.5 parts out of 7 parts can be expressed as 0.07143 parts to 1
: This is essentially the same result of simply dividing the num. by the den.

Ex. If a fast runner can run 1 mile in 4 minutes, how many miles an hour did they, or could they potentially run?

First, so as to have like units in the denominators, lets convert the units of hours to minutes:
1 hour = 1 hr = 60 min

$\dfrac{1\ mi}{4\ min} = \dfrac{x\ mi}{60\ min}$, since: 60 min / 4 min = 15,   and also multiplying the numerator by this value:

x mi = (1mi)(15) = 15mi

1 mile per 4 minutes = 15 mi per 60 minutes = 15 mi / 60 min = 15 mi / 1 hour = 15 mi / hour

Another way is to mathematically solve for x mi:

$$x\ mi = \left(\dfrac{1mi}{4\ min}\right)(60\ min) = \left(\dfrac{1mi}{\cancel{4\ min}}\right)\overset{15}{(\cancel{60\ min})} = (1mi)(15) = 15\ mi$$

Ex. A basic and "generalized" (with non-specific units) recipe for hard candy is shown below. It also shows how to make an "equivalent proportions" recipe so as to make a bigger or smaller batch (amount, quantity) of the candy. Even though a batch can be a different size, weight or volume, the percentage, portion or fraction of each ingredient in the new batch will still be the same percentage, portion or fraction as that of each ingredient in the original recipe. It is sometimes noted that since the ingredient amounts used in a new sized batch are the same portion as that used in the original recipe, that the new batch and ingredients are said as being "proportional" (same portions, or "within proportion(s)") to the original recipe and-or batch. This ensures that the new sized batch will still taste the same as that of the original recipe and-or batch.

Main Ingredients: 2   units sugar   : here "units" could be cups, or some other possible container
                              3/4 units corn syrup   : 3/4 = 0.75 units   used for the measurement(s)
                              2/3 units water
Other Ingredients: A few drops of flavor oil, and a few drops food color.

Mix the main ingredients in a bowl and then pour the mixed ingredients into a cooking pot. Cook (without stirring) till all the water is evaporated and to a temperature of 300°. Do not let the mixture turn dark yellow-brown and burn. Remove from heat, mix in flavor and color. Pour mixture onto a greased cookie sheet and let it cool. For the purpose of this mathematical example, the actual units of cups for the main ingredients is not explicitly indicated since you should use or expect any possible units for these measurements; perhaps you'll have some homemade units of and for your measurements such as: a jar, or spoon full, etc. Let's say you want to make a a different quantity (more or less) of candy. To do so, you need to adjust all the measurements (amounts of units used) for each ingredient equally by the same (magnification) factor value. Let's say you also want to make a more basic or general ("generalized") recipe so as to make batches of any quantity of candy that is based on a (1) unit of sugar, then for each additional unit of sugar added in, you will simply add in the new "general recipe's" measurement for all the other ingredients. Since there are 2 units of sugar used in the original recipe, you can divide these 2 units by 2 to have a recipe based on just using 1 unit of sugar. To keep the same ratio (ie. relative size amts.), or numerical proportions (consistent portions) of the amounts used for the ingredients in the recipe, you must then also divide all the other ingredient measurements by the same value, here, 2, and this will ensure that they are, and have, the same proportion (proper (same) and consistent fractional or portion value) as that of the original recipe, and that the ratios (essentially the quotient value of their division) of each two ingredients amounts is the same as that of the original Even though the specific quantity of an ingredient is now more or less, it is still numerically the same recipe. fractional or portion value as that of the original recipe. Expressing the basic concept of this mathematically:

$$\frac{\text{units of ingredient A}}{\text{units of ingredient B}} \;=\; \frac{\dfrac{\text{units of ingredient A}}{2}}{\dfrac{\text{units of ingredient B}}{2}}$$

: still has the same ratio value among , of, or between these two amounts of ingredients. Essentially created an equivalent fraction. Divide the amount of all ingredients by 2.

The ratios, or relative (relational) sizes of one ingredient value to another, will still be the same as that in the original recipe. The total mixture or completed item being made, here candy, will also be divided in half since all of its parts (quantities) were divided in half, or vice-versa.

Dividing 3/4 by 2 we have 3/8 (division of fractions will be explained later), and dividing 2/3 by 2 we have 2/6 = 1/3, and the main ingredients and measurements of the new generalized recipe will become:

Main Ingredients: 1   units sugar   For each 1 unit of sugar used, use:
                              3/8 units corn syrup   : 3/8 = 0.375 units which is half of 0.75 = 3/4 units
                              1/3 units water

Since 1/2 = 0.50, you could have multiplied each ingredient amount by 0.50 instead of multiplying by 1/2 which is also equivalent to dividing by 2 as seen above. For some extra verification of all this, consider:

$$\frac{3}{4} \times \frac{1}{2} = \frac{(3)(1)}{(4)(2)} = \frac{3}{8}$$: note that 3/4 and 3/8 are not equivalent fractions since the numerator and denominator were multiplied by different values.

0.75 x 0.5 = 0.375   also:

If we divide a fraction, say 3/4, by 2 (same as 2/1), it is the same as multiplying both the entire new num. and den. by (1/2) as seen below. This procedure also shows why they are no longer equivalent fractions, since then num. (3), and den. (4) of 3/4 will not actually be multiplied by the same value. In short, dividing or multiplying a fraction by a value does not create an equivalent fraction, however, dividing both the num. and den. of a fraction by the same value does. For example if we have 4/8 = 0.5, and multiply this fraction by 2 or doubling (adding itself) it, we have (4/8)(2/1) = 8/8 = 1 which indicates that this is not an equivalent fraction (0.5 and 1 are not equal) but it is a magnification (ie. growth) of the fraction to where it does not represent the same part of the whole any longer, and is a completely new (different) fractional part or value.

_____ same as multiplying the entire num. and den. by 0.5 = 1/2

$$\frac{\frac{3}{4}}{\frac{2}{1}} = \frac{\frac{3}{4}\frac{(1)}{(2)}}{\frac{2}{1}\frac{(1)}{(2)}} = \frac{\frac{3}{8}}{\frac{2}{2}} = \frac{\frac{3}{8}}{1} = \frac{3}{8}$$: = 0.375

Checking (note that 3/4 = 0.75):

$$\frac{0.75}{2} = 0.375$$: as seen above, 3/4 = 0.75, and 3/8 = 0.375 are not equivalent fractions

We also see above, that a quick way to divide one fraction by another is to simply invert ("turn upside down") the denominator fraction and then simply multiply both the numerator and denominator by this value.   Here, 2/1 inverted is 1/2. As another example, if you were to divide a fraction by 5/6, you could simply multiply both the num. and dem. of that fraction by 6/5 for the resulting quotient value. This process is also a good way to "clear" or "rid" the fractions, where possible, during a simplification (of an expression) process so as to have just one value (the quotient) to continue working with.

For completeness of the new recipe, note also that (2/3)unit / 2 = 2/6 unit. Dividing this num. and den. by 2, we have 1/3 unit , such as 1/3 of a cup, and which is usually easier to understand and measure in the kitchen, than 2/6 units or "two - sixths" of a cup.

When two or more things are said as being linear in value, they are also said as being proportional in value. That is, when one thing increase by some factor, the other thing will increase by the same rate or factor, and this is much like a magnification and-or a demagnification. This is also very much like an equivalent fraction concept, and can be solved as such. This book will sometimes mention that things such as this can be solved using a "proportional equation" or "equivalent fraction equation". Expressing all this in a pseudo-mathematical analysis and expression for when thing1 and thing2 are said as being proportional, or linear in value(s) and-or magnifications:

$$\frac{thing1}{thing2} = n = \text{magnification}$$, mathematically: thing1 = (n) thing2   , $$\frac{thing1}{thing2}\frac{(factor1)}{(factor1)} = \frac{thing1magnified}{thing2magnified} = n$$
                                              and: thing1 / n = thing2

ADDING FRACTIONS

Adding fractions is not as difficult as you might think. Actually, we can think that we have always been working with fractions when we were just using whole values. For example, we know that 4+3=7, but we also know that we can make fractions by placing these values over one, which we know will not change their values.

$4 + 3 = \dfrac{4}{1} + \dfrac{3}{1} = \dfrac{7}{1} = 7$: Add the numerators (the 4, and 3) of fractions with like denominators, and keep the denominator. The like denominators here are 1.

Checking:

$4 + 3 = \dfrac{(4 + 3)}{1} = \dfrac{7}{1} = 7$: checks

This leads to a general rule for combining (adding) fractions: Add the numerators of fractions with like, same or common denominators, and keep that same denominator. This is commonly known as the process of "adding like fractions".

If a circle (or 1 whole circle) is divided into two equal (ie. evenly divided) parts, each and every single part will numerically be (1/2), and we know that the sum of these parts (all the fractions or fractional parts) should equal to one (the whole circle) or the entire value of which they were taken out from and are in reference to. Since the fractions in question have like denominators (here, 2 or "halfs"), which makes them "like-fractions", simply add the numerators and keep the (same) denominator. [FIG 7]
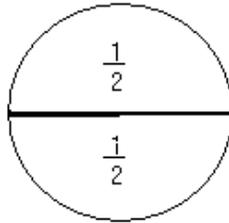
$\dfrac{1}{2} + \dfrac{1}{2} = \dfrac{(1 + 1)}{2} = \dfrac{2}{2} = 1$: A half (of a circle), plus another half, equals one (1) full circle.

If a circle or some quantity was evenly divided into 3 parts, each part is numerically (1/3). [FIG 8]
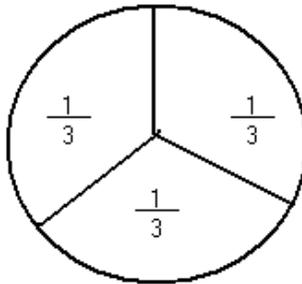
Summing up just two of those fractional parts or values, we have: [FIG 9]
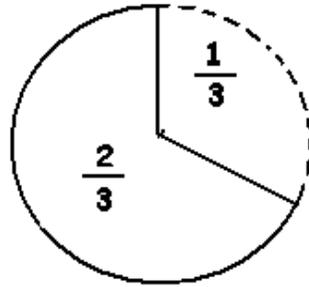
$$\frac{1}{3} \ + \ \frac{1}{3} \ = \ \frac{1+1}{3} \ = \ \frac{2}{3}$$: "two thirds of a circle"

If we were to then add in one more of the parts, so as to have all 3 parts in the sum, we should then have one whole circle:

$$\frac{2}{3} \ + \ \frac{1}{3} \ = \ \frac{2+1}{3} \ = \ \frac{3}{3} \ = \ 1$$

As a check to this, expressing the sum of all three of the parts:

$$\frac{1}{3} \ + \ \frac{1}{3} \ + \ \frac{1}{3} \ = \ \frac{1+1+1}{3} \ = \ \frac{3}{3} \ = \ 1$$: checks

Also, expressing and performing the repeated addition above , with multiplication:

$$\frac{1}{3} \ x \ 3 \ = \ \frac{(1)}{(3)} \frac{(3)}{(1)} \ = \ \frac{(1)(3)}{(3)(1)} \ = \ \frac{3}{3} \ = \ 1$$: checks

Ex. Add 4 tenths, 2 tenths, and 1 tenth

Notice that all the values have units of tenths. Hence, there is no "apples and oranges" problem with unlike units. Simply add the numerical values (or variables as in algebra) of quantities with like units for the sum, and keep the same units.

4 tenths + 2 tenths + 1 tenth = (4+2+1) tenths = 7 tenths

We can also do this problem using the addition of fractions method. Here, the fractions will be "like fractions" which have the common or like denominator of 10 or "tenths" (tenths of one) and therefore can be added.

First: 4 tenths   = 1 tenth x 4   = 4 x 1 tenth   =   4 (0.1)   =   0.4

4 tenths $= \ \frac{1}{10} \ + \ \frac{1}{10} \ + \ \frac{1}{10} \ + \ \frac{1}{10} \ = \ \frac{4(1)}{(10)} \ = \ \frac{4}{10} \ = \ 0.4$

And for the specific problem mentioned:

$$\frac{4}{10} \ + \ \frac{2}{10} \ + \ \frac{1}{10} \ = \ \frac{(4+2+1)}{10} \ = \ \frac{7}{10}$$: "seven tenths" , 7/10 = 0.7 as a basic decimal number.
         A check by converting each fraction to its decimal value:

0.4 + 0.2 + 0.1   = 0.7   : "seven tenths of one (1)"

Ex. You have a certain recipe that you only want to make one-half of its resulting amount. How much of each ingredient or part must you then use to make that recipe?

Here is the original recipe:

amount_of_ingredient1 + amount_of_ingredient2 + amount_of_ingredient3 = resulting_recipe

If we divide the right hand side of the equation by 2, so as to make half of the resulting_recipe, the equation will be unbalanced. If we have done nothing to the amount of the ingredients used, surely the resulting_recipe will still be the same, and yet we would be expecting half the results. What we need to do is divide both side of this equation by the same value of 2 so as this equation, recipe or formula, is still correct and in balance or equivalence (equivalent, equal):

$$\frac{amount\_of\_ingredient1 \ + \ amount\_of\_ingredient2 \ + \ amount\_of\_ingredient3}{2} = \frac{resulting\_recipe}{2}$$

Now, what exactly is the sum of all the ingredients combined divided by 2? The sum, of all the ingredients or parts of the result, divided by 2 is equal to the sum of the amount of each ingredient or part divided by 2:

$$\frac{amount\_of\_ingredient\ 1}{2} \ + \ \frac{amount\_of\_ingredient2}{2} \ + \ \frac{amount\_of\_ingredient3}{2} = resulting\_recipe$$

Since $\frac{1}{2} = 0.5$, you can optionally multiply the amount of each ingredient by 0.5.

In simple words, the above can be described as: To make half the recipe, use half of all the ingredients, and therefore half of each individual ingredient. To make a certain or specific fraction of the recipe, use the same fraction of the amount of each ingredient.

Ex. If amount_of_ingredient1 is 1 cup: 1 cup , and to make half the amount of the original recipe, use:

$$\frac{1cup}{2} \ = \ \frac{1}{2} \ cup \ = \ \text{"one-half cup"} \ = \ 0.5 \ cups$$

Ex. If amount_of_ingredient2 is 7oz, and to make half (50% = 0.5 = 1/2) the amount of the original recipe, use::

$$\frac{7oz}{2} = 3.5 \ oz$$

ADDING MONEY AS FRACTIONS, AND A SHORT DISCUSSION ABOUT PERCENTS

When adding fractions which do not have a common denominator among them, one (or several) of them must be converted into an equivalent fraction so that all the fractions will then have the same (common) denominator. For example, fourths (ex. 1/4) cannot be added to thirds (ex. 2/3) unless there is some conversion of one of these fractions into an equivalent valued fraction and which will have the same denominator as the other fraction.

Here is a practical illustration of why the denominators must be the same for the addition (and subtraction) of fractions, and it is very similar to the concept of having like units or quantities for proper addition:

Given a number of pennies and nickels, you can't add the number of pennies (which is a certain fraction of a dollar, actually one-hundredth = 1/100 = 0.01 of a (1) dollar = "1 cent" = "1 percent") and the number of nickels (which is another certain fraction of a dollar, actually one-twentieth = 1/20 = 0.05 of a (1) dollar) together to find out how many total (or the total monetary value of) pennies you have, or to find out how many total nickels you have. A nickel is a different coin and monetary value than a penny.   Before adding nickels and pennies, you must first convert the number or quantity of pennies to an equivalent number of nickels, so as to express the sum in terms of nickels, or convert the number of nickels to an equivalent number of pennies, so as to express the sum in terms of pennies or cents.

For the conversions: 1 nickel =   5 pennies   :the known basic conversion reference,   or, after switching sides:
                    5 pennies =   1 nickel   :as an extra note, since 5 pennies is the same monetary value as a nickel,
                                       or 0.05. It is sometimes said that "there are 5 pennies in a nickel". A
                                       penny is also a fraction of a nickel:
                                       1 penny =1 cent = 1/5 nickel = "one-fifth of a nickel". If you divide both
                                       sides of the conversion expression by 5 you can verify this.

If given a (1) nickel, you can multiply it by 5 to find the corresponding number of pennies. If given twice (2) as many nickels, the total quantity of pennies will also be twice (2 times) as much:

                     1 nickel =   5 pennies   : basic conversion facts expressed as an equation
           1 nickels + 1 nickel = 5 pennies +   5 pennies   since multiplication is repeated addition :
                2 (1 nickels) = 2 (5 pennies)   multiplying or distributing the multiplying value:
                 2 x 1nickels = 2 x 5 pennies
                   2 nickels = 10 pennies

And in general, if given some quantity (Q) or number of nickels, the total quantity of pennies will also therefore increase by that times as much:
                 From: 1 nickel = 5 pennies   expressing the multi. of both sides by the same value, here Q:
                     Q (1 nickel) = Q (5 pennies)   or:
                   (Q x 1) nickel = (Q x 5) pennies   after some simplification:
                   Q nickels =   Q x 5 pennies

This might be spoken as: "multiply the quantity or number of nickels by 5 to find the equivalent number of pennies", or "The quantity of pennies is 5 times more than the quantity of nickels". In a reverse type of manner, we now know that if given a quantity of pennies, we can divide it by 5 to find the corresponding number of nickels.

Ex.   10 pennies + 2 nickels   : converting nickels to pennies , first, it is known that: 1 nickel = 5 pennies
        10 pennies + 2 x 5 pennies
        10 pennies + 10 pennies   performing the indicated addition of similar things, simply add their
          (10 + 10) pennies   numerical quantities and keep the same units, and this is much like
             20 pennies   adding the numerical coefficients (ie. factors) of variables as in algebra.
                                       Algebra is basically symbolic math where text letters represent, or are
                                       "placeholders", for (known or unknown) numeric values of mathematical
                                       expressions.

Ex.   10 pennies + 2 nickels   converting the number of pennies to its equivalent quantity
                                       of nickels, by dividing the number of pennies by 5:

(10 pennies / 5 pennies per nickel) + 2 nickels   or:
(10 pennies/1) / (5 pennies/1 nickel) + 2 nickels   or:

$$\frac{\dfrac{10 \text{ pennies}}{1}}{\dfrac{5 \text{ pennies}}{1 \text{ nickel}}} = \frac{\dfrac{10 \text{ pennies}}{1}\left(\dfrac{1 \text{ nickel}}{5 \text{ pennies}}\right)}{\dfrac{5 \text{ pennies}}{1 \text{ nickel}}\left(\dfrac{1 \text{ nickel}}{5 \text{ pennies}}\right)} = \frac{10}{5} \text{ nickels} = 2 \text{ nickels}$$

: or can use: 10 pennies/5pennies = 2

2 nickels + 2 nickels   performing the indicated addition:
  (2 + 2) nickels   : or (2 x 2) nickels , since 2 was added to itself twice and
      4 nickels   that multiplication can represent repeated addition

By using the fractional values (of a dollar) that a penny and nickel represent, and using letter Q for the quantity of each coin, we have expressions for the total monetary (in dollar units) values of a quantity of either pennies or nickels. Remember, multiplication is repeated addition. First, since each penny has a value of 0.01 of a dollar, for each next penny, add 0.01 to the total sum or monetary value. This repeated addition, can be represented and expressed with multiplication; here by Q which is the number or quantity of pennies, or nickels, and times it was added into the total sum:

Pennies   Nickels

$$Q\left(\frac{1}{100}\right) = Q(0.01) \qquad Q\left(\frac{1}{20}\right) = Q(0.05)$$

If P was used to represent the Quantity of pennies, and N was used to represent the Quantity of nickels, the above could be expressed as:
          Total pennies monetary value = P(0.01)   and   Total nickels monetary value = N(0.05)

For converting the monetary value of a quantity of nickels to the corresponding monetary value of pennies, we will multiply the corresponding monetary value of pennies by 5 since a nickel is defined as equivalent to the monetary value of 5 pennies = (0.01 x 5 = 0.05). Note that the fractional value of 1/100, or "one cent", of 1 dollar assigned to a penny coin is also assigned as the pennies monetary value of 1 dollar. The monetary value of a penny is often called a "cent", meaning one-hundredth of. The word percent mathematically means "per hundredths of", or "parts per hundredth of", for example 5 percent means 5 equal parts of something (considered as 1 whole or entirety) that was divided into 100 equal parts (each being 1/100), hence 5 one-hundredths = 1/100 + 1/100 + 1/100 + 1/100 + 1/100 = (5/1)(1/100) = 5/100 = 0.05 = 5% = "five percent". More will be said about this later in the topic of percents. Since a nickel is (monetarily) equal to 5 pennies, it therefore also represents the monetary value of 5 pennies which is 5 cents = 0.05 = 5/100 = "five hundredths of 1 dollar" = 1/20 of a dollar when 5/100 is reduced to a lower termed equivalent fraction. A nickel is 5/100 = 1/20 = one-twentieth of a dollar. Coins less than a dollar in monetary value are fractions of a (1) dollar's monetary value.

          Converting the monetary value of a quantity (Q) of nickels to that of pennies.

          Pennies   Nickels

$$(Q \times 5)\left(\frac{1}{100}\right) = Q\left(\frac{1}{20}\right)$$

: here, we let Q equal the quantity of nickels
  Perhaps you might use N instead of Q as
  mentioned previously. Remember that
  multiplication is essentially repeated addition.

$$Q \times 5 \,(0.01) \quad = \quad Q \,(0.05) \qquad : \text{or}$$

$$Q \times 0.05 \qquad = \qquad Q \times 0.05 \qquad : \text{checks}$$

Ex. Find the total monetary value of 2 nickels and 4 pennies.

2 nickels + 4 pennies

$$2 \left(\frac{1}{20}\right) \quad + \quad 4 \left(\frac{1}{100}\right)$$: denominators and not identical, and "unlike fractions" cannot be added or "combined". Can use: 2 (0.05) + 4 (0.01) = 0.10 + 0.04 = 0.14

$$\frac{2}{20} \quad + \quad \frac{4}{100}$$: denominators are not identical, so we have unlike fractions that cannot be easily added or combined as currently expressed so as to have just one fraction

Converting the monetary value of nickels to its' equivalent monetary value in pennies so that an addition (of fractions with a common, similar (same) or "like" denominator of 100) can take place, we have:

$$\frac{2 \times 5}{5 \,(20)} \left(\frac{1}{}\right) \quad + \quad \frac{4}{100}$$: the numerator and denominator in the first fraction were both multiplied by 5 to create an equivalent (valued) fraction of which has a denominator value of 100.

$$\frac{(2)}{(1)} \frac{(5)}{(100)} \quad + \quad \frac{4}{100}$$

$$\frac{10}{100} \quad + \quad \frac{4}{100}$$: "ten-hundredths plus 4-hundredths" equals:
: We now have "like fractions" with the same denominators that can be added.

$$\frac{14}{100} \quad = \quad 0.14$$: "fourteen-hundredths" or= 14 cents

If you wanted to express a quantity of coins in terms (ie. with similar variables or units) of dimes (which is a certain fraction of a dollar, actually one-tenth = 1/10 = 0.10 of a dollar), you would have to convert the number of pennies and nickels to their corresponding number of dimes. Again, this is a common "apples and oranges" type of problem, except here, it is possible to make numeric conversions. Since 1 dime = 2 nickels = 10 pennies, divide the number of pennies by 10 to find the corresponding number of dimes, and divide the number of nickels by 2 to find the corresponding number of dimes.

Ex.   10 pennies + 4 nickels   after dividing the quantity or number pennies by 10, and nickels by 2 for the conversion to it's (monetary) equivalent quantity of dimes:
       1 dime + 2 dimes   : we now have "like units" that can be combined
          (1 + 2) dimes   : expressing the sum of the quantities of the like or similar units
             3 dimes   : 3 dimes = 3 (tenths of 1 dollar) = 3 (0.10) =   3/10 = 0.30
                        This value can be spoken as: "thirty-hundredths" or "thirty-cents".

Also note in the above discussions, that when converting a quantity with larger units (of measurement) to a quantity with smaller units, that a multiplication is performed on the quantity with larger units. This is because there will be a greater quantity of these smaller units per each larger unit.   For example, when converting (a quantity of) nickels to a corresponding number (or quantity) of pennies, multiply by 5, and when converting (a quantity of) miles to feet, multiply by 5280. Likewise, in a inverse type of manner, when converting to a larger units, a division is performed on the quantity that has the smaller units (of measurement or reference) since there will be a fewer quantity of the larger units.

Here is an extra side note of fractions that you may occasionally encounter:

Ex. What is ppm and ppb?   ppm = parts per million parts = parts / 1,000,000 parts
　　　　　　　　　　　　　　1ppm is numerically: 1part/1,000,000parts = 0.000,000,1
　　　　　　　　　　　　　　10,000 ppm = 10000 parts / 1000000 parts = 0.01 = 1%
　　　　　　　　　　　　　　Seawater is about 10560 ppm sodium = ~ 1% , and about 19000 ppm chloride = ~ 2%
　　　　　　　　　　　　　　Seawater is often mentioned as being 3% salt. Salt is sodium-chloride molecules.
　　　　　　　　　　　　　　ppb = parts per billion parts = parts / 1,000,000,000 parts
　　　　　　　　　　　　　　ppm and ppb can be considered as the number of parts of a certain mass and-or weight.

COMMON DENOMINATOR FOR ADDING FRACTIONS

The common, same or "like" denominator found for the proper addition of fractions will be some multiple (including 1) of the largest denominator value of all the fractions in question. Hence, the common denominator will be greater than or equal to the highest denominator value.   It will be shown that a common denominator of two or more fractions can simply be the product of all their denominators, and that it is therefore evenly divisible by each of those denominators since it will contain a multiple of each denominator.

Ex. Add 1/4 and 4/8   : "add one-fourth, and four-eighths"

First, notice that the highest denominator value of these two fractions is 8, and that the fraction of 1/4 can be converted to an equivalent fraction of 2/8 if we multiply both the numerator and denominator by 2. Both fractions will then have the same denominator value and can be added as fractions:

$$\frac{1(2)}{4(2)} = \frac{2}{8}$$

We now have "like fractions" (with a common denominator) which can be added.

$$\frac{2}{8} + \frac{4}{8} = \frac{6}{8}$$

Note that this example may have been stated as: How many eighths are 1/4 and 4/8? The answer is six eighths.

For the two fractions being summed in the last example, the common denominator derived at from the two given and "unlike" denominators is called the lowest common denominator (sometimes and simply indicated as LCD, as in this book). The name comes from the fact that it is the lowest possible value for the common denominator that can be evenly divided by each denominator of the fractions in question. Note, if all the fractions to be summed are first reduced to their lowest or "simplest terms" (ie. Made into equivalent fractions - with a lower numerator and denominator) before the LCD is sought, that this resulting LCD of those equivalent fractions may even be lower in value.

If the numerator and denominator, each being considered as a product value, have factors that are common to both, then the fraction can be reduced or changed to an equivalent fraction of lower terms by dividing out, or "canceling" (ie. removing) these common factors. If you divide both the numerator and denominator by the highest or greatest common factor of both the numerator and denominator, the fraction will be reduced to its lowest terms. Remember, this processes is allowed since the fractions will still be equivalent in value. The greatest common factor can be found as the product of the smallest power of each different prime factor of the two numbers; prime numbers will be discussed ahead. Whereas multiplying both the numerator and denominator creates a "magnified (increased)" equivalent fraction that is said to be of "higher terms (of both num. and den. values)", dividing both the numerator and denominator creates a "demagnified (decreased, reduced)" equivalent fraction of "lower terms (of both the num. and den. values")".

Ex. For the fractions below, reduce each to an equivalent fraction of lowest terms before the addition.

$$\frac{2}{8} + \frac{4}{8} = \frac{6}{8}$$

The fraction of 2/8 can be reduced to lower terms by dividing both the numerator and denominator by 2. The value of 2 was chosen since it is the highest factor common to both the numerator and denominator. This is seen when the numerator and denominator of the fraction are first factored:

$$\frac{2}{8} = \frac{(2)(1)}{(4)(2)}$$: factored numerator and denominator, showing the common factor(s) of 2

Dividing both the numerator and denominator by the highest common factor of 2, or simply "cross canceling" (crossing out, to reduce and-or possibly eliminate to just a factor of 1) these factors with a line or hash mark. You can also write a small one near these values, indicating the result of a division, here by 2, and that the result was 1. After canceling common factors, the result in an equivalent fraction of lowest terms.

$$\frac{2}{8} = \frac{\frac{(2)}{(2)}}{\frac{(8)}{(2)}} = \frac{1}{4} \qquad : 2/2 = 1 \quad \text{and} \quad 8/2 = 4$$

or: $\frac{2}{8} = \frac{(2)(1)}{(4)(2)} = \frac{1}{4}$: showing the "cross canceling", "canceling out" or "dividing out". Also,
: remember that 2/2 = 1 and anything times 1 is that same value. A multiplier or factor of 1 need not be shown always, but it can always be implied and-or removed so as to simplify equations as much as possible.

Likewise, for the fraction of 4/8:

$\frac{4}{8} = \frac{(4)(1)}{(4)(2)}$: showing the factoring of both the numerator and denominator, and the highest common factor of 4

Note that both the numerator and denominator also have a common factor of 2, but this is not the highest common factor since 4 is, and this will help create a fraction of lowest terms.

$$\frac{4}{8} = \frac{\frac{4}{4}}{\frac{8}{4}} = \frac{1}{2}$$

hence, $\frac{2}{4} + \frac{4}{8} = \frac{1}{4} + \frac{1}{2}$: results in an equivalent expression, after we reduced each fraction to lower terms.

After converting all the fractions to their corresponding equivalent fractions (with lower terms), their sum should still be equivalent to the sum of 6/8 as shown above. First consider that we need "like fractions" (have the same denominator) for proper summing:

$$\frac{6}{8} = \frac{\frac{6}{2}}{\frac{8}{2}} = \frac{3}{4} \qquad : \text{"six-eights" reduced or expressed as "three-fourths".}$$

Now, a common denominator possible between the two fractions is 4, which is also the LCD. By multiplying both the numerator and denominator of the fraction (1/2) by 2, we can get "like fractions" which have a common denominator for their addition.

$$\frac{1}{4} + \frac{1}{2} = \frac{1}{4} + \frac{1(2)}{2(2)} = \frac{1}{4} + \frac{2}{4} = \frac{3}{4} \qquad : \text{which checks}$$

       As mentioned previously, the easiest way to find a common denominator when adding or subtracting fractions is to use the product of all the denominators. This new and common denominator will then contain a factor of each single denominator. This denominator is usually larger than the LCD possible. The LCD is more desirable since reducing the result to lowest terms is already or more easily done. A common way to start finding the LCD of all the fractions is to factor the denominators into prime factors. Prime numbers are discussed next as a helpful part of this current topic.

PRIME NUMBERS

Prime or primary numbers are numbers that cannot be evenly divided by any other (whole, integer or counting) number except itself and 1, in other words, a prime number being considered as a product or not, cannot be factored into two other factors that are whole numbers. For example, 2 is a prime number since it is only divisible by 2 (itself) and 1. 4 is not a prime number since it is evenly divisible by 2. 5 is a prime number. 6 is not a prime number since it has factors of 2 and 3. A composite number, such as 6, is composed of factors that are either composite and-or prime. The first 10, and perhaps the most commonly used prime numbers are:

1, 2, 3, 5, 7, 11, 13, 17, 19, 23 , **. . .**

Note that 2 is the only even prime number. An even number will have its most rightward digit being either: 0, 2, 4, 6, or 8, and the entire number is therefore divisible by 2, that is, the number can be divided in half or evenly into two identical whole or integer numbers. An odd number cannot be divided in half evenly and expressed as two identical integers. Any even number will contain 2 as one of its factors, hence even numbers other than 2 cannot be a prime or "unfactorable number".

To factor (factorization, factorize) is where you are given a product value (perhaps a numerator or denominator as is the case of working with fractions) and you are to find its factors (multiplicands and multipliers) that compose or construct that given product value. You could say that these factors are "unseen" or "hidden" because they are now effectively "contained within", "composing", "making up", or "part of" that product value.

Some numbers or product values may have more than one set of possible factors, for example:

18 = 6 x 3 =   9 x 2

Some of these factors are composite numbers that can be factored further to only prime numbers. After doing this, we see that the given value (here 18) has only one corresponding and specific set of prime number factors:

18 = 6 x 3   =   9 x 2
18 = (3 x 2) x 3 = (3 x 3) x 2   by rearranging or regrouping we can see that they are the same value(s):
18 = 3 x 3 x 2   = 3 x 3 x 2   :18 completely factored to prime number factors, or simply: "primes"
                                        : The different prime factors of 18 are 2 and 3.

Factoring a factor further can be considered and expressed mathematically as:

product = (factor1)(factor2)   If anyone of these factors is considered as a product that can be factored,
                                        such as (factor2) which can be factored into (factor3) and (factor4):
product = (factor1)(factor3)(factor4) **. . .**

When adding or subtracting fractions: Considering all the completely factored (to "primes") denominators, each different valued prime factor is chosen only once, and specifically the highest multiple or power it occurs in a given denominator in the entire group of denominators, and these selected values are then multiplied together, and this product is the LCD of all the fractions.

Ex.   Given the denominators of 4, 8 and 3, what would be the LCD?

The factors of 4 are 2 and 2, and both are prime. The factors of 8 are 4 and 2. 4 is not prime since it has factors of 2 and 2. Hence, the prime factors of 8, when completely factored to prime factors, are 2, 2 and 2. The factors of 3 are 3 and 1, hence it is already a prime number. The different prime factors are 2 and 3. The prime factor of 1 is always understood. Since 2 appears a maximum of three times in a single denominator, and 3 appears only once, the LCD is:

```
(2 x 2 x 2)  x  (3)
     8       x   3
         24               : 24 is the lowest common denominator (LDC) that each denominator can wholly ("evenly",
                            entirely, completely) divided into without a remainder left over.
```

Again, notice that the highest power or multiple of each prime factor must be chosen so that each denominator, more specifically, the largest of the group, can (evenly) divide into the LCD at least once, and hence the LCD must be at least equal to or greater than the largest of the denominators.

One might now ask: Given a fraction, how is the numerator of an equivalent fraction found once the denominator (such as a common denominator for adding or subtracting fractions) of the equivalent fraction is found? The answer is found by dividing the "new" denominator by the "old" denominator. This effectively finds the multiplier (factor of the product) value that the "old" denominator was multiplied by to get the "new" denominator of the equivalent fraction. This is the case when the equivalent fraction being created is of higher terms. It is possible that a multiplier value can be less than one, such as is the case if the equivalent fraction is of lower terms. Once the multiplier is found, the same value must also be multiplied to the numerator so as to make an equivalent fraction that still represents the same numeric value, part or portion, that the original fraction does.

Ex. $\dfrac{2}{3}$ + $\dfrac{1}{4}$

A common denominator to add these fractions is simply the product of those denominators.

(3)(4) = 12   substituting 12 for all the denominators of the equivalent fractions being created:

$\dfrac{}{12}$ + $\dfrac{}{12}$ Now we are to find the proper or correct numerators of these equivalent fractions being created:

Since the first fraction's denominator was multiplied by 4 (from 12/3 = 4), we must also multiply the numerator of that fraction by 4 to keep it an equivalent fraction. Likewise, since the second fraction's denominator was multiplied by 3 (from 12/4 = 3), we must also multiply the numerator of that fraction by 3 to keep the fraction equivalent in value:

$\dfrac{2\,(4)}{3\,(4)}$ + $\dfrac{1\,(3)}{4\,(3)}$ performing the indicated multiplications:

$\dfrac{8}{12}$ + $\dfrac{3}{12}$ = $\dfrac{11}{12}$: we have "like fractions" (have the same denominator), and can
                                           therefore be added, and keeping that same or common denominator

Ex. $\dfrac{10}{5+2}$ + $\dfrac{4}{7+1}$

This is an example of adding or combining fractions where the denominators are actual expressions themselves. Notice that each denominator can easily be simplified first, but here, the order of operations will not be strictly followed in order to show another method. Frequently, there are times in algebra (where letters are used to represent unknown values or numbers yet to be found) where simplification is not preferred or possible. The fractions can still be combined (added) by considering each entire denominator expression as just like one single or unique value to work with:

$\dfrac{numerator1}{denominator1}$ + $\dfrac{numerator2}{denominator2}$: expressing a generalized or "formula" form, here, of adding fractions

A common denominator, and the lowest common denominator if a denominator is not evenly factorable (divisible) or composite is simply the product of those denominators. The fractions are combined in exactly the same manner as the last example:

Here, a common denominator, expressed in a factored form, is: common denominator = (denominator1)(denominator2).

$$\frac{\text{numerator1 (denominator2)}}{\text{denominator1 (denominator2)}} \quad + \quad \frac{\text{numerator 2 (denominator1)}}{\text{denominator 2 (denominator1)}} \quad \text{: expressed as the sum of the equivalent fractions}$$

When combining (adding) like fractions, add the numerators over the same denominator:

$$\frac{(\text{numerator1})(\text{denominator2}) + (\text{numerator2})(\text{denominator1})}{(\text{denominator1})(\text{denominator2})}$$

Using the example given:

$$\frac{10}{(5+2)} \quad + \quad \frac{4}{(7+1)} \qquad \text{creating equivalent (to each fraction given) and like denominator fractions:}$$
Using the product of (5+2) and (7+1) as a common denominator:

$$\frac{10 \ (7+1)}{(5+2)(7+1)} \quad + \quad \frac{4 \ (5+2)}{(7+1)(5+2)} \qquad \text{combining (adding) these like fractions we have:}$$

$$\frac{10(7+1) + 4(5+2)}{(5+2)(7+1)} \qquad \text{: a sum of several fractions, expressed as one fraction}$$

Checking: $\quad \dfrac{10}{(5+2)} \ + \ \dfrac{4}{(7+1)} \ = \ \dfrac{10}{7} \ + \ \dfrac{4}{8} \ = \ 1.42857 \ + \ 0.5 \ = \ 1.92857 \qquad$:approximately

$$\frac{10(7+1) + 4(5+2)}{(5+2)(7+1)} \quad = \quad \frac{80 + 28}{(7)(8)} \quad = \quad \frac{108}{56} \quad = \ 1.92857 \qquad \text{: checks}$$

Sometimes when working with fractions, an equivalent fraction of (expressed as) lower terms or values of the given numerator or denominator must be created. To find out what the common divisor to both the numerator and denominator is, divide the "old" numerator or denominator by the corresponding "new" numerator or denominator. Note that the method indicated above for finding an equivalent higher term fraction will still work as a method for finding an equivalent lower term fraction.   If that method is continued to be utilized, the effective multiplier will always be less than one.

Ex. $\dfrac{2}{3} = \dfrac{\quad}{12}$ $\qquad$:Create an equivalent fraction of higher terms (because the denominator of 12 of the fraction being created is higher than 3) by finding the numerator that correctly corresponds to this denominator so as the fractions are equivalent in value.

The "new" numerator to be found will also be higher. By dividing 12 by 3 we get the value of 4. Since the "old" denominator was essentially multiplied by 4 to get this "new" denominator.   The "old" numerator must then also be multiplied by 4 to create the "new" numerator of the equivalent fraction:

$\dfrac{2}{3} = \dfrac{2(4)}{3(4)} = \dfrac{8}{12}$ $\qquad$: It may help in the understanding of this by remembering that (4/4) = 1, and multiplying anything, such as a fraction or quotient, by 1 will not change its value, and here, the quotient or value of both (equivalent) fractions is the same: 2/3 = 8/12 = 0.666666... = about 67% of the whole or entirety (= 100% = 1).

Ex. $\dfrac{8}{20} = \dfrac{2}{\quad}$

Here, the equivalent fraction being found will be of lower terms since the "new" numerator is less than the "old" numerator. 8 divided by 2 is 4 (that is, 2 and 4 are two factors of 8). Therefore, a factor of 4 was divided out of 8. To make an equivalent fraction, the same must also be done to the denominator.

$$\frac{8}{20} = \frac{\frac{8}{4}}{\frac{20}{4}} = \frac{\frac{2}{1}}{\frac{5}{1}} = \frac{2}{5}$$

hence, $\frac{8}{20} = \frac{2}{5}$: checking, after dividing the numerators by the denominators, both fractions equal 0.4

Here is the same example as above, but the method for finding an equivalent higher term fraction is utilized:

$$\frac{8}{20} = \frac{2}{}$$

Finding the effective multiplier of the numerator 8 that equals a product of 2: $\frac{2}{8} = 0.25$

Checking: $(8)(0.25) = 2$

Now multiplying this same value to the denominator to create the equivalent fraction: $20 \times 0.25 = 5$

Hence: $\frac{8}{20} = \frac{2}{5}$

The example above, for creating an equivalent fraction, can also be solved by the basic facts about division and multiplication:

$\frac{8}{20} = \frac{2}{}$ simplifying by performing the indicated division on the left side of this equation:

$0.4 = \dfrac{2}{\text{denominator}}$ Here we can consider 2 as the dividend of a division operation, and the factors of that dividend are equal to the quotient (here 0.4), and denominator or divisor (to be found).

From: $\dfrac{\text{dividend}}{\text{divisor}} = \text{quotient}$ we have:

dividend = divisor x quotient   : the dividend as a product. Dividing a product by either of it's factors (the divisor or quotient) will yield the other corresponding factor of that product:

$\dfrac{\text{dividend}}{\text{quotient}} = \text{divisor} = \dfrac{2}{0.4} = 5 =$ the denominator being found of the equivalent fraction

ADDITION OF MIXED NUMBERS

When adding mixed numbers, you can first change them to (improper) fractions and then add those fractions. You can also add the whole (integer) parts, and then add this sum to the sum of the fractional parts. This is the common method taught, and this method produces another mixed number.

Ex. $5\frac{2}{3} + 2\frac{1}{4}$, This can also be expressed as: $\frac{5}{1} + \frac{2}{3} + \frac{2}{1} + \frac{1}{4}$

Using the first method discussed:

$5\frac{2}{3} + 2\frac{1}{4} = \frac{17}{3} + \frac{9}{4} = \frac{68}{12} + \frac{27}{12} = \frac{95}{12}$ or $= 7 + \frac{11}{12}$ when you divided 95 by 12, the result is 7 with a dividend remainder of 11

Note that for example: $5\frac{2}{3} = 5 + \frac{2}{3}$: "five and (+) two-thirds". Expressing 5 as a sum of 1's:

$1 + 1 + 1 + 1 + 1 + \frac{2}{3}$ since any number divided by itself equals 1, this can be expressed as this when considering the denominator value of 3, so as to make "like fractions", that have "like" denominators, which can then be added:

$\frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{3}{3} + \frac{2}{3}$ combining or adding the first 5 fractions:

$\frac{15}{3} + \frac{2}{3} = \frac{17}{3}$: after combining these two fractions, and : this fraction was indicated above

Using the second method discussed:

$5\frac{2}{3}$
$+ 2\frac{1}{4}$

OR :

$5\frac{2}{3} = \frac{8}{12}$
$+ 2\frac{1}{4} = \frac{3}{12}$

$(5+2) + (\frac{2}{3} + \frac{1}{4})$: horizontally

$7 \qquad \frac{11}{12}$: vertically

$7 + (\frac{8}{12} + \frac{3}{12})$

$7 + \frac{11}{12}$ or simply : $7\frac{11}{12}$: checks

Another way to add (or subtract) improper fractions is to convert the fractional parts to their decimal equivalents. This was also mentioned previously.

Ex. $5\frac{2}{3} + 2\frac{1}{4}$

first,
```
            0.66                      0.25
        3│ 2.00                   4│ 1.00
           18                        8
           20                        20
           18                        20
            2                         0
```

Note that 2/3 produces a repeating decimal. You may "round-off" (truncate or shorten) the quotient where needed.

$$5 \frac{2}{3} + 2 \frac{1}{4} = (5 + 0.66..) + (2 + 0.25) = 5.66 + 2.25$$

```
    1
   5.66          basic format is:        whole . fractional
 + 2.25                            +    whole . fractional_
   7.91                                 whole . fractional
                                             sum
```

Comparing this to the previous example: $7 \frac{11}{12}$

```
         0.916
7 + 12│ 11.000          ~ 7.91 (not rounded)  ,  ~ means about or approximately equal to the following value.
        108                Note also that 11/12 also produces a repeating decimal as seen above.
         20
         12
         80
         72
          8
```

SUBTRACTION OF FRACTIONS

The rules and concepts regarding basic addition and subtraction, and for the addition of fractions, also hold true for the subtraction of fractions. The main issue is to have like denominators so the subtraction of fractions can take place.

Ex. You have 1 whole circle divided into 3 parts, hence each part is a fraction of that circle and has the numeric value or representation of (1/3). The sum of all of these 3 parts is therefore: (1/3) + (1/3) + (1/3) = 3 (1/3) = 3/3 = "three thirds" = 1. If you take away (ie. remove or subtract) one of these parts, what value will remain of the entire circle?

$$\frac{3}{3} - \frac{1}{3} = \frac{(3-1)}{3} = \frac{2}{3} \qquad \text{: "two-thirds" of the circle remains}$$

Below is an example using mixed numbers, and then a discussion about the procedure will follow.

Ex. Subtract 2 6/7 from 3 5/7

$$
\begin{array}{cccc}
3\,\frac{5}{7} & & \overset{2}{\cancel{3}} + \frac{(7+5)}{(7\quad7)} & 2\,\frac{12}{7} \\
-\,2\,\frac{6}{7} & = & -\,2 + \frac{6}{7} & = \quad -\,2\,\frac{6}{7} \\
& & & 0 + \frac{6}{7} = \frac{6}{7}
\end{array}
$$

Considering the fractional parts of the mixed numbers, 6 cannot be taken from 5, hence we need to borrow. One might ask: where did the 7/7 come from? The answer is that borrowing 1 from the 3 is essentially borrowing 7/7 since 7/7=1. The denominator of 7 was chosen since the other fraction used in the subtraction operation also has a denominator of 7. This makes subtracting (or adding) the fractions easy since they will all have common or "like" denominators.

Ex. $\frac{10}{3} - \frac{4}{7}$

We must first find a common denominator for the subtraction. It is usually best to find the LCD. Here, the LCD has a value of 21 since 3 and 7 are both prime and they can simply be multiplied together. Now convert each of the fractions to equivalent fractions that have the same denominator of 21 :

$$\frac{10}{3} = \frac{70}{21} \qquad \text{since } 21/3 = 7 \text{ , and then} \quad \frac{10(7)}{3(7)} = \frac{70}{21}$$

$$\frac{4}{7} = \frac{12}{21} \qquad \text{since } \frac{4(3)}{7(3)} = \frac{12}{21}$$

$$\frac{70}{21} - \frac{12}{21} = \frac{58}{21} \qquad \text{: Since both of these operands were expressed in a fractional form, the result is arbitrarily expressed in a fractional form rather than as a mixed number.}$$

As a mixed number:

$$\frac{58}{21} = 2\,\frac{16}{21} \qquad \text{: after performing the indicated division of: 58/21 and expressing the result as a mixed number. Remember that you can check a subtraction problem by using addition.}$$

DECIMAL VALUES AND FRACTIONS

The basic way to change a decimal value that is less than 1 to its equivalent fractional value is to first find the positional weight of the least significant digit, and then remove the decimal point and make a product of this value and weight. From the discussion of positional weights, this weight will be a power of 10 with a negative (basically, conceivable numbers that are less than 0) number as the exponent. More advanced mathematics (besides what has already been discussed in this book, and that which will be shown ahead) will show that multiplying by a value with a negative exponent is actually dividing by that value with a regular (positive, even if it's not explicitly indicated) exponent of the same value.

Ex. $0.5 = 5(10^{-1}) = \dfrac{5}{10^1} = \dfrac{5}{10}$: "five-tenths of one". Note that the last or "least significant" digit, the 5, is located in the "tenths" position.

Note that 0.5 was made larger to a value 5, so it should be clear, as seen above, that to equal the original value (here 0.5), a division would then be necessary to reduce it to it's original value. We also know that a division by 10 will essentially moves the decimal point one position to the left toward the more significant digits.

Checking by performing the indicated division:

```
        0.5
   10 ) 5.0
        5 0
          0
```

You may also consider the decimal point being moved leftward in any value without using division, for example:

  $5 = 0.5 (10^1) = 0.5 (10)$: one place leftward , and the indicated exponent of 10 is 1, and 10 has 1 zero
  $0.5 = 0.05 (10^2) = 0.05 (100)$: two places leftward, and the indicated exponent of 10 is 2, and 100 has 2 zeros

Note here that 5 was made smaller to 0.5, so it should be clear that for it to equal the original value (here 5), a multiplication applied to 0.5, to increase this value, would then be necessary.

Ex. Represent 0.47 as a fraction

  $0.47 = 47(10^{-2}) = \dfrac{47}{10^2} = \dfrac{47}{100}$: "forty-seven hundredths". Note that the last or "least significant" digit, the 7, is located in the "hundredths" position.

Checking by using the concepts of positional notation, and the sum of positional products:

  $0.47 = 4(10^{-1}) + 7(10^{-2}) = \dfrac{4}{10^1} + \dfrac{7}{10^2} = \dfrac{4}{10} + \dfrac{7}{100} = \dfrac{40}{100} + \dfrac{7}{100} = \dfrac{47}{100}$

  The last fraction is "forty-seven hundredths".

Another question about decimals you might ask is: What is the reasoning of moving the decimal point during "long" division? In decimal division, to have the nice ability to work (divide) with an integer divisor (no fractional part) and to automatically adjust the decimal point of the result, the decimal point of the divisor is moved completely to the right of the least significant digit (note: include all trailing zero's in the whole portion) and then the decimal point of the dividend is moved to the right the same number of digit positions (including any possible new trailing zero's placed as least significant digits). This process is essentially creating an equivalent fraction by multiplying the divisor (denominator) and dividend (numerator) by a common (same) value such as when creating an equivalent fraction. Specifically, whenever the decimal point is moved rightward, this resulting value is, and was caused by, some multiple or power of 10. Some of these concepts have been previously mentioned, and in particular for "long" or manual division.

Ex. $1/0.5 = \dfrac{1}{0.5} = 0.5\overline{\big)1.0} = 5\overline{\big)\substack{2.0 \\ 10.0}} = 2.0$: 5/10 is where the decimal point is moved 1 position
$$\phantom{Ex. 1/0.5 = \dfrac{1}{0.5} = 0.5)1.0 =} \underline{10}$$
$$\phantom{Ex. 1/0.5 = \dfrac{1}{0.5} = 0.5)1.0 =====} 0$$
rightward in both the numerator and denominator

Here, since the decimal point was "shifted" right one digit position, in the divisor (or denominator), the common multiplier to both the dividend and divisor (or numerator and denominator) was essentially 10.

checking: $\dfrac{1}{0.5} = \dfrac{1\,(10)}{0.5(\,10)} = \dfrac{10}{5} = 2$: checks

MULTIPLICATION OF FRACTIONS

As like with the addition of fractions, we can essentially say that we have always been multiplying fractions if we just place each whole (entire) value over 1, and the relatively simple process of multiplying fractions becomes apparent.

Ex. $5 \times 3 = \dfrac{5}{1} \times \dfrac{3}{1} = \dfrac{(5)\,(3)}{(1)\,(1)} = \dfrac{15}{1} = 15$: division by 1 does not change any value

The basic rule for multiplying fractions is to simply multiply the numerators (N) together, and multiply the denominators (D) together. Unlike for the addition and subtraction of fractions, the denominators when multiplying fractions need not be the same. The product of all the numerators being multiplied is the resulting numerator of the product. The product of all the denominators being multiplied is the resulting denominator of the product.

Symbolically: $\dfrac{(N1)}{(D1)} \times \dfrac{(N2)}{(D2)} = \dfrac{(N1)(N2)}{(D1)(D2)} = \dfrac{(N3)}{(D3)}$: a symbolic form of the multiplication of fractions

Ex. If you had half a circle (1/2) and added this to the other half (1/2) of that circle, which is the same as multiplying the first half by 2 since multiplication is repeated addition, the result should be one (1) whole circle. This example also verifies the rules for multiplying fractions.

$\dfrac{1}{2} + \dfrac{1}{2} = \dfrac{2}{2} = 1$ Since this is repeated addition of the same value, we can express this with multiplication :

$\dfrac{1}{2} \times \dfrac{2}{1} = \dfrac{(1)(2)}{(2)(1)} = \dfrac{2}{2} = 1$

Consider if you have the fraction:

$\dfrac{3}{5}$ which, by factoring the numerator and denominator, can be expressed as:

$\dfrac{(3)(1)}{(1)(5)}$ by grouping fractions, this can be expressed as the multiplication of two fractions:

$\dfrac{(3)}{(1)}\dfrac{(1)}{(5)}$: with the understanding that these fractions can be multiplied.

Checking: $\left(\dfrac{3}{1}\right)\left(\dfrac{1}{5}\right) = 3 \times \left(\dfrac{1}{5}\right) = \left(\dfrac{1}{5}\right) \times 3 = \dfrac{1}{5} + \dfrac{1}{5} + \dfrac{1}{5} = \dfrac{1+1+1}{5} = \dfrac{3}{5}$

For more verification of this multiplication of fractions rule, first consider that there is nothing intuitive about this rule. Why should the following be true:

$$\frac{2}{3} \times \frac{4}{5} = \frac{(2)(4)}{(3)(5)}$$ or if this was (somewhat oddly) explicitly expressed in terms of "hand or long division":

$$3\overline{)2} \times 5\overline{)4} = (3)(5)\overline{)(2)(4)}$$

To make this mathematical discussion "less wordy", let's label the first numerator and denominator as N_1 and D_1, and the second numerator and denominator as N_2 and D_2 (Note, this does not mean N_{x2}, D_{x2}, etc, but these are unique or different (symbolic) identifiers or names assigned to a value that is either known, unknown, or being solved for).

First consider this where like fractions are combined, and the numerator (2) is repeatedly added (ie. multiplication):

$$\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{2+2+2+2}{3} = \frac{2(4)}{3} = \frac{2(4)}{3(1)} = \frac{8}{3}$$

Or by repeatedly adding the same fraction 4 times, and expressing this with multiplication:

$$\frac{2}{3} \times 4 = \frac{2}{3} \times \frac{4}{1} = \frac{8}{3}$$

The two must therefore be equal, and expressing this we get: $\frac{2}{3} \times \frac{4}{1} = \frac{(2)(4)}{(3)(1)} = \frac{8}{3}$

In general, we have $\frac{N_1}{D_1}$ to be repeatedly added (ie. multiplication) to itself say N_2 times.

Then we have: $\frac{N_1}{D_1} \times N_2 = \frac{N_1 N_2}{D_1}$

If N_2 was shown as initially divided by some value (including 1 as shown above), say D_2, this can be expressed as:

$$\frac{N_1 \left(\frac{N_2}{D_2} \right)}{D_1} = \frac{\frac{N_1 N_2}{D_2}}{\frac{D_1}{1}}$$: think of this numerator as the result of repeatedly adding (N_2/D_2) a total of N_1 times. Remember, in adding fractions, the like or common denominator is also expressed in the sum.

By multiplying both the numerator and denominator by D_2 (creating an equivalent fraction):

$$\frac{\frac{N_1 N_2 D_2}{D_2}}{D_1 D_2} = \frac{N_1 N_2}{D_1 D_2}$$: after canceling, and therefore:

This same result could also be had by multiplying both the num. and den. by (1/D1). In conclusion we have:

$$\frac{N_1}{D_1} \times \frac{N_2}{D_2} = \frac{N_1 N_2}{D_1 D_2}$$

A FRACTION OF A FRACTION

Below is a visual or graphical example of having a fraction of a fraction, or having only a part of a part of something. As you might imagine, a part, of only part of something, is a smaller part and is represented by a smaller numeric value. The method to solve for the numerical result of a fraction of a fraction is to simply multiply the fractions together. The example below will demonstrate why, but first a graphical illustration of a fraction of a fraction. [FIG 9A]

B, C and D are fractions of A.
D is less than C, and C is less than A.
D < C < A

A = whole thing =
100% = 1.0

B is a fraction of A
C is a fraction of A
B < A
C < A

D is a fraction of C , hence
D is a fraction of a fraction
D is a fraction of both C and A
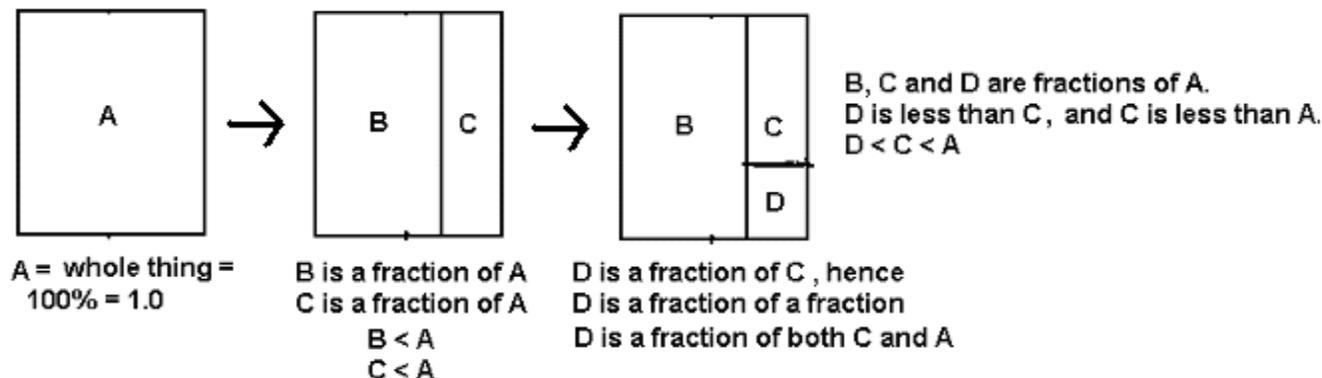
Ex.   What is one-fourth of one-half? This may also be mathematically expressed as:

   1/4 of 1/2 = ?   : 1/2 is a fraction, and 1/4 is a fraction, hence we are to find a fractional value of
                                       another fractional value.

The question in general is: "What is the value that represents this fraction of another fraction?"
At first, you might be tempted to do something like divide, but then the question would have
been stated as something like: "How many times can 1/4 go into (or divide into) 1/2?", or "How
many times can 1/2 be divided by 1/4?" So solving the original question by division is not
to be used. Can multiplication be the answer? Yes. A keyword to look for is the word "of".
as indicated in 1/4 of 1/2. The word "of" is often associated with multiplication problems. You
can also consider reading the word "of" used here as meaning: "of this", or "of this value".

Let's start at a basic example. If you have half of 10, you have 5 of the 10. Whenever you
want to find half of some value, you divide that value by 2 to find out the numeric value.

$$\frac{10}{2}$$ half, or one-half of 10. This can be expressed as:

$$\frac{10 \times 1}{2 \times 1}$$ using the commutative rule in the denominator:

$$\frac{10 \times 1}{1 \times 2}$$ which can be expressed as:

$$10 \times \frac{1}{2}$$: a whole number times a fraction, which can be expressed as:

$$\frac{10}{1} \times \frac{1}{2}$$: clearly, a fraction times a fraction.   Since this is "one-half of ten",
                    the times symbol (x) is clearly associated with the word "of".
                    For possible clarity and for how "one-half of ten" is spoken, this can
                    also be mathematically expressed (with the commutative rule) as:

$$\frac{1}{2} \times \frac{10}{1} \quad = \quad \frac{1}{2} \text{ of } \frac{10}{1} \quad = \quad \frac{10}{2} \qquad : \text{"one-half of ten"}$$

If the whole number (10 as in this ex.) was divided by a number, say 4, we would then have a fraction times (multiplied by) another fraction:

$$\frac{10}{4} \times \frac{1}{2} \qquad : \text{this could be expressed, read or spoken as: "one-half , of one-fourth of ten".}$$
Here is some more verification of this:

$$\left(\frac{10}{1} \times \frac{1}{4} \right) \times \frac{1}{2} \qquad : \text{"one-half, of one-fourth of ten". For clarity, this could be expressed as:}$$

$$\frac{1}{2} \times \left(\frac{1}{4} \times \frac{10}{1} \right) \qquad : \text{"one-half, of one-fourth of ten".}$$

So to solve for 1/4 of 1/2, "one-fourth, of one-half", simply multiply the two fractions:

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \qquad : \text{"one-fourth, of one-half is one-eighth" or}$$
"one-half, of one-fourth is one-eighth".   If you know how to divide fractions, you can divide this product by either factor so as to find the other (associated or corresponding) factor as a check.

Here is a check using pure decimal numeric values one-fourth of one-half is:

one-fourth of one-half is = 1/4 of 1/2 = (0.25)(0.5) = 0.125 = 1/8

Note also that one-half of one-fourth has the same result, particularly because the factors are the same:

one-half of one-fourth is = 1/2 of 1/4 = (0.5)(0.25) = 0.125 = 1/8

Here are some other basic examples for more verification to this process:

Ex.   What is one-half of one?

$$\frac{1}{1} \times \frac{1}{2} = \frac{(1)(1)}{(1)(2)} = \frac{(1)(1)}{(1)(2)} = \frac{1}{2} \qquad : \text{"one-half", also you can divide any value, including}$$
a fraction, by 2 to find the half-value of it.

Ex.   What is one-half, of one-half?

$$\frac{1}{2} \times \frac{1}{2} = \frac{(1)(1)}{(2)(2)} = \frac{(1)(1)}{(2)(2)} = \frac{1}{4} \qquad : \text{"one-fourth". As mentioned above, this can also be solved}$$
by dividing the value by 2.   (1 / 2) / 2 = (1 / 4)
Dividing by 2 is the same as multiplying by (1 / 2) = 0.5

Here is a graphical representation of this example (FIG 10): The circle or square was first divided into two equal parts (ie. in half), and then each of these parts was divided into two equal parts (ie. in half). The entire (1) square or circle has effectively been divided into 4 equal parts, and each part will therefore have a numeric value of (1/ 4).

[FIG 10]

half = 1/2
of 1
circle

half, of the half = 1/4

half = 1/2
of square
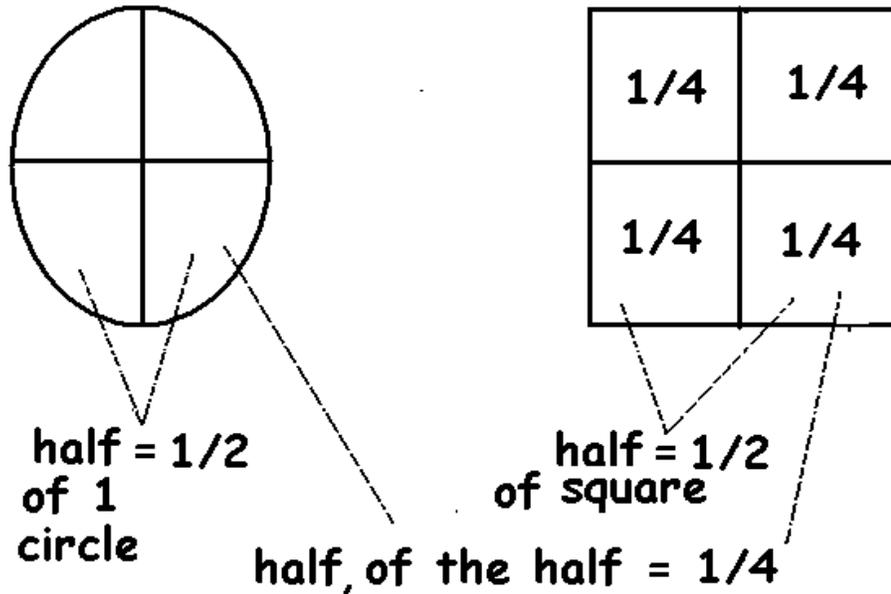
If you were to then divide each quarter section (which are already existing fractions, of the entirety (=1)) into two equal parts, the square or circle would then contain a total of 8 equal parts or sections (fractions). Each of these new sections is a half (1 / 2) of any give quarter (1 / 4) section, hence mathematically: (1 / 2 of 1 / 4), and this new section only represents a smaller value, of (1 / 8) of the entire whole structure since it is a fraction (part) of another fractional (part less than 1 whole) value, and can therefore only be less in value:

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$: one-half of one-fourth   or =   one-fourth of one-half   = one-eighth

Before moving on to the next topic, the concept and wording of a fraction of a fraction may need to be stressed a little more carefully. Sometimes what is actually meant or being discussed can be subtle and different. Consider this:

"five-tenths of ten"   and   "five-tenths out of ten"   : the key difference is the words "of" (= x), and "out of" (= /)

$\frac{5}{10} \times 10$!= $\frac{(5/10)}{10}$!= or $\neq$ or /=/ means: "is not equal to"

0.5 x 10   != $\frac{0.5}{10}$

5   !=   0.05

DIVISION OF FRACTIONS

A fraction divided by another fractional or whole value is called a complex or compound fraction. That is, both the numerator and denominator of these fractions will also be fractions. A compound fraction is basically a "nested fraction" where the numerator and denominator fractions are the innermost fractions of the larger compound fraction. Some examples of compound fractions have already been seen in this book.

$$\text{A compound fraction:} \qquad \cfrac{\dfrac{2}{3}}{\dfrac{4}{8}} \qquad \begin{array}{l} : \text{numerator} \\[2em] : \text{denominator} \end{array}$$

Actually, you can think that you have always been working with compound fractions when working with just simple fractions by dividing both the numerator and denominator by the same value (as used when creating equivalent fractions) of 1:

$$\frac{2}{4} \;=\; \cfrac{\dfrac{2}{1}}{\dfrac{4}{1}}$$

Clearly, by simplifying both the numerator fraction and denominator fraction, of the compound fraction, the compound fraction is simplified to a simple fraction. Often, the simplified form of a compound fraction is expressed in terms of the numerators and denominators. Before a working example is shown, you should know more about canceling, or "to cancel", and what a reciprocal means.

Canceling is a process to possibly eliminate fractions to a single numeric value or number, essentially ridding the fraction completely. Remember, this was also stated in the order of operations second rule: clear/cancel grouping symbols. To cancel is to "divide out", "cancel out" or remove factors that are within and-or common to both the numerator and denominator. The result is either an equivalent fraction of lower or reduced terms, or a single numeric value or number. Canceling is basically like a "pre-division" and simplification step before the order of operations is followed more correctly where factors, if possible, are first multiplied together before the division. Sometimes a "hash mark" (/) type of symbol is shown when some division or canceling has been performed. A good approach for simplifying a compound fraction is to first try to rid the compound fraction of its denominator fraction. The method to do this will be shown below, and it is essentially done using the equivalent fraction concepts.

The reciprocal or "inverse" of a number is that number divided into one (1). Sometimes this is said as "one over the number (or value)". The word reciprocal is related to the word reciprocate which means "to move or change back and forth". The reciprocal of 5 or 5/1 is 1/5 as a fraction, or 0.2 when the indicated division is performed, which is its decimal equivalent. Likewise, the reciprocal of 0.2 is 1/0.2 = 5 which is the same value we started with. A reciprocal value of any integer (ie. counting) number is unique, that is, it can never be equal to the reciprocal of some other value. The product of two corresponding reciprocals is always 1. For example (1/4)(4/1) = 4/4 = 1 = (0.25)(4). One might ask: What is the reciprocal of a fraction? The answer is to simply "invert", or "turn upside down" the fraction. The numerator will become the denominator, and the denominator will become the numerator. As indicated in this discussion, the reciprocal of 5 or 5/1 is 1/5. The reciprocal of 2/3 is 3/2, and this is verified below:

$$\text{The reciprocal of (2/3) is mathematically noted or expressed as:} \qquad \cfrac{1}{\dfrac{(2)}{(3)}} \;=\; \cfrac{\dfrac{1}{1}}{\dfrac{2}{3}} \qquad \begin{array}{l} : \text{these are compound fraction} \\ \;\; \text{expressions} \end{array}$$

Given this compound fraction, we note that if we multiply the denominator's fraction by its reciprocal value of 3/2, that we can effectively cancel it out.. The denominator will then be equal to one (1) after the canceling ("out", or "ridding") of each common factor of the entire numerator and entire denominator has been performed. Even though it has a value of 1, the denominator is still effectively eliminated since any value (here the numerator) divided by, or "placed over", 1 or not divided by any indicated value, is still equal to that value. Also, in order not to change the value of the (compound) fraction, that which is done to the denominator (fraction) must also be done to the numerator (fraction). This is necessary to create a (numerically) equivalent (compound) fraction.

$$\frac{\frac{1}{1}}{\frac{2}{3}} = \frac{\frac{1(3)}{1(2)}}{\frac{2(3)}{3(2)}} = \frac{\frac{3}{2}}{\frac{6}{6}} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$$
: verification that 3/2 is the reciprocal of 2/3
The product of two corresponding reciprocals is always 1.

Note, it is best to attempt to cancel-out factors common to both the (entire) numerator and (entire) denominator before performing the indicated multiplication. This was also mentioned in step 2 of the order of operations.

$$\frac{\frac{1}{1}}{\frac{2}{3}} = \frac{\frac{1(3)}{1(2)}}{\frac{2(3)}{3(2)}} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$$
: expressing the canceling with hash marks. If the values are not the same, such as for this example, a 2 in both the num. and den., you can reduce them both by dividing each by a common factor or value and indicate the quotients next to those factors that were reduced.

Hence, multiplying the denominator of a compound fraction by its reciprocal will easily cancel out or rid it of its denominator, leaving only a "simple" fraction in the numerator to work with. Here is a simple verification: How many times will one-half (1/2) of a circle divide into one (1=1/1) full circle? Clearly, the answer is 2, let's check:

$$\frac{\frac{1}{1}}{\frac{1}{2}} = \frac{\frac{1(2)}{1(1)}}{\frac{1(2)}{2(1)}} = \frac{\frac{2}{1}}{\frac{2}{2}} = \frac{\frac{2}{1}}{1} = \frac{2}{1} = 2$$
: note also: 1 / (1/2) = 1 / 0.5 = 2

Ex. $$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{\frac{1(4)}{2(1)}}{\frac{1(4)}{4(1)}} = \frac{\frac{2}{1}}{1} = \frac{2}{1} = 2$$
: one-fourth will go into one-half, twice

Checking using the decimal equivalents of the fractions:

$$\frac{\frac{1}{2}}{\frac{1}{4}} = \frac{2\overline{|1.0}}{4\,|\,1.0} = \frac{0.5}{0.25} = 2$$
: checks

Ex. $$\frac{\frac{1}{4}}{\frac{5}{8}} = \frac{\frac{1(8)}{4(5)}}{\frac{5(8)}{8(5)}} = \frac{2}{5}$$

Checking, by multiplying the denominator (divisor) by the resulting fraction (the quotient) to see if the product is equal to the numerator (dividend):

$$\frac{(5)(2)}{(8)(5)} = \frac{10}{40} = \frac{\overset{1}{\cancel{10}}}{\underset{4}{\cancel{40}}} = \frac{1}{4} \qquad \text{: checks}$$

Ex. A certain cooking recipe requires 1/3 of a cup of water, however you decide to only use enough of each ingredient to make half of the total quantity the recipe will make, how much water will you then need?

The answer is to use one-half (half = the whole or given part evenly divided by 2) of each ingredient.

$$\frac{\dfrac{1}{3}\ \text{cup}}{\dfrac{2}{1}} = \frac{1(\ 1\ )}{3\ (\ 2\ )}\Big/\frac{}{2\ (1\ )}{1\ (\ \cancel{2}\ )} = \frac{1}{6}\ \text{cup} \qquad \text{: one-sixth of a cup of water.}$$

As mentioned previously, product of a number and its corresponding reciprocal is always equal to 1. Reciprocals are also the reciprocals of each other.

$$5 \times \frac{1}{5} = \frac{(\ 5\ )(\ 1\ )}{(\ 1\ )(\ 5\ )} = \frac{5}{5} = 1 \qquad \text{or since (1/5) = 0.2 :}$$

$$5 \times 0.2 = 1.0$$

We see that in the above expressions for 1, that in one factor, the number is a multiplier, and in the other factor, that same number is a divisor. Since multiplication and division are inverse operations of each other, you may sometimes see reciprocals being mentioned as a "multiplicative inverse".

Most scientific calculators have a reciprocal-key or "calculator function" for the reciprocal operation. This key or button is usually labeled as: [1/x]   . If your calculator does not have this convenient key, then simply divide 1 by the number in question that you want to find the reciprocal of.

Note: Canceling can only be done on one factor at a time of both the entire numerator or denominator. Once a value is reduced to 1, it is therefore essentially eliminated (canceled out) and what is left of its initial value cannot be used for any more possible canceling.

Ex. $\dfrac{(8)(4)}{2} = \dfrac{\overset{4}{\cancel{(8)}}(4)}{\cancel{2}}$ or $= \dfrac{(8)\overset{2}{\cancel{(4)}}}{\cancel{2}} = 16$ $\neq$ $\dfrac{\overset{(4)(2)}{\cancel{(8)(4)}}}{\cancel{2}} = 8$: once an initial factor is canceled or factored, it cannot be reused as that same initial factor value , but any factor remaining of an initial factor can be used for further canceling.

Checking: $\dfrac{(8)(4)}{2} = \dfrac{32}{2} = 16$

              Donate

CANCELING ENTIRE EXPRESSIONS

Since an expression essentially, or eventually, represents one specific quantity or value in question, often indicated within a surrounding grouping symbol for clarity, it can be canceled-out just like one single numeric values if it's a (common) factor to both the entire numerator and denominator of the fraction.

Ex.

$$\frac{5(3+2)}{7(3+2)} = \frac{\overset{1}{5\cancel{(3+2)}}}{\underset{1}{7\cancel{(3+2)}}} = \frac{5}{7}$$

Checking by varying the order of operations, which can be performed wherever possible as long as the result is not caused to be made incorrect. Here, for some simplification of the expressions, addition is first performed within the grouping symbols as part of the effort to clear the grouping symbols in the numerator and denominator of the fraction. Canceling (division or dividing out) common factors is then chosen over multiplication, so as to first simplify the numerator and denominator) and help prevent the need to reduce the resulting fraction to lower terms:

$$\frac{5(3+2)}{7(3+2)} = \frac{\overset{1}{5\cancel{(5)}}}{\underset{1}{7\cancel{(5)}}} = \frac{5}{7}$$

: or = 25/35 when "unreduced" (to simpler terms), and if both the numerator and denominator are divided by 5, the equivalent fraction created is 5/7

PERCENTAGES

The word cent (or centi) in the word percentage means "one-hundredth of" which is 1/100 or 0.01 numerically. Therefore, percent means how many "(per) hundredths of something" or "parts per hundred of something, that was (physically, and-or mathematically, numerically) divided into one-hundred parts". The concept of percent was briefly mentioned in the topic of: A Short Discussion About Percentages.

A percentage, or more correctly, a percentage rate is simply a numerical measure of a part, fraction or portion of something, and is in reference and-or comparison to that whole or entire something. Symbolically:

  Part   = numeric representation of how much Part and-or its numeric portion or representation
  Whole   is in reference to, or with respect to, the Whole thing. We see that this expression or
               mathematical representation of a percentage rate is the same as that of an expression
               for a fraction, and this is so since we are numerically finding out what is the relative size
               (ie., between 0 to 1, or 0% to 100% = "zero percent to one-hundred percent") that this
               portion or part is in reference to the entire or whole thing. The value of this expression is
               a rate or comparison of two values, and is called the portion rate or percentage rate, and
               the actual numeric value of Part is technically the actual percentage value (without the
               rate or comparison meaning) in question that was used to find the (percentage) rate:

  Part / Whole = percentage value / total = percentage rate   therefore, mathematically:
  (Whole) (percentage rate) = percentage value = Part

Below are some examples to help demonstrate as to how we may use the concepts of percentages.

1. The monetary value of a common penny coin is one cent, or "one-hundredth" of a dollar, hence 1/100 = 0.01 of a dollar. To find a percentage is to find the total number of hundredths in question of any value or quantity that was divided (or imagined or conceived as divided) into 100 equal parts or fractions. The accepted symbol that indicates the amount of hundredths in question, or that hundredths are being considered, is %. It is called the percent symbol. Percent values are mostly used to easily generalize or indicate the basic comprehensible, relative value or size of a portion or fractional value of something, rather than use exact specific values, especially when the size or value of that something can change or vary, yet the (percentage) rate of the portion value in reference or respect to the whole value, remains constant, say for example, as being always 50% of any current value or total at hand or being considered.   For some simplified examples of relative values and their percentages: "none" - such as 0 percent, "tiny" - such as 1 percent, "small" - such as 10 percent, "medium" - such as 50 percent, "big" - such as 70 percent, "large" - such as 90%, all, "entire" or "whole" - such as 100%.

2.   When using a fixed or constant percentage value, the actual or specific result value automatically adjusts for any changes in the overall size of that something.   For example, a store may indicate that the selling cost of any item(s) is 10% cheaper or less than its previous selling price rather than list the actual amount(s) reduced and-or the previous selling price of the item(s). A store may even have a "store-wide (all items) sale for a day", where everything in the store is 20% cheaper than the indicated price on the price-tag, rather than make many new and temporary price-tags for all of the store items.

3. By some contract or agreement, a person might own or be allotted a specific percentage, say 10%, of all (entire) the grain produced for the year at a farm. rather than say the allotment is to be specifically 500 pounds only, or 500 pounds maximum. In a good growing year, 10% of all the grain might be 600 pounds, and in a bad growing year, 10% of all the grain might only be 200 pounds. This way, all the people who were allotted or assigned to that grain produced at that farm will also share in the gains and losses in the actual total amount of grain produced during that growing season. Clearly, if the farm only produced 200 pounds of grain, and if a person was to be allotted a specific value of 500 pounds, it is no even possible for that person to receive that specific amount of grain.

4. Sales tax, often some low fixed percent, say 5 or 6 percent (5%, or 6%), at a store to purchase a specific item(s) will be

based on the selling price of the item(s). The higher the price of the item(s), the more the total sales tax money value will be, and yet the indicated percentage or sales tax rate is a constant or fixed value. At this point in this book, it could be obvious or intuitive that if the cost of the items doubles (2), or goes up by some factor, then the sales tax money needed for that item will also double or go up by that same exact factor (here 2), and yet the indicated percentage or sales tax rate is still the same constant or fixed value. When changes correspond in such a direct manner as this, the relationship of the resulting values, or the change values themselves, is said to be a direct, constant, straight, "one to one", or "linear" (line-like) mathematical relationship.

5. In a hospital, sometimes the amount or dosage of a certain medication to give a patient will be a specific (fixed, constant) percentage (ie. a fractional value or portion) of a persons weight. The more someone weighs, the more the specific amount of medicine they will need and receive, and yet the percentage value remains the same or "fixed" for patients that have various weights.

6. If a certain drink mixture is described as containing 10% fruit juice, regardless of the volume of that fruit juice, and hence for any specific volume (such as a spoonful, glassful, or a liter) of the mixture considered to be put into several glassfuls to drink, each glassful will contain 10% fruit juice. If you know the specific volume of the entire mixture used, and after solving for 10% of that volume, you have found the specific volume or amount of fruit juice in that specific volume of the mixture.

If you have 5% (commonly spoken as "five percent" or= a "five percent rate") of something, you have 5 one-hundredths of it:

$$\frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100} = \frac{5}{100}$$: "five one-hundredths" or simply: "five hundredths"

Or, since we have a repeated addition of adding the same value 5 times, we can express this with multiplication as:

$$\frac{(5)}{(1)} \times \frac{(1)}{(100)} = \frac{(5)(1)}{(1)(100)} = \frac{5}{100} = 0.05 \text{ in decimal form.}$$ or:

$$5(1/100) = 5(0.01) = 0.05$$

This is often indicated as: 5 hundredths = 5 percent = 5% when percents are considered. The percent symbol and the value preceding it are sometimes, and more formally, called the rate. The word rate is rooted in the words of: reason, rational (reasoning, logic), and ration (an allotment, fraction, or portion of something, and in the numerical sense, it similar to the word ratio). In this example, the (percent) rate is 5 percent = 5% = (5/100) = 0.05.

Ex. What is five percent of three hundred apples? This can be expressed as: 5% of 300 apples is how many apples? According to the definition of percent, 5% of 300 apples is = 5 hundredths of 300 apples = (5/100) of 300 apples. 300 represents the whole (entire, all = 100% = 1.0) lot, amount or quantity that is to be divided into 100 even (or equal, similar) parts when percents are being considered.

First of all, 1% (= one-hundredth = 1/100 = 0.01) of 300 apples is 300 apples divided into 100 equal parts:

$$\frac{300 \text{ apples}}{100} = 3 \text{ apples}$$: (a group of) 3 apples physically compose, and is mathematically, 1% of 300 apples. Checking by multiplying the quotient and divisor, the dividend is: (3 apples)(100) = 300 apples, and this expression also indicates that 300 apples is 100 times bigger or more than 3 apples. Likewise, it also expresses that if you have 3 apples, and increase, magnify (mathematically, a multiplication = multiples or many of) or multiply it by 100 (or 100 times, or multiples of), you will have 300 apples.

If we mathematically compare 3 apples to 300 apples, perhaps to find the "magnification size or factor", "how big one is compared to the other", or "how many times bigger one is to the other", we can get this expression below, which can also

be mathematically derived from the above equation by dividing both sides by (300 apples).

$\dfrac{3 \text{ apples}}{300 \text{ apples}}$ = (1/100) = 0.01 = 1%   :This shows why a percent is really a rate or plain unitless numeric ratio of two values. Extra: the reciprocal of this expressed fraction is: (300 apples) / (3 apples) = 100. This value of 100 indicates that 300 apples is 100 times more or bigger than 3 apples.

This indicates that three apples out of, or compared to 300 apples, is (only) 1% of them. Three apples is a portion (part, or fraction (less than all = the entirety = 100% = 1.0)) of the 300 apples.   Mathematically, this portion is 0.01 = 1/100 = "one-hundredth" = 1%.   The rate can be said as: "3 out of 300", or "3 per 300", or more commonly, basically or fundamentally (in terms of hundredths) as: "1 out of 100", or "1 per 100", and numerically, this is expressed as: 1/100 = 0.01 = 1%.

["Technically", the indicated rate is equal to the indicated percentage, and here, its 0.01 or 1%, and that the actual or specific value that this percentage rate represents (of the entirety) for this example is (3 apples). (3 apples) is therefore called the (actual or specific) percentage or percentage value.   This may seem a bit confusing at first, but it is something to consider. It would help if the mathematical (rate) value (in terms of hundredths of the entirety) of a portion is called the percentage rate, or the indicated percentage, rather than simply and commonly as just the percentage. As we have seen, we can have a fixed percentage rate of say 6% sales tax, but the actual tax value or money needed, or the actual percentage value that this 6% rate indicates or is associated with, can vary widely since it depends on, and is mathematically in reference to, or based on, and calculated using the "total" (all, entire) cost or money required during a purchase or sale. Some more will be said about this further below.]

1% of anything or value is a hundredth of it, this can be mathematically expressed as: (value /100) or (total/100). 1% of 300 can therefore be expressed as (300/100) = 3 if we perform this indicated division. This leads to the conclusion that 3 apples is 1% of 300 apples . If we multiply this specific 1% value by 5, we will then know the number of apples equal to 5% of 300 apples, since 5 percent is obviously 5 times more than 1 percent. (5 percent/1 percent = 5). This could be stated in a question as something like: "If 3 apples = 1% of all the apples, how many apples are 5% of all the apples?".

(1% of apples) x 5 = (5 x 1%) of apples = 5% of apples :

$\dfrac{300 \text{ apples}}{100}\dfrac{(5)}{(1)}$ = $\dfrac{\overset{3}{\cancel{300}} \text{ apples}}{\underset{1}{\cancel{100}}}\dfrac{(5)}{(1)}$ =   15 apples   or: 3 apples (5) = 3 apples x 5 = (3 x 5) apples = 15 apples

checking: $\dfrac{15 \text{ apples}}{300 \text{ apples}}$ = (15/300) = (5/100) = 0.05 = 5%

We showed that 5% = 5/100 = 0.05 = 5 (1/100) = 5 x 0.01 = 0.01 x 5 = 1% x 5 , and note that:

$\dfrac{300 \text{ apples}}{100}\dfrac{(5)}{(1)}$ can be written or expressed as as:

$\dfrac{300 \text{ apples}}{1}\dfrac{(5)}{(100)}$: due to multiplication being commutative.   This is equal to:

(300 apples)(0.05) = 15 apples = (300 apples)(5/100) = (300 apples)(5%) =   5% of 300 apples

Therefore, to simply get the actual (rather than the rate or indicated) percentage value (ie., the specific or actual quantity or value that this rate actually represents), multiply the total value or quantity given by the indicated rate expressed in decimal or fractional form. The general "formula" for finding percentages is usually written as:

(total)(rate) = percentage   **: the common and basic "percentage formula"**

This percentage formula stems from (ie. is derived from) the basic form of any ratio. Here, rate is a ratio value (a mathematical representation of the comparison of two values which is essentially a numeric (and unitless) quotient) value of the actual percentage value (typically only some amount of the total amount) in question divided by the total or Whole value in question. Note the similarity of the word rate and ratio. Whereas two numbers can be mathematically compared by their resulting (linear) distance separation or difference (ie. a simple change, or mathematical measure of the separation), two numbers can also be mathematically compared by their resulting ratio value (ie. a multiplier or magnification change of one value [as a numerator] as compared to, or in reference to, the other value [such as a denominator]). Ratios will be given a more general discussion further ahead in this book.

Remember, to find the unknown factor in a product, you simply divide by the other given factor. Considering this using the formula above, this results to the following which is also equivalent to dividing both sides of the equation by (total), and then canceling out common factors in the entire numerators and denominators.

$\dfrac{percentage}{total}$ = $\dfrac{Part}{Whole}$ = rate   : Showing that the rate or "percentage rate" value is the mathematically or numerically "indicated percent" of the actual percentage value with respect to the total value.   Likewise, the given formula can be expressed as:

$\dfrac{percentage}{rate}$ = total   : total is sometimes called the base, base value, or working value being considered

From: (total) (rate) = percentage value

(300 apples)(5%) = (300 apples)$\dfrac{(5)}{(100)}$ =   (300 apples)(0.05)   =   15 apples

The largest possible part or fraction of something is numerically: 100% = 100 parts out of, or of, 100 parts = 100/100 = 1 of something which is the entire, "all of it", or whole thing. It is possible to have percentages, or more correctly, rates, that are higher than 100% as when the that something is now larger than before, hence that higher than 100% value is in reference to a previous value, since the largest possible part or fraction of any value is still only 100%. If something is claimed to be 100% larger, or a 100% increase or bigger, it generally means that it has increased or been magnified (multiplied) by a factor of: (original value + increase) / original value = (100% + 100%) /100% = 200% / 100% = 2, and may be stated as being two times more or greater than the previous or original (reference) value, or now being twice as big.

Ex.   What is 3.75% of $1,530.35 to the nearest penny?

This can also be expressed as: What is 3.75% of $1,530.35 rounded to the nearest penny.

"To the nearest" means **rounded** or mathematically adjusted and truncated (reduced or limited to) at that decimal position or "precision" to reflect if the next least significant digit (possibly rounded itself) was 5 or greater, and if so, it will be increased or "rounded up", by 1. Any and all digits having lesser numerical significance will be discarded as unneeded. Hence, the result, when rounding was used or considered, can only be assumed to be an approximation (not exact) or close value due to this process. **Rounding** is used to eliminate often unnecessary (lengthy. least significant or practically meaningless in value, that are of low effect or concern) digits in a number and to make further calculations shorter in length and easier to process, and to have nicer looking result values. The small error and effect of, and after discarding even some small (fractional) part of a value(s) can eventually "add up", or multiply in significance and therefore have growing influence on the results of further calculations, and in such a manner that the error or the difference from the actual or true result can be significant enough to consider and not ignore and the result should then be considered and explicitly stated as being an approximation.

To use a percentage value as an operand of an expression, first convert it to its equivalent decimal form.
This can be done visually by moving the decimal point two digit positions leftward. For example:

5% = 0.05   : "five percent" equals 0.05 in decimal, or strict numeric form.

3.75% = 0.0375   : since 3.75% = 3.75/100 = 0.0375
                             The percent symbol, %, numerically means: (1/100).

Using a calculator: $1,530.35 x 0.0375 = $57.388125

Rounding to the nearest penny (or hundredth of a dollar) we get: $57.3<u>9</u>   The underlined number, here for 9, is an expression or symbol to express that the actual, true or entire number given was rounded to and truncated this this position or precision ("precise", smallest part of).

Ex. In a coupon you received, a store claims that a certain items selling price of $7.99 will be reduced (decreased, discounted) by 10% (or "10% off") when you give them your coupon. What will be the new selling price?

From: (total)(rate) = percentage

          ($7.99)(10%) = percentage   Often spoken as: "10 percent of $7.99" is:
          ($7.99)(0.10) = $0.799   : Which is $0.<u>80</u> when rounded to the nearest penny. In rounding, as with addition, you must still consider carries. The cost of the item will be 80 cents less than the previous value.

New price = Old price - Reduction
New price = Old price - (10% of Old price)
New price = $7.99 - $0.80
New price = $7.19   : new selling price. Any sales tax should now be considered on this new and lesser value.

This problem may have been solved knowing that if 10% is removed, (100% - 10%) = 90% of the price remains, and 90% of $7.99 is:

          ($7.99)(90%)   = $7.19

        You can easily find 10% of a numeric decimal value if you move the decimal point one place to the left. This effectively multiplies that value by (1/10) = 0.10, or divides that value by 10.

          Ex. 10% of 4.57 = 0.457

Just the same, to find 1% = 0.01 of a value, simply move the decimal point two places leftward, which is the same as multiplying that value by 0.01, or dividing it by 100.

Ex. 1% of 103 = (103)(0.01) = 103/100 = 1.03

Ex. 1% of 47 = 0.47

If you double this 1% value of 0.47, you then can find 2% of 47. Showing some helpful intermediate steps:

2% of 47 = (1% of 47) + (1% of 47) = 2 x (1% of 47) = 2 (0.47) = 0.47 + 0.47 = 0.94

The last equation was probably the easiest or quickest way to find the result, but it may not have been so (initially) obvious of just adding the 1% value twice so as to have the 2% value :   0.47 + 0.47 =   0.94

If you know 10% of a value, you can double that percentage value to find 20% of that same value.

Ex. If 10% of 100 is 10 ,   20% of 100 = (10% of 100) + (10% of 100) = (10% of 100)(2) = 2(10) = 10 + 10 = 20

It is somewhat easy to find 50% of a value since 50% = 0.50 = 50/100 = 1/2, that is, you can simply divide a value by 2 to find 50% or "one half" (or simply: "half") of it.

Ex.   50% of 60 = (60/1)(1/2) = 60/2 = 30   : Likewise, if you divide a value by 2 or "in half", you have the 50% portion and value of it.

Ex.   An item is on sale for $5.20 and this value is indicated as being 30% cheaper. What was the original selling price?

There is a straight-forward algebraic approach to solving this, but since this book has not yet covered much of algebra yet, a simple mathematical method will be shown first:

If 30% was taken away (the reduction) from the 100% value (the original/starting or previous value), this leaves us with a value equivalent to:

 100% - 30% = 70%   of the original value (price). Hence, the ratio (fractional or percentage value if you will) of the current selling price (with respect) to the previous price is 70% or 0.70 in decimal form:

  $\dfrac{\text{current-price}}{\text{previous-price}}$ =   0.70   : "the current-price is 70 percent of the previous-price".
                                                                  Also, 100% - 30% = 70%   =   1.0 - 0.30 = 0.70

  $\dfrac{5.20}{\text{previous-price}}$ = 0.70   therefore, mathematically, as when checking a division problem we multiply the quotient and the divisor together to find the dividend, or an expression for it:

  5.20 = previous-price x 0.70   and therefore, dividing the product by one factor to find the other factor. as when solving or checking a multiplication problem:

  $\dfrac{5.20}{0.70}$ = previous-price   simplifying this,we find that:

  previous-price = $7.43   : original selling price

You may also consider this check:

current-price = previous-price - reduction in price
current-price = previous-price - 30% of the previous-price
current-price = 7.43   - (7.43)(0.30)
current-price = 7.43   - 2.23
current-price = 5.20

Since: previous or original price x percent reduced = reduction   , we can get:

  $\dfrac{\text{reduction}}{\text{previous-price}}$ =   percent reduced

  $\dfrac{2.23}{7.43}$ = 0.30 = 30%   : "2.23 is 30% of 7.43"   or: 2.23 = 30% x 7.43   or: 2.23 = (30%)(7.43)
                                              In these above expressions, you may also use 0.30 for 30%, such as for actually doing or calculating those mathematical equations.

It could be stated, as indicated in the equations, that:

30% of the previous or original price is the reduction in price.

The current price is 70% of the previous or original price.

A similar, but a more algebraic approach, using the concepts of signed numbers that are yet to be discussed, to solve this problem would be:

previous-price - reduction = current-price   using the concept of substitution to substitute the values we know:
(1.0) previous-price - (0.30) previous-price = 5.20   Combining the like terms of: previous-price:
(0.70) previous-price = 5.20   After dividing both sides by 0.70 and canceling:

previous-price = _5.20_ = 7.43
                    0.70

The most straight-forward way to solve this problem above is to use the "percentage formula". Assigning $5.20 as the actual or specific value of the percentage (left or remaining) of the Total value, and expressing that (100% - 30%) = 70% of the (unknown) Total is $5.20:

   Total x Rate = Percentage
    ?   x   70% = $5.20   Solving for Total = ? , dividing the product by one factor to find the other factor:

   Total = _$5.20_ = $7.43
              0.70

Ex. If you were to receive 50% of only 1/4 of something, how much of that something would you then receive?

First, note that what we are actually trying to find is a fractional value (here, expressed as a percentage; 50%) of another fraction (here 1/4), in short, we are to find a fraction of a fraction. This topic has been discussed previously in this book. (1/4) of something is 0.25 = 25 percent of something. You are to receive 50% of 25% of something.

Since 50% is 0.50 in decimal form, and we know that we can multiply this decimal form to the total value in question so as to find the actual percentage value:

  (0.50)(1) = _0.50_ = 0.125 = 12.5% of something
    1 (4)   4

  Or, since 1/4 = 0.25 (or 25%) in decimal form:

   (0.25 of something) (0.50) = (0.25)(0.50) of something = 0.125 of something , or = 12.5% of something

Note that since fractions and indicated percentages of a value are only relative to a specific value in question, "something", as in this example, can actually be any thing or value, and is considered as 100% = entirety or Whole.

  50% of 25% = _(50)_ _(25)_ = (1) of (1). = (1)(1). = _1_ = 0.125 = 12.5%
                 (100) (100)   (2)   (4)   (2)(4)   8

Ex.   If 4 is 20% of something, (or same as: "if 20% of something is 4"), what number (ie. the actual percentage value) corresponds to 30% of that something?

We can easily solve this by setting up a proportion type of problem. Even though the concept of proportions has not been covered up to this point, let's first consider the common (formula) percentage type of equations that can be created with the information stated in the problem:

From: total x percentage rate = percentage

Equation 1:   (total)(0.20) =   4
Equation 2:   (total)(0.30)   = (number)

Since the indicated percentage increases from 20% to 30%, the (number) to be found will be greater than 4.

Taking each equation and considering the two factors (that make the product) in each, and solving for (total):

   total   = $\frac{4}{0.20}$: Note, this here could also be solved by division, and then substituting this quotient
                                                     value into the following, derived from the second equation:

   total   = $\frac{number}{0.30}$

Since the right-hand side of both of these equations is equal to: total , they can both be equated as being equal. The resulting equation is an example of what is known as a proportion (same or equivalent portions, parts, or fraction of the whole) type of expression:

   $\frac{4}{0.20}$ = $\frac{number}{0.30}$: a proportion type of problem,   solving for number :
                                                     : This might be read or spoken as "4 is to 20%, as (equal to) number is to 30%".
                                                     : Note also that the left hand side could of been simplified here to: 4 / 0.20 = 20

If may not be apparent to you at this point, but notice that since each side is a fraction, and that both sides are equal, then the fractions can only be equivalent fractions and the problem could be solved as such. Much more will also be said about proportions later in this book. The two equations above for: total , are also known as a "system of equations" or as "simultaneous equations" since the value for "total" will solve both equations simultaneously (at the same time) or together, or "at once" when it is found just once, and since it has the same value for both equations.

Considering the entire left hand side of the equation as a quotient value of a division problem, such as on the right hand side, we know that to check a division problem, you multiply the quotient and divisor together and check if it equals the dividend. Indicating (expressing) this with the stated values:

number = (0.30) x $\left(\frac{4}{0.20}\right)$ = $\frac{(0.30)(4)}{0.20}$ after simplifying the right-hand side :

number = 6

Extra: Here is an example of how the value of: "total" can be found which could then be used to solve any of the original two equations:

From: $\frac{4}{total}$ = 0.20   =   20%   : mathematically expressing that "four is (only) twenty percent of "total"

Multiplying the quotient and divisor together should yield the dividend:

   4 = 0.20(total)   : as was stated in the problem, and the equations: (Eq. 1)

Dividing 4 by one factor should yield the other factor, hence solving for "total", and this is also the result of dividing both sides of the equation by the same value of 0.20:

   total = $\frac{4}{0.20}$

total = 20

Checking the value of number: (total) (0.30) = (number)
                                                  20 x 0.30 =   6.0   : checks

Here are some more examples of using percentages:

Ex. If 26% of something (perhaps an amount of money) is given to a person A, and then person A promises 35% of his share to person B, what percent of the original total (the 100% = 1, entire, all, full value, of the money) will person B actually receive?

First, note that this is a problem pertaining to a fractional value of another fraction. Person B will receive a fraction of Person A's fraction of the total.

Let's start by converting the percentages to decimal values:

     26%   =   0.26

     and 35% of this is: 0.26   : (0.26)(0.35) , a fraction, of another fraction
                                   x 0.35
                                      130
                                   + 780
                                   0.0910   =   9.1%   : Person B will receive 9.1% of the total money

Notice that the problem above is a relative or relative values (ie. with respect to one another, and not something else) type of problem and that the values are all relative (ie., not specific in actual values which could be any values conceivable) in the form of percentages. No actual amounts such as the total money value was used. The total money value could be any value, say from 1 dollar up to 1000 dollars, and the actual value does not matter for this type of problem, and the same (indicated) percentages (more specifically, the percentage rates) would result regardless of the actual values being considered.

Ex. If you are given $1.25 from a total of $7, what percent of this total did you receive?

First, consider that you received only a part or fraction of $7 since $1.25 is less than $7. This problem is solved by placing the part you received over the total, hence making a proper fraction:

     1.25   =   0.17857   =   17.857%   : You received $1.25 which is 17.857% of $7.
     7.00   Again, formally, $1.25 is the (resulting) percentage value
                                                                due to applying the percentage rate of 17.857%

CONVERTING A FRACTION TO A PERCENTAGE

If you have 5/10, or "five-tenths" of something, what percentage of that something do you have? Here, we have 5 parts of something divided into only 10 equal parts. Since a percentage is the result wanted, we really want to know how many parts we would have if we divided that something into 100 equal parts instead of 10. Considering both of these fractions, the denominator of 100 is ten times more than the other denominator of 10. Hence, we can make an equivalent fraction where the numerator and denominator are both ten times larger:

$$\frac{5}{10} = \frac{}{100} \qquad \text{creating an equivalent fraction:}$$

$$\frac{5}{10} = \frac{5(10)}{10(10)} = \frac{50}{100}$$

$$\frac{50}{100} \quad \text{can be written as:} \quad 50\left(\frac{1}{100}\right) = 50\% \qquad \text{: If you have five-tenths of something, you have fifty-hundredths or } 50\% \text{ of something}$$

If we perform the indicated division and multiplication, we get:

$$\frac{50(1)}{1(100)} = 50(0.01) = 0.50 = 50\%$$

If we perform the indicated division on the original fraction, we should also get this result without the need for creating the equivalent fraction. After all, equivalent fractions represent the same value, and if you find the value of one fraction, you have found the value of the other equivalent fraction:

$$\frac{5}{10} = \frac{0.5}{10\,|\,\overline{5.0}} = 0.50 = 50\%$$
$$\underline{50}$$
$$0$$

This method can also be verified using factorization, and then canceling out common factors (to both the entire numerator and entire denominator):

$$\frac{50(1)}{1(100)} = \frac{5(\cancel{10})(1)}{1(\cancel{100})} = \frac{5}{10} = 0.50 = 50\% \qquad \text{or:} \quad (50/100) = (\cancel{50}/\cancel{100}) = 1/2 = 0.5 = 50\%$$

We see that the decimal equivalent of a fraction is equal to the fractional value or percentage (rate) that the numerator is (in reference, or respect) to the denominator, or it can be stated as the decimal equivalent of a fraction is equal to the rate that the numerator is to the denominator. The decimal equivalent of a fraction is also equal to the "magnification factor" or "multiplication, multiplier, or factor" of how many times bigger (or even smaller, if its less than 1, hence it's only a fractional part) that the numerator is in reference to the denominator. Consider:

numerator x magnification = denominator   therefore: $\dfrac{\text{numerator}}{\text{denominator}}$ = magnification or fractional value

factor1 x factor2 = product   therefore: $\dfrac{\text{factor1}}{\text{product}}$ = factor2 = $\dfrac{\text{dividend}}{\text{divisor}}$ = quotient

Given: $\dfrac{5}{10}$ = 0.50 = 50%,   we can then say:

   "5 is 50% of 10"   or = :   "50% of 10 is 5"   : If you substitute the equals symbol for "is", you will see that these two statements are the same, but the expressions on both sides of the equals sign were switched.

Ex. 7 is what percent of 8 ?

Assuming there are 7 parts in question of something divided up into 8 equal parts, and expressing this as a fraction, we get:

   $\dfrac{7}{8}$ = 0.875   .

                                                                                                  1

0.875 is the decimal equivalent of the percentage

Note, if we divide both the numerator and denominator by the denominator, we get: (7/8) / (8/8) = (7/8) / 1 = $\dfrac{0.875}{1}$ = 0.875

To show this value with the percent symbol, multiply it by 100 or simply move the decimal point 2 digit positions rightward:

   0.875   = "eighty-seven point five, hundredths" = 87.5/100 = 87.5 (1/100) = 87.5%

We can then say that: "7 is 87.5% of 8"   or=:   "87.5% of 8 is 7"

To convert 87.5% back to its full decimal equivalent, show that 87.5% is 87.5 parts of 100 equal parts. This is easily done remembering that % = (1/100):

   87.5%   = $\dfrac{87.5(1)}{1\ (100)}$ = $\dfrac{87.5}{100}$ =   0.875

Hence, to convert an indicated percent to a strict, single numeric decimal value, divide it by 100 or simply move the decimal point 2 digit places leftward and remove the percent symbol.

Ex. A certain mixture contains 4 ingredients. The units of measurement for each ingredient could be ounces, teaspoons, tablespoons, cups, ounces, liters, milli-liters, and for this example, they will simply be indicated as (generic, generalization, unspecific) units so as to have a non-specific aspect. Regardless of the specific units (of measurement or reference) used, the percentage values would be the same. Here is the list of ingredients for the recipe (ingredients or substances, and instructions) of the mixture:

   3 units of ingredient A
   2 units of ingredient B
   1 unit of ingredient C
   0.75 units of ingredient D

   What percent of the entire mixture is ingredient B?

   Summing up all the units included in the entire mixture:

   3 units + 2 units + 1 unit + 0.75 unit = 6.75 units   : total units of ingredients included in the entire mixture

$$\frac{\text{amount of ingredient B}}{\text{total amount of all ingredients}} \;=\; \frac{2 \text{ units}}{6.75 \text{ units}} \;=\; 0.296\underline{3} \;=\; 29.63\% \qquad \text{: Ingredient B is about 30\%}$$
of the entire mixture.

If you were to calculate each ingredients percentage of the total mixture, the sum of these percentages will be 100% = 1.0

Ex. An expected or ideal value of 10.0 is typical for a certain thing, but its measurement is found to be slightly over at 10.81. Express this difference in values as a percentage.

Difference = measured value - ideal value = 10.81 - 10.0 = +0.81

Expressing the difference as a percentage from the ideal value: $\dfrac{+0.81}{10.0}$ = 0.081 = 8.1% over or increased

Note also for relative values: $\dfrac{10.81}{10.0}$ = 1.081 = 1.0 + 0.081 = 100% + 8.1% and 1.081 - 1.0 = 0.081 = 8.1%

FINDING A VALUE OR ITS' PERCENTAGE BETWEEN A RANGE OF VALUES

Given a range of values from 7 to 15, what percentage (%) from 7 (ie., seven will be considered as the start, or 0% vallue within the range specified) is 10, in other words, 10 is what percent of the "total numeric distance = 100% range of values" between 7 and 15? Also, if given a certain percentage (rate) between a range of values, what value between that range corresponds to that percentage (rate)?

Expressing some of this in number form, we are NOT finding this: $\dfrac{10}{15}$ = 66.$\underline{7}$ = 66.$\underline{7}$% :Though this is mathematically right, this is not what we are solving for.

This last concept was mentioned because the denominator would then actually represents a range of 0 to 15, that is 15 is in reference to, and is the distance from, 0 and not some other value such as 10 as first mentioned.

We can use line segments (a small portion of a line that extends to infinity) an-or a number (ie., numbered, indicated) line as a graphic aid for analyzing these types of problems since a length value is essentially the difference between the start and end values on a line.

Line Segment 1:
Line Segment 2:

Line segment 1 is obviously less in length than line segment 2, and is therefore only a certain fraction or percentage of line segment 2.

Given two values, say 2 and 4, 2 is 0.50 = 50% of 4 since: 2/4=0.5 and 4 x 0.5 = 2 or:

$$\% \;=\; \frac{number1}{number2} \qquad \text{: percentage that number 1 is with respect, or in reference to, number 2}$$

$$\frac{2}{4} \;=\; 0.5 \;=\; 50\% \qquad \text{: number1 is equivalent to a 0.50=50\% portion of number2}$$

These values can be thought of as the lengths of line segments:

```
    ---------------                    : length1
          2
```

```
    -------------------------          : length2
              4
```

$$\% = \frac{length1}{length2} \quad : \text{percentage that length1 is with respect to length2}$$

Each line segment can also be considered as a part of the same or different number lines where the difference between the maximum and minimum values is the length of that line segment.

Since: max. = min. + length   : You can also think of length as distance. Algebraically:
             length = max. - min.

```
        ----|---------------|----       : length = max. - min. = 10 - 8 = 2
    . . .    8             10    . . .    max. = maximum, here the maximum value
                                          min. = minimum, here the minimum value
```

We see that length and-or the difference or change between two lengths or values is actually calculated as a difference of the two value or locations.   This difference or length between two values is sometimes called their "range" or "extent".

```
        ---|----------------------|---   : length = max. - min. = 7 - 3 = 4
    . . .   3                    7  . . .   We also see that length is actually a "difference"
                                            which is a measure of separation.  This difference
                                            could also be called the "change" from one value
                                            (here the min. value) to another (here, the max. value).
```

Below, a smaller line segment (line1) with a length of 0.5 is superimposed (placed on or over) on a larger line segment (line2) with a length of 2.0 that is within (bounded by) the range from 3 to 5.

```
        ---|--------|-------------------------|---   : numbered line
    . . .   3      3.5                         5   . . .
              |— 0.5 —|
              |———————— 2 ————————|
```

Using the above drawing, 3.5 is what percent (%) of this range from 3 to 5? This is very similar to the original problem stated above.   Again, considering the numbers as line segments is a good method for such problems.

$$\% = \frac{length1}{length2} = \frac{(max.1 - min.1)}{(max.2 - min\,2)} = \frac{distance\ in\ question}{total\ distance} \quad : \text{CORRESPONDING PERCENTAGE A}$$
VALUE IS WITHIN TWO OTHER VALUES

Note that here in this specific example, that min.1 of line1, and min.2 of line2 are actually the same value of 3.

$$\% = \frac{3.5 - 3.0}{5.0 - 3.0} = \frac{0.5}{2.0}$$

$$\% = 25\% \quad : \text{It could be said that 3.5 is 25\% of and-or along the way of the distance or length from 3.5 to 5.}$$

Now solving the original problem stated:

$\dfrac{(10 - 7)}{(15 - 7)} = \dfrac{3}{8} = 0.375$ or 37.5%   : This could be stated as "10 is at the 37.5% point in the range from 7 to 15". Or, "37.5% of the range from 7 to 15, is 10".

Likewise, given a certain indicated percentage (ie. the percentage rate or simply the rate) between some range of values, the corresponding value to that percentage can be found:

Solving for max.1 that is in the above formula:

max.1 = min.1 + (%) (max.2 - min.2)   : CORRESPONDING VALUE TO A
PERCENTAGE RATE WITHIN TWO VALUES
(The minimum values are also the same.)

You can think of max. 1 as the number that is somewhere between the range of values of min. 2 and max. 2.

```
        0%                                      100%
    -------|--------------|----------------------------------|-------
. . .     min.2                                max.2     . . .
          min.1        max.1
```

Ex. Using values from the above example, what value corresponds to 25% (0.25 = 1/4 = "one-fourth") of the range from 3 to 5?

value = max.1 = 3 + (0.25)(5 - 3)
value = max.1 = 3 + (0.25) 2
value = 3 + 0.5
value = 3.5

Note also that if the min. values, or the "starting points", were at 0, then the equation simply reduces to:

max.1 = min.1 + (%)(max.2 - min.2)
max 1 = 0   + (%) (max. 2 - 0)
max 1 = (%) (max.2)

A NOTE ON COMBINING PERCENTAGES

The ingredients for some manufactured products identified as Product A and Product B are to be combined to produce Product C. If 50% of Product A is water, and 25% of Product B is water, what percent of Product C is water (w)? [FIG 11]


```
              A ==== |      w        |
              B ==== |  w    |        |
                     |                     |        C
                     0%      50%      100%
```
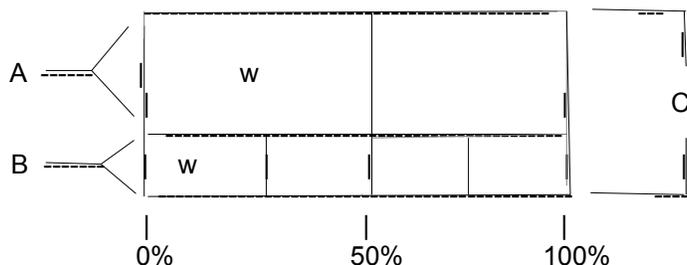
To begin with, the answer is **not** the sum of the percentages: 50% + 25% = 75%   : a wrong method and solution

In general, the answer is also **not** the average percentage: $\dfrac{50\% + 25\%}{2} = \dfrac{75\%}{2} = 37.5\%$

This is so since the percent values are in reference, or with respect to, two different things, here Product A, and Product B, rather than just one thing in question.

That is, in general, relative values (ie. percentage values such as 50% and 25% here) cannot be combined by addition or averaging to produce another relative value. The only exception is when the units (perhaps weight or volume) of both (or more) products being combined have the same actual value or measurement.

Now consider product B getting smaller and smaller in weight, but still, 25% of it is water, even if its an negligible amount. The entire amount of units of Product C, including the amount of water, will also get smaller and smaller due to Product B getting smaller and smaller. As the actual units or amount of water in Product B becomes very small and meaningless, the percent of water in Product C will approach nearer and nearer to that of Product A. Hence, Product C will then contain only the water from Product A which is 50% water during that condition, and it is obvious that the percentage values cannot be simply combined to find the total percentage value for this kind of problem: [FIG 12]
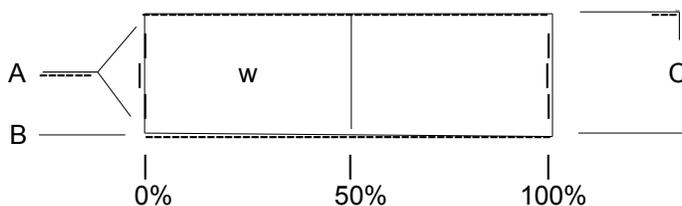
A ------< | w | | C   : If Product B was was very
B ----------   small negligible amount, it
       | | | would still consist of 25%
      0%   50%   100%   water.

The following is correct, but it is unsolvable given just the values mentioned:

      Product C = product A + product B

This is also correct, but it is unsolvable given just the values mentioned, because we need their specific amounts.

      Product C water /=/ 50% of product A + 25% of product B

To find the actual percentage of water in Product C, you must first know the actual amount (not just a relative value(s)) of water in both Product A and Product B. If Product A weighs 200grams, then 50%, or 200g x 0.50 = 100g is water. If Product B weighs 60grams, then 25% or 60g x 0.25 = 15g is water. When combined, Product C will contain:

Amount of water in Product C = Amt. of water in Product A + Amt. of water in Product B
Amount of water in Product C =   100g   +   15g
Amount of water in Product C =   115g

The total weight of Product C is:

Weight of Product C = Weight of Product A + Weight of Product B
Weight of Product C =   200g   +   60g
Weight of Product C = 260g

Technically, grams is a measurement of the mass (ie. a measurement of the amount of physical matter, atoms) of a substance and not the weight (a force due to gravity) of that substance. However, mass and weight are physically directly related, and mathematically proportional to each other. The ratios of two values with units of mass (such as grams), will be equal in numeric value to their corresponding ratio of their weights (such as units called Newtons).

If 115g of Product C is water, then 115g is: $\underline{115g}$ = 0.4423   = 44.23% of the total weight of Product C is water.
                                                                          260g

If it was initially indicated that the amount of water in Product B is equivalent to 30% of the water in Product A, an equation for the relative amount of water in Product C in relation to only Product A, can be derived without knowing any specific values (amount of water, or weight) for either Product A or Product B. The amount of water in Product B is now indicated in reference to the amount of water in Product A, hence the amount of water in Product B is a fraction (here 30% = 0.30) of the amount of water in Product A and which is a fraction (0.50 = 50%) of Product A. Hence, the amount of water in Product B is a fraction of another fraction. The basic solution here is to substitute the mathematical equivalent of the amount of water in Product B with a value mathematically expressed in relation or respect to, or in reference of (ie. "in terms of") Product A. This will help create a simpler and-or a solvable equation:

water in Product C = water in Product A   +   water in Product B
water in Product C = (50% of Product A) +   (30% of the water in Product A)
water in Product C = 50% Product A   +   30% of (50% of Product A)
water in Product C = 0.50 Product A   +   (0.30 x 0.50) of Product A
water in Product C = 0.50 Product A   +   0.15 Product A   combining like values with the same units or "terms":
water in Product C = (0.50 + 0.15) Product A
water in Product C = 65% Product A   : note, Product C still contains the amount of water in Product B,
                                                          even though Product B is not included in this last expression. Also,
                                                          the original description, only 50% of Product A is water, but in this
                                                          expression here, the amount of water in Product C is expressed
                                                          as being 65% of the water in Product A since this value includes,
                                                          or takes account of, the percentage of water in Product B.

Ex. A mixture of 8 ounces weight (rather than volume or space measurement) contains 10% sugar by weight. If 1 more ounce of sugar is added into the mixture, what total percent of the mixture is now sugar?   Let's first find out the number of ounces of sugar that were originally in the mixture, and then add 1 more ounce to that so as to find the total weight of sugar in the new or modified mixture:

   total amount of sugar in the original mixture = (8 ounces)(10%) = (0.10) 8oz = 0.8oz

   total amount of sugar in the new mixture = original amount + added in amount = 0.8oz + 1oz = 1.8oz

   percent of sugar in the new mixture = $\dfrac{\text{total amount of sugar}}{\text{total amount of mixture}}$ = $\dfrac{1.8oz}{8oz + 1\ oz}$ = $\dfrac{1.8oz}{9oz}$ = 0.2 = 20%

PERFORMING OR EXPRESSING DIVISION WITH MULTIPLICATION

The concept and process can be useful for re-expressing or simplifying expressions, and is as follows:

$$\frac{\text{dividend}}{\text{divisor}} \quad \text{or} = \quad \frac{\text{numerator}}{\text{denominator}} \qquad \text{equals these expressions:}$$

$$\frac{(\text{dividend})}{(1)}\left(\frac{1}{\text{divisor}}\right) = (\text{dividend})(\text{reciprocal of the divisor})$$

Hence an alternative to division is to multiply by the reciprocal value of the divisor.

The process of finding the reciprocal is somewhat defeating to the entire process since it in itself normally requires division. However, the concept is important since it shows how to express or convert a division problem to one of a multiplication problem. Most people also find it easier to see and-or to perform multiplication than division, hence it is helpful to express an equation that uses some multiplication rather than division.

Ex. $\quad \dfrac{1234}{5} \quad = \quad \dfrac{1235\,(1)}{1\,(5)} \quad = \quad 1234\,(0.2) \qquad : = \ 246.8$

If you make a table of the reciprocals of the prime numbers, you can then solve for (non-prime) composite number reciprocals (which may or may not contain a fractional portion) by factoring the number into prime factors whose reciprocals are already known. Solving a reciprocal is therefore converted to a problem of multiplication. For example:

number   reciprocal of number = 1/number

number	reciprocal of number = 1/number
1	1
2	0.5
3	0.333333333
5	0.2
7	0.142857142 : clearly, the larger the number, the smaller it's corresponding reciprocal is

Ex.   Find $\dfrac{5}{6}$

$$\frac{5}{6} = \frac{5}{1}\left(\frac{1}{6}\right) = \frac{5\,(1)(1)}{(2)(3)} = \frac{5}{1}\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = 5\,(0.5)\,(0.333333333) = 0.833333333$$

Note above, that the prime factors of 6 are 2 and 3. If you multiply the reciprocals of 2 and 3, you will have the reciprocal of 6:

$$\frac{1}{6} = \frac{1}{(2)(3)} = \frac{(1)(1)}{(2)(3)} = \frac{1}{2} \times \frac{1}{3} = 0.5 \times 0.333333.. = 0.1666666667$$

Ex.   Find $\dfrac{1}{0.6}$

First using an equivalent fraction method:

$$\frac{1}{0.6} = \frac{1\,(10)}{0.6(10)} = \frac{(1)10}{(6)} = \frac{(1)(1)}{(3)(2)}\,10 = (0.333333333)\,(0.5)\,(10) = 1.666666666$$

Now using the concepts of "scientific" and exponential notation:

$$\frac{1}{0.6} = \frac{1}{(6)(10^{-1})} = \left(\frac{1}{6}\right)(10^{+1}) = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)(10) = (0.333333333)(0.5)(10) = 1.666666666$$

AVERAGE

An average is a single numerical value that can be used to represent an entire set of group numerical values. The average will always be a value between the minimum and maximum values of the specific set of data values given or used, and may, or may not, be equal to any member of that set of values. The mathematical definition of average is (initially, or loosely) based upon the "half-way" or "center-point" point between two values of which you can imagine as the "end-points" on a line-segment (a small part or section of an infinite line) which bound that half-way or "middle" point:

```
<------------------|-----------------------------------|----------------------------------|------------->
          min. value                          mid. value                          max. value
                                             "half-way" value
```
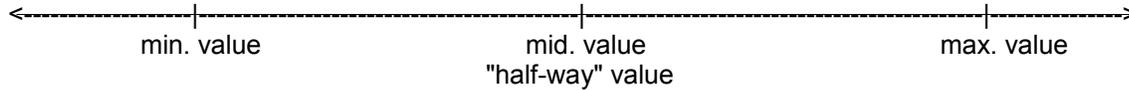
$$\text{"half-way" value} = \frac{\text{sum of two values}}{2}$$

:Halfway (or "mid-point" between two values) Value Formula
The two values in question can be called the extreme values or "end points".

The value or distance from both of the two end points (ie. the minimum and maximum extreme values) to the "half-way" point is exactly the same. The total distance or value between those defining, bounding, or end points is equal to the length or difference between those two points or values:

difference = maximum value - minimum value

: a (numeric) difference is the amount of (numeric) separation, and is also known as the change in one value with respect to, or in reference to, another value.

The halfway-value formula is only slightly intuitive and may need some further explanation. First, the formula is based on the half (or half-value) of any given number, and that half of a value is equal to the given value divided by 2:

$$\text{half-value of a number} = \frac{\text{number}}{2}$$

: here number is actually equal to the max. value, and 0 is equal to the min. value in relation to the "half-way" value formula. This could be expressed as:

half-value of a number or "halfway" from 0 to number $= \frac{0 + \text{number}}{2} = \frac{0}{2} + \frac{\text{number}}{2} = 0 + \frac{\text{number}}{2} = \frac{\text{number}}{2}$

The value of half of a number, is also (equivalent to) the "halfway-value" between 0 and that given number. This "halfway-value" is half of the "distance" or amount from 0 to number. This value is exactly between those two given values, and is therefore said as being "equidistant" (equal, or equivalent, in distance) from them. Here, 0 can be considered as the starting or minimum value of this distance (or range of values), and number can be considered as the ending or maximum value of the distance (or range of values).

Note, for ex., if given 6 things, it is incorrect to consider the average as being a value between the third and fourth thing:

Between 3 and 4 , is 3.5 , = the value between 3 and 4, and is not the average of 6.
3.5 is actually the average value of 3 and 4 = (3 + 4)/7 = 7/2 = 3.5

(6 things)
```
* * * * * *        * * *   * * *
                  1 2 3   4 5 6
```
: showing each thing or item enumerated (assigned with an increasing number, with the last number being the count or total of all the items)

The average (point or location, the "halfway-point) is a numerical value (formally defined as) between two values or

numbers is considered as being between a min. value, and a max. value, and for here, it is: 0 and 6. That is, 0 is actually the (unspecified) min., or "start value" (to consider, or include in the calculation, so as to consider the distance or values between 0 and 1), and it is actually the reference value of what 6 is in reference to, and therefore, 0 must also be considered for the average. (6 + 0) / 2 = (6/2) = 3 (and not 3.5) . 3 is technically the equidistant value from 0 and the max value 6. Consider that length or distance values start at 0, and not 1, that is, there is still some distance to consider between 0 and 1 which cannot be ignored. This will also account for the average of fractional values (ie. less than 1). A single value, such as 6, technically, does not have an average among several values to consider. Still, you can divided any number or value into two equal valued parts by dividing it by 2. Consider that If you had some data values all being near (about, close to), or even the same value of 8, and therefore, the average, common or typical value would be said as being about 8, and not (8+0)/2 = 8/2=4. In short, the average value of a set of data or values, may not even be the "half-way" or "mid value" as when 0 is considered, and therefore, it is generally incorrect to consider an average or representative value as the same thing as the "half-way" value.

Sometimes the word "range" or "range of values" is used as a generalized indication of the distance or separation (difference) between the min. and max. values. This may be spoken as something like, for example: "the range of the data values or numbers start at 300, and end at 400", or "the (range of) values go from 300 to 400". The specific size of this specific range (of data values) is the difference between the maximum and minimum values, and here, it is:
(400 - 300) = 100. If you had several various data values, of say having values that are between or within 500 and 600 in value, the (entire) range of values is still 100 since (600-500 = 100). It could also be said that the rage of values extends from, and-or, are within the values of, or from, 500 to 600.

Don't confuse the actual "half-way" point value with half of the distance or physical length (ie., a difference) between the two points. For example: Given values of 103 and 105, the half-way point or value is 104, and the difference or length between these defining, bounding, or extreme points is only 105 - 103 = 2, and half of this distance is just 1. Clearly, the "half-way" point or (data) value (of all the data values), and the half-length (ie., half of the difference or (range)) distance between the max. and min. extreme values, or "end points", is not the same concept and-or value.

Given 0 and 10, what is the "half-way" or "middle" value?

$$\frac{10 + 0}{2} = \frac{10}{2} = 5.0$$

In the example.

difference = max. - min = 10 - 0 = 10. This also verifies the "halfway-point", or "point-halfway", value formula above. 5 is both the halfway value (between 0 and 10) and is also equal to the difference (here, 5) that this halfway values is from both of the extreme or end values, here 0 and 10. Clearly, as expressed, 5 equal to just half of the difference (here indicated as: max - min. = 10 - 0 = 10) in the two given values, as seen expressed as: 10/2 = 5. The distance from 5 to the two "end points" is the same, so therefore, twice (2) of the "halfway" value" (here 5) that is between the two given extreme or end values is equal to the sum of those two values (here, 0 and 10 = 10).

(halfway-value) x 2 = sum of the two values.   Therefore, mathematically:

halfway-value = sum of the two values / 2   :as expressed in the initial formula.

At this point in the discussion, we can consider this illustration given these two sets containing just two numbers:
(10 and 20), and (0 and 10): [FIG 13]
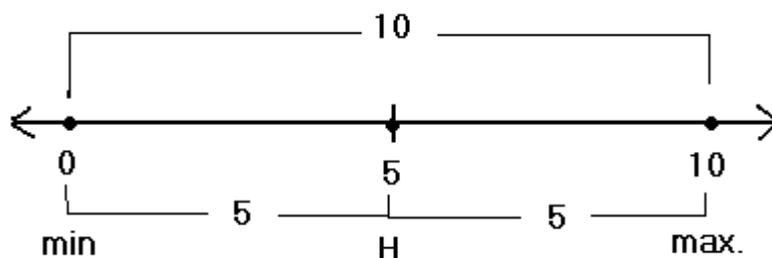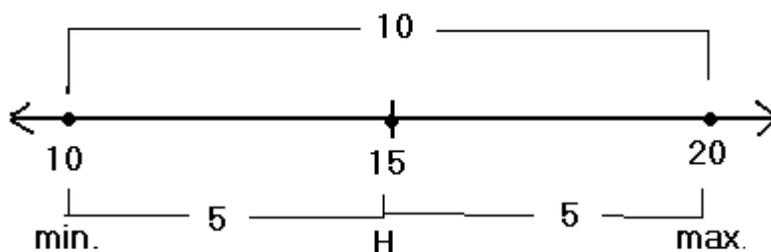
In both cases, the difference (D) from, or between, the minimum value to the maximum value is (max. - min) = 10, and half of this difference is 5. H is the halfway point and value between the minimum and maximum values. Also:

H - min. = max - H

$:\quad = D/2\ =\ $ (max. - min.)/2
Adding min. to both sides of this equation:

H - min + min = max - H + min
H - 0 = max. + min - H
H + H = max. + min = - H + H
2H = max. + min.

since a value added (here, min), and then subtracted is equal to adding 0:
: adding H to both sides:
 combining 1H + 1H, we get (1 + 1)H = 2H
 expressing the division of both sides by 2, and canceling common factors:

H = _max + min._
           2

: formula for the halfway point or value between two values

   Also: H = min. + (D/2)   =   max. - (D/2)   : since (H-min) = (D/2)   and   (max-H) = (D/2)

However, even with all that has been mentioned, an average value is only based on the "half-way" value concept, and it is incorrect to use this as a general way or formula to find the average of any set of values, especially where there is more than two values in the set to consider. An average is only equal to the half-way value if there are only two values involved. When more than two values are involved, another more generalized formula for an average must be used, and this will be derived (developed) in the discussion that follows.

Given a set that contains (data) values which are identical, it is common sense that the representative or average value of those values is equal to any one member of that set.

Ex. A set contains 100 and 100, what is the average?

average = 100   : since a value of 100 is said to, or can represent this set (all the values) very well.
                You could say something also like: "The typical or representative value is 100".

Considering the last example, if one or more of the values is only slightly different, it is fair to say that the average will still be about the same value of 100.   For example, if this new set contains values of 100 and 101, what is the average? You can make an estimate of the average here at being slightly higher than 100. The actual average is 100.5

Ex. A set contains 5, 5, and 5, what is the average?

average = 5   :5 is said to represent this set very well.

        Even though the average of the above sets is quite obvious, what is a numerical formula for this average given any set which contains all the same values? Notice that the sum of the values ("elements", "members", or "data") of the set (list or group of things related) divided by the number of individual values or members (of that set) is equal to the average value of that set of values:

$$\frac{100 + 100}{2} = \frac{200}{2} = 100$$

$$\frac{5 + 5 + 5}{3} = \frac{15}{3} = 5 \qquad \text{now expressing this into a general formula:}$$

$$\text{average} = \frac{\text{sum of values}}{\text{number of values}} \qquad \text{: AVERAGE FORMULA (when all the data values are the same)}$$

Given values that are not identical, say 2 and 4, what is their average? Does the formula above hold true for different values? It should be obvious that the average should not be either 2 or 4, since the values are not the same and are different to each other. As you may have guessed, after viewing the previous discussions, that the average is somewhere or a value between those two values.

Here is a representation of all the values from 0 up to any value that we can call or indicate as B.   Letter (or mathematical variable) A represents the half-way point and average (A) of these two values:

0-------------------A--------------------B

Here, the value of $A = \frac{B+0}{2} = \frac{B}{2}$ since there is only two values involved and one value is 0.

If we assign the variable of C to represent the value of 0 above, we get:

C-------------------A-------------------B

$A = \frac{B + C}{2}$: formula for average of any two values

If we let both C and B increase by 1, you would expect their half-way point and-or average to increase by 1 also, and so that it is still "centered" or "midway" between them.

The average of 0 and 2 is 1. If the members of this set are increase by 1, the average (ie, the "center" of the two values) of this set would naturally be expected to also increase by 1. If the values increased by (1), such as values (0) and (2) increased by (1): (0+1) and (2+1), or= the new values of (1) and (3), their average is now equal in value to the average of (0) and (2), and also increased by (1). The new average value is: (old average + 1) = (1 + 1) = 2.

Let's verify this concept a bit more mathematically in equation forms:

Let's see if this equation is correct: $(A + 1) = \dfrac{(B+1) + (C+1)}{2}$ simplifying the right side:

$A + 1 = = \dfrac{B + C + 2}{2}$

Since we know that $A = \dfrac{B + C}{2}$ we can substitute this value for A into the previous equation:

$\dfrac{B + C}{2} + \dfrac{1}{1} = \dfrac{B + C + 2}{2}$ creating like fractions, as needed for their addition, on the left side:
(: note 1 = 1/1 = 2/2, and then combining fractions of the left side:)

$\dfrac{B + C}{2} + \dfrac{2}{2} = \dfrac{B + C + 2}{2}$ combining fractions on the left side:

$\dfrac{B + C + 2}{2} = \dfrac{B + C + 2}{2}$:checks

Ex. The average of 2 and 4 is: $\dfrac{2 + 4}{2} = \dfrac{6}{2} = 3$, now consider:

If one member increased by 1, and the other member decreased by 1, you would expect the average to remain the same in order to still remain "centered" between the two. This is verified below, where C is decreased by 1, and B is increased by 1:

$A = \dfrac{(C - 1) + (B + 1)}{2} = \dfrac{C - 1 + B + 1}{2}$ simplifying using the commutative law of addition:
Also note that: $-1 + 1 = +1 - 1 = 0$

$A = \dfrac{B + C}{2}$: checks , the average remained the same

Ex. Using the last examples values: $\dfrac{2 + 4}{2}$

$\dfrac{(2 - 1) + (4 + 1)}{2} = \dfrac{1 + 5}{2} = \dfrac{6}{2} = 3$

Now, let's give some verification to general average formula when the average of more than two different values is to be found:

Average 100 and 100. According the above discussions:

$\dfrac{100 + 100}{2} = \dfrac{200}{2} = 100$

If we average 99 and 101, the average should be, and somewhat obviously a value between those two values: = 100

$\dfrac{99 + 101}{2} = \dfrac{200}{2} = 100$

Likewise, now average 98 and 102, and we should expect the result to be the same:

$$\frac{98 + 102}{2} = \frac{200}{2} = 100$$

Now with three values: 100, 100, and 100:

$$\frac{100 + 100 + 100}{3} = \frac{300}{3} = 100$$ which can be mathematically written or expressed as:

$$\frac{100}{3} + \frac{100}{3} + \frac{100}{3} = 33.333... + 33.333... + 33.333... = 100$$ combining (adding) the first two fractions we get:

$$\frac{200}{3} + \frac{100}{3} = \frac{300}{3} = 100$$

Now with other values: 99, 100, and 100:

$$\frac{99 + 100 + 101}{3}$$ which can be mathematically written as:

$$\frac{99}{3} + \frac{101}{3} + \frac{100}{3}$$ by combining (adding) the first and second fractions:

$$\frac{200}{3} + \frac{100}{3} = \frac{300}{3} = 100$$: and this is the same equation expressed above

Now, perhaps when it's less obvious:

Average 1, 2 and 297

$$\frac{1 + 2 + 297}{3} = \frac{300}{3} = 100$$ which can be written as:

$$\frac{1}{3} + \frac{2}{3} + \frac{297}{3}$$ which can be written or expressed as:

$$\frac{1}{3} + \frac{2}{3} + \frac{(200 + 97)}{3} = \frac{1}{3} + \frac{2}{3} + \frac{200}{3} + \frac{97}{3}$$ combining some terms:

$$\frac{200}{3} + \frac{100}{3}$$: same fractions as in the previous examples above, and this can even be expressed as:

$$\frac{100}{3} + \frac{100}{3} + \frac{100}{3} = \frac{300}{3} = 100$$ Giving verification to the general average formula, for when the values are not the same.

When finding the average, a data value of extreme value, either large or small, can effect the value of the average in such a manner that this average, if used, does not fairly represent most of the data values. For example:

The average of: 1, 3, 3, 5, 6, 7, 8   is = (1 + 3 + 3 + 5 + 6 + 7 + 8) / 7   = 33/7 = 4.71

Now here is the same set of data, but instead of a data item with a value of 8, there is a value of 300:

The average of: 1, 3, 3, 5, 6, 7, 300   is 46.43   : this new average is not even close in value
                                                                                      to most of the values in this set.

Often, "extreme" values, when not to be expected and/or considered, are not to be used when finding the average. These "abnormal" values could be an indication of some type of measurement and-or communication error. Where the average value is concerned, they would be discarded (removed, ignored) and not considered (here added) in the sum of the set members, and in the number of the total members in the set.

The word average is sometimes referred to as the arithmetic mean ("in common with", or "about") or simply "mean". An average value of a set of data values, say a data set represented or identified as x, is frequently shown as an x with a bar-line over it:

$\bar{x}$: expresses and represents the average value a the set of values, here identified as set x

A common use for average is in finding a single value that can represent several values such as test scores or measurements. Average has widespread use in the branch of mathematics called **statistics**. In brief, a statistical value is a single and a particular representative value of many data values of a group ("ie., set) of data. A statistic provides a quick numeric grasp or understanding of a situation and can be used for estimates, expectations and-or predictions. Average will be discussed again further ahead in this book during the topics of average deviation and standard deviation, and due to that the topics of square roots and signed numbers need to be discussed first.

COMMON MONEY PROBLEMS

Typical monetary (money) problems involve finding the total money value when given some whole dollars and coins (which essentially are fractions of a dollar or dollar unit), and how much change there is after a purchase. Here, change is the difference between the total amount given to pay for a product and the actual price of that product. When a product is purchased, often some extra money, such as paying with an even amount of dollars, is used to pay for the item(s), hence some value (ie., "change" or the difference) needs to be given back to you. Change in the monetary sense is often spoken as something like: "What you will get, or be given back, after a purchase". Some of the concepts of money and fractions, particularly with coins, has been previously discussed in this book during the topic of: adding money as fractions.

amount given - price = change   : change essentially means the difference in, or of, values, hence it
                                              is mathematically equal to the amount of separation. When a value
                                              has increased or decreased, it is said to have changed since it is
                                              now a different value. The amount that is different is called the change
                                              value. The difference between any two values is also then called the
                                              change value, even if none of the values actually changed in value.

A good method to understanding money (particularly coins) problems is to understand the penny. For instance, someone might ask, "How many pennies are in a dime"? In the physical sense, the answer is none, since a penny is a coin, not a monetary value, and a dime is a separate coin unto itself. There is however the equivalent of 10 cents, not pennies, in a dime since a dime's 10 cent monetary value is equivalent to, or can represent, the monetary value of 10 pennies which is also 10 cents. Even though this question is probably "understood" by most, it may of been better to say, "How many pennies are equivalent (in monetary value) to a dime"?

The basic unit in our monetary system is the dollar. A penny's monetary value is defined as one-hundredth (= 1/100) of 1 dollar, and this value or "derived" (and fractional) unit is commonly called a cent. A cent is often symbolized as letter c, or c with a vertical line through it. The numerical representation of 1 cent is therefore:

1 cent = $\dfrac{1\ dollar}{100}$ = 0.01 dollar   : a cent is one-hundredth of a dollar ($1). Again, dollar is a unit of
                                                              of measurement here, and not a specific numeric value or variable.
                                                              Cent is the name of a (smaller, fractional) unit of measurement
                                                              derived from the (base of the monetary system) units of dollars.

A dollars' monetary value is therefore equivalent to 100 cents. Since (0.01)(100)=1.00, or (cent)(number of cents) = (total dollars), given some total number of cents, simply move the decimal point 2 places leftward (the same as dividing by 100) to find the decimal equivalent in dollars.

Ex. 365cents is equivalent to: $\dfrac{365}{100}$ = 3.65dollars.

Likewise, given a total dollar value, simply move the decimal point two places rightward to find the equivalent number of cents.

Ex. $1.79 is equivalent to: 1.79 x 100cents = 179 cents   : $ means dollars, and an indicated value is therefore
                                                                        in reference to, and has units called **dollars**
                                                                        $ is called the "dollar symbol".

The monetary value of a nickel coin is equivalent to that of 5 pennies:

1 cent + 1 cent + 1 cent + 1 cent + 1cent   =   1 cent x 5   =   0.01 x 5   =   0.05   : "five cents"

Here are the monetary values of the common coins:

COIN	CENTS (c)	DECIMAL EQUIVALENT (in dollars, $)	FRACTIONAL EQUIVALENT (of a dollar, $)
PENNY	1	0.01	1/100 = "one-hundredth of a dollar"
NICKEL	5	0.05	5/100 = 1/20 = "one-twentieth of a dollar"
DIME	10	0.10	10/100 = 1/10 = "one-tenth of a dollar"
QUARTER	25	0.25	25/100 = 1/4 = "one-quarter of a dollar"

Given several coins, to find the total monetary value, simply sum up the corresponding cent values for all the coins. If some coins are the same, multiplication can be used instead of repeated addition.

Ex. How much money is 1 quarter, 3 dimes, 4 nickels and 2 pennies?

COINS	MONETARY VALUE ($)	CENTS (c)	
1 quarter	0.25	25	: 0.25 x 1 = 0.25 = 25 cents x 1 = 25 cents
3 dimes	0.30	30	: 0.10 x 3 = 0.30 = 10 cents x 3 = 30 cents = 0.30
4 nickels	0.20	20	: 0.05 x 4 = 0.20 = 5 cents x 4 = 20 cents = 0.20
2 pennies	0.02	2	: 0.01 x 2 = 0.02 = 1 cents x 2 = 2 cents = 0.02

+ ---------   + ----------
$ 0.77   =   77c

If you were to give 77c to someone, you could use the coins mentioned in the last example, however, to give the least number of coins, first find the highest number (multiple) of the larger (in monetary value) coins first and of which the total monetary value of those coins does not exceed the required change. As a simple example, rather than give 25 pennies for 25c, you could simply give 1 quarter for 25c. For the 77c example above, 25 cents (from 1 quarter) can go into (divide into) or be subtracted from 77 cents a total of 3 times with a 2 cent remainder:

```
         3
 25  | 77
      75     : 3 quarters is (equivalent to) 75 cents  =  0.25 x 3  =  $0.75  =  75c
       2
```

Since both dimes and nickels have monetary values larger than 2 cents, the only coin left to consider is the penny, and it takes two pennies to account for two cents monetary value. For this example, we find that the minimum number of coins to give is only 5 coins consisting of 3 quarters, and 2 pennies, instead of the 10 total coins listed above.

Ex. If you gave a 5 dollar bill to purchase an item that cost only $1.99, how much change should you receive?

amount given - price = change
    $5.00   - $1.99 = $3.01

At the time of writing this book, pennies, and other coins, are still in widespread use, but they may be obsolete in the future, and hence this topic could then be considered a historical review of the times when they were available. At the time of this writing, a basic loaf of sliced sandwich bread, about 1 pound in weight, can be purchased at a thrift (low item cost, bargain) grocery store for about the price of: $1.25

Some Common Currency Exchange Rates

At the time of writing this book edit and edition at May 7, 2020, know that these values can vary by a few percent in just 1 day. Even if these values change, you can still use it for practical and historic comparison purposes.

1 English or Great Britain Pound (GBP) = $1.24 USD   : **USD = United States Dollars , 1 cent = 1 penny = $0.01 USD**
$1 USD = 0.81 GBP or "sterling" = 106.26 Japanese Yen = 0.93 Euro (EU Union) = 7.10 Yuan (Chinese)
$1 USD = C$1.41CAD (Canadian Dollar) =   24.33 Peso (Mexico) =   74.43 Ruble (Russian) = 0.98 Franc (Swiss)
$1 USD = 50.56 Philippine Peso = 5.72 Brazilian Real = 15.73 Egyptian Pound   = 7.75 Hong Kong Dollar
$1 USD = 3.52 Israel New Shekel = 1.66 New Zealand Dollar   = 4.22 Zloty (Poland) = 18.74 Rand (South Africa)
$1 USD = 1225.54 Won (South Korea) = 6.70 Trinidad And Tobago Dollar = 23445.50 Dong (Vietnam)
$1 USD = 607.69 CFA franc (West Africa) = 33.09 Birr (Ethiopia) = 76.05 Rupee (India) = 42105.00 Rial (Iran)
$1 USD = 1.55 AUD (Australian Dollar) = 4.47 Leu (Romania) = 7.21 Lira (Turkey) = 55.30 Sudanese pound
$1 USD = 582.0 Somali (Somalia) Shilling = 931.79 Rwandan (Rwanda) franc = 159.01 Pakistani Rupee

The unit of money in the United States of America is the dollar. 1 dollar = $1 , A fraction of a dollar is usually expressed in the number of cents (ie., from the word percent) of a dollar. 1 cent is 1percent of a dollar. 1 dollar = $1 = 100 cents = 100c. A price of an item expressed as: $dollars.cents = dollars + cents of a dollar = dollar + cents
Ex. $1.75 = 1 dollar and 75 cents = $1 + 75c

Common United States Coins: quarter = a quarter of 1 dollar = $1/4 = $0.25 = 100c / 4 = 25c
                             dime = a tenth of 1 dollar = $1/10 = $0.10 = 100c / 10 = 10c
                             nickel = one-twentieth of 1 dollar = $1 / 20 = $0.05 = 100c / 20 = 5c
                             penny = one-hundredth of 1 dollar = $1 / 100 = $0.01 = 100c / 100 = 1c

The unit of money in the England is the pound (L). A pound is sometimes called a "quid".
1L = 20 shillings = 20s   , A shilling is sometimes called a "bob".
1 shilling = 12 pennies = 12d
1L = 20 shillings = (20s)(12d / 1s) = 240d
Ex. L5.3s.2d   = L5 , 3s , 2d = 5 pounds + 3 shillings + 2 pennies

(USD) Metal Prices Per Ounce (On May 7, 2020, silver and gold also show the price at February 20, 2023) ,
Note: 1 oz of weight = 1 dry ounce = 1 oz = 28.35 g   , 1g = 0.035274 oz
(General for plain bullion (bulk) and-or scrap (waste, recycled) metals, and not for specific manufactured products):

Gold = $1684 USD / 1 oz   and $1845 / 1 oz on February 20, 2023, In terms of bartering and receiving change when using gold or silver, it is perhaps best to get small weights of it, such as 1/10 oz, 1/20 oz , and-or 1 to 5 gram sizes.

Silver = $14.81 USD / 1 oz   and $21.84 / 1 oz on February 20, 2023 and this is an increase of,
        (new / old) = $21.84 / $14.81 = 1.475 or about 0.47 = 47% more than 1.0 = 100%   , or calculated as:
        change or difference = (new - old) = (21.84 - 14.81) = 7.03 and
        (change from old / old) = (change from reference value / reference value) = 7.03 / 14.81 = 0.475

        A 90% = 0.90 silver USD quarter weighs 0.22 oz, and therefore has (0.22 oz)(0.90) = 0.198 oz of silver , and
        has (price per ounce)(number of ounces) = ($21.84 / 1 oz)(0.198 oz) = $4.32 worth of silver as of
        Feb., 20, 2023. 5 of these old and worn-out "junk silver" quarters will weigh about: (5 coins) (0.198 oz / coin) =
        = 0.99 oz =~ 1 oz.

Platinum = $752 USD / 1 oz   ,   Palladium = $ 1712 USD / 1oz
Copper = $ 0.14 / 1 oz , Brass = $0.13 / 1 oz ,   Bronze = $0.07 / 1 oz , Aluminum = $0.045 / 1oz , Lead = $0.05/oz
Nickel = $0.38 / 1 oz   ,   Tin = $0.47 / 1 oz   ,   Cobalt = $0.90 / 1 oz   ,   Zinc = $0.06 / 1 oz
Steel = $0.028 av / 1 oz   , Stainless Steel = steel with about 11% chromium = $0.038 / 1 oz

Metals such as gold, silver, platinum, palladium are relatively rare and are known as "precious metals" and which mostly means means that they are valuable in some way(s). These metals are often needed for some modern technology such as electrical contacts so as to reduce corrosion and have good electrical power and signal conduction. Gold has been used for plating metals(s) to make them look impressive and to prevent rusting and-or corrosion.

WERE THINGS REALLY CHEAPER "BACK THEN"?

Consider this example so as to find out if things were really cheaper (less expensive) in the "old days" (previously, in the past, years ago, decades ago). This method of analysis will compare, in a relative manner (by using percentages; %), the price of the same product to the minimum labor wage (income) per hour. This method of analysis can also be used for the current situation of one or more locations (cities, states, countries, etc), and which depends upon the current local (supply, demand, etc) economy there.

Price of a product as a % of Minimum Wage = (Price of product / Minimum Wage) x 100

Year	Price	Minimum Wage	Price of a product, as a % of Minimum Wage	
1930	$0.15	$0.75 per hour	20.00%	: 15 cents or hundredths of a dollar or 100 cents = 15/100
2017	$1.25	$6.50 per hour	19.23%	: 125 cents or pennies

This analysis shows that the price of a certain product in the year 1930 was about the same percentage, and in fact, it was slightly higher "back then", and hence, it was slightly cheaper in 2017.   15c is certainly cheaper than $1.50 or 150 cents, but the percentage and-or actual cost of that item for the consumer is about the same as it was "back then". Often, modern production methods and the greater availability (distribution, transportation, selling price, fair competition) of some resources have, or can, effectively reduce the average selling price of an item(s) for consumers (people purchasing the product).   This analysis does not mean that all items were effectively cheaper "back then", and the analysis must be considered for each specific item in question.

Associated with the above concept is the topic called **value shopping.** For example given two similar quality products for sale that may have a different quantity and-or different price, how can we value shop so as to find the best value in terms of having the lowest cost? To do this we can find the cost per ounce or pound of each item and then compare them:

__total cost of item 1_ , and compare this to: _ total cost of item 2__   : both fractions will find the cost per
  total ounces of item 1   total ounces of item 2   ounce of the associated item, when
                                                     the division, "reduction" (to lowest
                                                      terms) or cancellation takes place.

POWERS

Whereas on a fundamental level, multiplication is the sum of the repeated addition of the same number, the **power** of a number is the product of the repeated multiplication of the same number. The number (ie. the factor) used for the repeated multiplication is formally known as the **base** of the power or resulting value of it. You could say that the base of a power is what the power value to be found is based upon, sourced, rooted upon, or derived from. A power of a number is the result of the repeated multiplication of the base value.

The "shorthand" notation used to represent this repeated multiplication is called exponential notation. You can basically view and think of it as the "power operation". This notation or expression of a power uses the base (effectively the multiplicand) value, and the number of times (ie. the multiplier) the repeated multiplication of it by itself is to occur is indicated by a value formally called the **exponent** (or the exponent of the base, and-or indicated (expressed) power). For example, 10 x 10, or = 10^1 x 10^1, can be represented or expressed using an exponential or power form as 10^2. 10 is the base, and 2 is the exponent of the base. 10 x 10 = 10^2 = 100. Sometimes an exponent is called the power indicated or the indicated power of a number. In a more technical definition, the exponent indicates the number of times that the base is to be multiplied to an initial value of 1. Every value has a factor of 1, and multiplying any value by 1 will not affect it.

Also note this similarity to addition:

Since multiplication = repeated addition, and powers = repeated multiplication (a multiple, of a multiplication), in a logical kind of way, powers could be considered as (multiples of) repeated addition. This concept, or the link to addition, is generally not considered much in applied or everyday mathematics, but it is a deeper understanding if you need it.

Here is the formal or general expression and equation for a power value:

$$\text{power} = \text{base}^{\text{exponent}}$$: GENERAL POWER EXPRESSION OR "FORMULA"

or:

power = base^exponent   : GENERAL POWER FORMULA
This book mostly uses the carat symbol = ^
to indicate an exponent, rather than use
superscript values (numbers or letters).

If you were to call the power value as "number", and use N as a shorthand or symbolic identifier for it, and likewise, use (b) for base, and (e) for exponent, we have this:

N = b^e   : GENERAL POWER FORMULA
An common alternate notation is to use (x) for (e):
N = b^x

10 = 10^1   : "plain 10" without any exponent indicated is actually "ten to the first power"
10 x 10 = 10^1 x 10^1 = 10^(1+1) = 10^2   : "ten to the second power" = "the second power of 10"
10 x 10 x 10 = 10^1 x 10^1 x 10^1 = 10^(1+1+1) = 10^3   : "ten to the third power" = "the third power of 10"

Due to the associative law, this can be expressed or checked as:

        10 x 10 x 10   : an "expanded" form of the expressed power of 10^3
    (10^1 x 10^1)(10^1)
            (10^2)(10^1)   = 100 x 10
                10^(2+1) = 10^3 = 1000

Clearly, when the base value (or variable as used in algebra) of the powers are the same, and are being multiplied, you can simply add their exponents and keep that same base for the simplified expression:

10 x 10 x 10 x 10 =   10^(1+1+1+1)   =   10^4
10 x 10 x 10 x 10 =   10^(1+1+1) (10^1)   =   10^3 10^1 = 10^(3+1) = 10^4

Ex.   What is the third, and fourth power of 2 ?

    2^3 = 2 x 2 x 2 = 8
    2^4 = 2 x 2 x 2 x 2 = 16

Note that the last answer could also be found by multiplying the preceding power of 2 that was 2^3, by 2 = 2^1:

    2^4 = 2 x 2 x 2 x 2 = (2 x 2 x 2) x 2 = 2^3 x 2^1 = 8 x 2 = 16

Note, frequently the words "exponents" and "powers" are exchanged for each other. For example, "ten to the second power", or "the second power of 10", really means a base of 10 with an exponent of 2. 2 is not the power, but the exponent or indicated power. (10^2) or 100 is the actual power (value, result) of 10 in question. Some older texts may use the word "index" (meaning the indicated power) instead of the word "exponent".

Given a number raised to a (integer or basic number) power, how many times greater is the next power? The answer is that the next power of a number is that number times greater:

Ex. 10^3 / 10^2 = 1000 / 100 = 10 = 10^1   : "one-thousand is ten times greater than one-hundred"

Ex.   How many times greater is 4^3 = 64, than 4^2 = 16:   64/16 = 4   : "sixty-four is four times greater than sixteen"

    Later in this book, all this can be generalized, or generally expressed as:
    (Number^(n+1)) / Number^n   =   Number^(n+1 -n) = Number^1 = Number = N   , and
     N^(n+1) = N^1 N^n
    Also, the difference between two successive integer powers of a number can be simply expressed as:
    (N^(n+1) - N^n   ,   Ex. 10^3 - 10^2   = 1000 - 100 = 900
    A more formal formula for the difference between successive (ie., N^n and N^(n+1)) integer powers of a number is:
    (N-1) (N^n)   , Ex: With N=base=10, and n=exponent=2, difference = (9)(10^2) = (9)(100) = 900

Powers with any base, and an exponent of 2 are often called "squares". It gets this name from the fact that area is defined and measured (with its units) in reference or with respect to a basic square shape and unit of measurement for areas, which is a two-dimensional (ie., planar (plane) or "flat, level shape", having. a length and width dimension only, with no height or thickness), four sided enclosed figure, where each of those two dimensions is equivalent in length and having the same units of measurement. This is true even if the area being measured is triangular, rectangular, circular or any other possible bounded, limited or defined region of a (infinite) plane. The area of these shapes will be measured with respect to their equivalent number (ie. a multiple or repeated sum) of square units. A "square unit" is one unit long and (by) one unit wide.   The units of any particular area shape are therefore that of the units of the two dimensions of a square (or "unit square") of which it (the area) is to be measured in reference (or compared to) to. Hence, an area value will have units of: units x units = units^2 or "square units". For example, an area might be stated simply as one value of 10 square feet, or 10 sq. ft., rather than say something like the area is 5 feet (units) long, by 2 feet (units) wide. Though the values of 5 feet and 2 feet are important measurements for something like a rug or piece of metal that you may need, the store may sell the rug or metal as priced per "square foot", rather than in the particular or specific sections or sizes that only you may need for a certain project.

Note: 10^2 is often referred to as "ten square"   : here the (base) value of 10 is squared , 10^2=100   , but
        10ft^2 is "ten, square feet"   where only the unit of feet is squared, and not the base value of 10, that is,
                it is not equal to 10^2 ft = 100 feet or 100 square-feet

Ex. If you had a rectangular area that was 2 units long, and 2 units wide, the total area is:
        square units in row 1 + square units in row 2
        2 square units + 2 square units   Since multiplication is repeated addition:
        2 (2 square units) = 4 square units   = 4 units square = 4 units x units = 4 units^2
     [FIG 14]
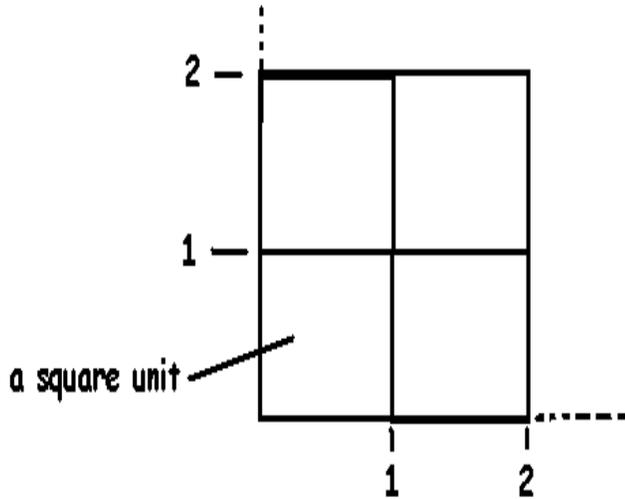
Clearly, the area is 4 square units

Ex. If you had a rectangular area that was 2 units long, and 3 units wide, the total area is:

        2 square units + 2 square units + 2 square units
        3 (2 square units) = 6 square units   = 6 units square = 6 units^2

        Note also that: 3 units x 2 units = (3) (2) (1 units) (1 units) = 6 (1 units^2) = 6 units^2

        We also see that the area of a rectangular shape is simply the product of the size (or
        measured length if you will) of each dimension (length and width) of the rectangle. This
        is so since multiplication (ie. for a product) essentially represents repeated addition.
        Here, the repeated addition is of similar squares or "square units" since each dimension
        of these squares have the same length and same units of measurement for that length.

[FIG 15]
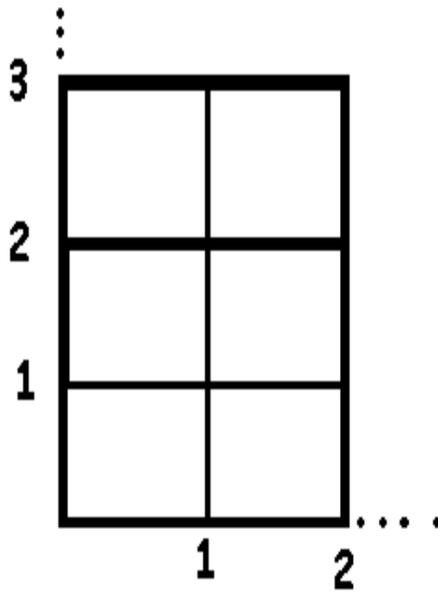
Clearly, the area is 6 sq. units or 6 units^2

Ex. 5^2 can be called "five to the second power" or "five square" and equal 25.

The third power of a number is often called the "cube" of that number. This is from the fact that a certain (reference of measurement) volume called a cube (ie. a cube of space or volume) has three dimensions, each having the same basic units of measurement The dimensions are called the length, width, and height. A cube is very similar to a square area with its length and width units, but the cube also places a height of 1 unit above that flat planar square, and creates a (3-dimensional) "space" or volume. All possible volumes are measured in reference to a "cube unit" or "cubic unit" of volume where the (sides) dimensions are all 1 unit of length in value.

Note: number^3 = a number raised to the third power is sometimes called the cube of that number.
        Ex.: 10^3 is sometimes called "ten cubed" , 10^3 = 10 x 10 x 10 = 1,000

If a value doubles (multiplied by 2), such as the side length of a square shape, it's square (ie. square area) will increase by four times. This is an example of a non-proportional or non-linear relationship of two values, where as when one value increases or changes by a factor, the other value increases or changes by some other constant factor. For the example shown below, the side lengths of a square change in a linear manner, but the corresponding area changes in an exponential manner, and the values of the area increase or change rapidly due to the squared value of its side length.

Ex. 1^2 = 1   if 1 is doubled to 2, its square is:
     2^2 = 4   an increase of a factor of (4/1) = 4.   If 2 is doubled to 4, its square is:
     4^2 = 16   an increase of a factor of (16/4) = 4,   if 4 is doubled to 8, its square is:
     8^2 = 64   an increase of a factor of (64/16) = 4

     Here is a mathematical verification of this increase, or magnification, factor, of 4:

     4^2 = (1 x 4)^2 = 4^2 = 4 x 4 = 16

     8^2 = (2 x 4)^2 = (2 x 4)(2 x 4) = 4 x 4 x 2 x 2   : this factor of 2 x 2 = 2^2 = 4 will always be present
                                           = 4^2 x 4

If a value doubles (multiplied by 2), such as the length of a cube shape, it's cube (ie. volume space) will increase by a factor of 8.

Ex. $1^3 = 1$: a cube with a side length of 1 = a cube or volume of 1 cubic unit
        $2^3 = 8$: a cube with a side length of 2 = a cube or volume of 8 cubic units = 8 units^3.
                                8 is 8 times more than 1
        $4^3 = 64$: 64 is 8 times more than 8

   Here is a mathematical verification of this increase or magnification factor of 8:

   $2^3 = (1 \times 2)^3 = 2^3 = 2 \times 2 \times 2 = 8$

   $4^3 = (2 \times 2)^3 = (2 \times 2)(2 \times 2)(2 \times 2) = (2 \times 2 \times 2)(2 \times 2 \times 2)$: this factor of $2^3 = 8$ will always be present
                                                                $= 2^3 \times 8$ when the side length doubles, consider:

   Given :   area1 = side^3 ,   if side length doubles:
                area2 = (2 side)^3 = (2^3)(side^3) = 8 side^3 = 8 area1

NOTE: It is incorrect to think that a cube is the second power (or numeric square) of an area such as that of a square. Give the side length (s) of a square area, its area is: Areas of the square = As = (s)(s) = s^2 . If this area was the side of a cube, its volume is NOT equal to Vc /=/ (As)^2 = (s^2)^2 = s^4, but is rather equal to Vc = (Vs)(s) = (s)(s)(s) = s^3. The reason for this is that given a square or square area with two dimensions, a three-dimensional cube only has one more physical or real dimension to be considered into the equation for an area.

AN OBSERVATION ON SQUARING VALUES

When a value (such as a base of a power) greater than 1 is squared or "raised" to any indicated power (ie. the exponent) greater than 1, the result is always a greater value than the value (the base) that was "raised" or increased to that power value.

Ex. $2^2 = 4$
$2^3 = 8$
$3.5^2 = 12.25$
$1^{1.01} = 1$
$1^{73} = 1$
$1.1^3 = 1.331$
$4^{1.0001} = 4.000554556$

Now note that when a value less than one is squared or "raised" to any other indicated power greater than 1, the result is smaller than the number itself. The greater the indicated power (ie. the exponent that the number is raised to), the smaller the resulting power value. Actually, multiplying any value by a (proper, less than 1) fractional value always results in a lower value. This is similar to the concept to having "a fraction, of a fraction", and the result is smaller than the original fraction.

Ex. First, $0.5 = \frac{5}{10} = \frac{1}{2}$: "one-half. or one-half of one (whole)" , squaring this value we have:

$(0.5)^2 = (0.5)(0.5) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = 0.25$ and:

$0.25 = \frac{25}{100} = \frac{1}{4}$: "one-fourth or one-quarter of one (whole, all or entirely)".
0.25 is less than 0.5 , and sometimes this
is indicated or mathematically expressed
as: $0.25 < 0.5$, $<$ is the mathematical
notation symbol for "less than", or "not
equal to, but less than the following number".

Even though 0.5 is half of the length (1st dimension) on 1 unit, this value will only represent one-fourth of the area (a two dimensional concept) of one square, as indicated above when the value is squared.

A geometric representation of this is shown below. The largest square shape represents $2^2 = 4$. The shaded square represents $(0.5)^2 = 0.25 = 1/4$ (= one-fourth of one square. One square $= 1^2 = 1$). A unit square (or a squared unit) is the basic unit of measurement for measuring (2 dimensional) surface or planar (ie.,flat) areas. [FIG 16]
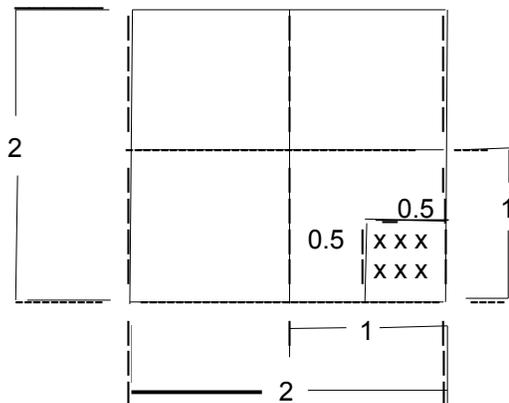
0.5 is half (ie., 1/2) of 1, but when 0.5 is squared (ie., 0.5^2), it becomes smaller and is only one-quarter (ie., 1/4) of 1.

The third or cubed power of the value of 0.5 will be considered below. As mentioned previously, a unit cubed is the basic unit of measurement for (3 dimensional) volumes. You can consider a volume as moving a flat area (a plane, or planar area, having a length and width dimension only) through a height distance. Area and volume will be discussed to a greater depth further ahead in this book.

(0.5)^3 = 0.125 = 1/8   : One-eighth of one cube. One cube = 1^3 = 1.
                                             It will take 8 of these cubes, each having a side
                                             length of 0.5, to fill a single cube of 1 unit length.

Even though 0.5 is half of the length (1st dimension) of 1, this value will only represent one-eighth of the volume (a 3 dimensional concept) of one cube. In a reverse type of manner, it could also be said that if a length value (such as 0.5 here) doubles, the volume (or cubed value) will increase by eight times. If the length values is halved (divided by 2), the volume will decrease (a demagnification) by eight times.

Ex.   0.5^3 = 0.125   : 0.5 is half of 1, and its cube or volume is 8 times less than that of 1
         1^3 = 1   : 1 is double or twice that of 0.5, and its cube is 8 times more than that of 0.5
         2^3 = 8   : 2 is double or twice that of 1, and its cube is 8 times more than that of 1
         4^3 = 64   : 4 is double or twice that of 2, and its cube is 8 times more than that of 2

These kind of relationships of the length to the areas and volume is not a constant or linear (mathematical, or numerical) relationship. For example, if the length doubles, the area and volume will increase, but they more than double. Area will increase by 4, and volume will increase by 8. This kind of non-linear (non-corresponding or the same growth or magnification [ie. multiplication of]) relationship is called an exponential relationship. Things that are said to be linear, are linearly related, that is, their mathematical relationship of their values is a linear or "line-like" relationship. Specifically, a mathematical relationship of or between two values is linear when the ratio (or rate) of the change of one variable with respect to the change of another variable is a constant value. Although this is of a topic to be discussed further ahead in this book, this can be mathematically expressing as: (change in variable1) / (change in variable2) = the rate of the changes, or simply the rate of change. For a linear relationship, the rate of changes between the two values or variables is a constant value. It can also be said that In a linear relationship, as one value, the independent variable changes in value by 1, the other corresponding or dependent value or variable will always change by the same constant value that is called the slope or "steepness" of the line. On a graph, a linear mathematical relationship between two variables can be shown as a line. In a non-linear relationship of values or variables, when one values changes by some factor, that the other does not change by some constant factor, but by a factor that is non-constant and is also changing, such as increasing or decreasing in value.   The rate of changes between the two values or variables is then not a constant value. On a graph, their relationship can be shown to be some type of curve.

Special note on dividing expressed powers: If given an expressed power value of a number that contains the base (b) and the exponent (e), and if that power value is divided by a value (d), it is incorrect to divide either or both the base and the exponent by that divider. The true result is found only by dividing the actual total or number value of that indicated power by that denominator.

Ex. $\underline{10^3}$ /=/ $\underline{(10)^3}$ = 5^3 = 125, or /=/   10^(3/2) = 10^1.5 = 31.62   ,   the true result is: $\underline{(10^3)}$ = $\underline{1000}$ = 500
         2   (2)   2   2

A CHART OF THE VALUES OF A POWER WITH RESPECT TO THE BASE AND EXPONENT

First,   N = 1^exponent = 1   : base is 1, and exponent is any value

N = base^1 = base   : N=number=actual resulting (power) value , and base is any value

Considering only positive (greater than zero , > 0) valued exponents:

When the base is greater than 1: (base > 1):   Examples:

base^(exponent < 1)   3^(0.5)   = 1.732<u>1</u>   : (N < base)
base^(exponent = 1)   3^1   = 3   : (N = base)
base^(exponent > 1)   3^2   = 9   : (N > base)

When the base is less than 1: (base < 1):   Examples:

base^(exponent < 1)   0.5^0.5 = 0.7071<u>1</u>   : (N > base)
base^(exponent = 1)   0.5^1   = 0.5   : (N = base)
base^(exponent > 1)   0.5^2   = 0.25   : (N < base) , and here due to a fractional value times a fractional value results in a smaller fractional value. 0.5^2 = (0.5)(0.5) = 0.25

MULTIPLYING LIKE VALUES

When multiplying like or similar values, variables (which can represent any value) or terms as in algebra, the result can be expressed as a power of that value; where the value becomes a base and whose exponent is the sum of the exponents of that value.

Note that a value without an exponent is considered or understood to have an exponent of one (1) For example: $3 = 3^1$. The explanation of this is that a value to the fist power is the first multiple by itself to an initial product of 1, and this is shown directly below, but may be skipped over if appears somewhat complicated for now:

For 3^1: $1 \times 3 = 3 \times 1 = 3 = 3^1$,   : the first power of a number is equal to the first multiple or instance of that number which is the same as multiplying that number by 1. The first power of a number is equal to itself.

Note that 1×3 is technically equal to 1 repeatedly added to itself 3 times: $1 + 1 + 1 = (1)(3) = 3 = (3)(1) = 3$ added to itself 1 time. It could be said that the first multiple of a number is equal to itself.

For 3^2: $1 \times 3 \times 3 = (1 \times 3)(3) = (3^1)(3) = (3^1)(3^1) = 3 \times 3 = 9 = 3^2 = 3^{(1+1)}$

Ex. $5 \times 5 = 5^1 \times 5^1 = 5^{(1+1)} = 5^2$: Note: $5^1 \times 5^1 \;/=/\; 2(5^1) = 5^1 + 5^1 = 10$
That is, you do not add the numerical coefficients of the same or like values when multiplying, but rather add their exponents.

$5^1 + 5^1 = ((1)(5^1)) + ((1)(5^1) = ((1+1)(5^1) = 2 (5^1)$
This is repeated addition of the like values. Add their numerical coefficients or factors for the resulting coefficient or factor of that value.

Remember that multiplication is associated with repeated addition, and powers are associated with repeated multiplication.

Ex. $(4^1)(4^2) = 4^{(1+2)} = 4^3$

checking: (4^1) (4^2) Or: $(4^1)(4^2)$
     (4^1) $(4^1)(4^1)$ $(4)(16)$
      $4^{(1+1+1)}$ 64
      4^3 $=$ 64 : $4 \times 4 = 16$, and $16 \times 4 = 64$

Ex . $2^2 + 2^2$

Since this is repeated addition of a value (here, 2^2) twice, we can express this with multiplication:

$2^2 + 2^2 = 2 \times 2^2 = (2^1)(2^2) = 2^{(1+2)} = 2^3$: add the exponents of like factors to simplify

Checking: $2^2 + 2^2 = 4 + 4 = 8 = 2^3$

As previously mentioned, try not to confuse the concept of adding like values with the concept of multiplying like values. When adding like values, their numerical coefficients are added and the variable is kept the same. When multiplying like values, their numerical coefficients are multiplied, and their exponents are added.

Ex.   First add, and then multiply these two values: 5x and 2x

    Note, variables, such as x, are used here. If you are somewhat worried or unfamiliar with using variables, you can replace (or substitute) it with a (constant, unchanging, known, fixed or solid) numeric value. For example, if we let x=3, we would have: 5x = 5(3) and 2x = 2(3). Their sum would be:

  5(3) + 2(3) = (5+2)(3) = 7(3)   : adding the values.   Checking:

  3(5) + 3(2) = (3+3+3+3+3)+(3+3) = 3+3+3+3+3+3+3

Using the concepts and expression for repeated multiplication, the result is: 3(7) = 7(3) = 21

  5x + 2x = (5 + 2)x = 7x   : adding the like terms.
                                    Here, you can see (x) is a common factor to all the terms (here 5x and 2x),
                                      hence they are called "like-terms".

Checking: 5x + 2x   Expressing the terms using repeated addition:
               (1x + 1x + 1x + 1x + 1x) + (1x + 1x)   Which can be expressed simply as:
               (x + x + x + x + x) + (x + x)   Now using the associative law of addition:
               (x + x + x + x + x + x + x)   Expressing this repeated addition, of x, using multiplication:
                       7 x

                              Note that for x and x^2, although being the same variable, they are not
                              like terms since their exponents are different and therefore these
                              (powers of the variable) represent two entirely different values. For example:
                              5x + 2x^2 !=, does not equal: (5 + 2)x, or (5 + 2)x^2, or (5+2)(x + x^2)

  (5x)(2x) = (5)(2) (x x) = 10x^2   : multiplying the like terms. Everything, both the numerical coefficients, and
                                       variables are to be multiplied together. This example is verified by the
                                        concepts of the associative law of multiplication that states that grouping
                                      (ie. associating) does not matter when multiplying, and the result will be
                                      the same.

      Before moving on to another topic, another good point must be made. If given an (indicated) power value, that value can essentially be factored by using the same base with exponents values that sum up to that given exponent of the power. Using an example previously given:

Ex.   Since: (4^1)(4^2) = 4^(1+2) = 4^3   we see that (4^1) and (4^2) are factors of the product 4^3, hence
                                              4^3 can be factored to:

       4^3 = (4^1)(4^2)   : the factors of 4^3 can be any indicated (with an exponent) power of that
                                     base of 4, just as long as the sum of the exponents of all the "like factors" sum
                                     up to 3

       4^3 = 4^(1.5 + 1.5) = (4^1.5)(4^1.5)

A POWER TO A POWER

If an indicated power of a value (base), or variable as in algebra, is itself raised to an indicated power, simply multiply the exponents for the resulting exponent (of that base) to simplify this type of expression. A simple verification of this is shown below:

First note: (5)^2 = $(5^1)(5^1)$ = (5^1)^2 = 5^$(1x2)$ = 5^2

Ex. (5^2)^2 = 5^$(2x2)$ = 5^4

checking: (5^2)^2 = $(5^2)(5^2)$ = 5^$(2+2)$ = 5^4   : this check uses "extending" or "expanding"
                                                                              the indicated expression to a fuller form

        or: (5^2)^2 = $(5^2)(5^2)$ = $(5^1)(5^1)(5^1)(5^1)$ = 5^$(1+1+1+1)$ = 5^4

        or: (5^2)^2 = (25)^2 = 625 = 5^4

To help understand this, we can remember that multiplication is a "shorthand" expression for a repeated addition expression.

Ex. (4^2)^3 = $(4^2)(4^2)(4^2)$ = 4^$(2+2+2)$ = 4^6   : showing the repeated addition of the exponents

   (4^2)^3 = 4^$(2x3)$ = 4^6   : showing using multiplication, instead of repeated addition, of the exponents

Ex. (0.7^2)^3 (0.7^2)^2 = (0.7^2)^$(3+2)$ = (0.7^2)^5 = 0.7^10

checking: (0.7^2)^3 (0.7^2)^2
              $(0.7^6)(0.7^4)$
              0.7^$(6+4)$
              0.7^10

A PRODUCT TO A POWER

This is very similar to a "power to a power" mentioned above, but the power that this product is being raised to will be "distributed" (shared or applied among, essentially multiplied) to the exponent of each factor of the indicated product when simplifying the expression.

Ex. $(4 x 2)$^2 = $((4^1)(2^1))$^2 = $(4$^$(1x2)$)(2$^$(1x2)$)$: Here, showing that the exponent was "distributed" (or applied)
                                                                              to each factor and its corresponding exponent..
                              = $(4^2)(2^2)$
                              = (16) (4) = 64

checking: $(4 x 2)$^2 = (8)^2 = 64   : Here, the expression in the grouping symbols was simplified first.

Ex. $(2$^2 x $3^2)$^2 = $(2$^$(2x2)$)$ $(3$^$(2x2)$)$ = $(2^4)(3^4)$ = 16 x 81 = 1296

checking: $(2$^2 x $3^2)$^2 = $(4 x 9)$^2 = (36)^2 = 1296

We also see a rule can be developed here due to the above example.   First, given a product of two (expressed) powers, and if the exponent is not indicated, it is understood as 1.   The product of two numbers having the same exponent is the

product of those numbers and whose exponent is that exponent. For example:

(2^2 x 3^2) =   (2 x 3)^2   here is a verification using the above discussion, distributing the exponent:
(2^2 x 3^2) =   (2^(1x2))(3^(1x2))
(2^2 x 3^2) =   (2^2 x 3^2)   : the result is the original expression on the left side, hence they are equivalent

Checking:

(2^2 x 3^2) =   (2 x 3)^2
(2^2 x 3^2) =   6^2   : another, somewhat obvious, rule to observe here is that you can factor a product
                                            (here 6) to where each factor (here, 2 and 3) has the same indicated power
  4   x   9   =   6^2
       36   =   36

Ex. (9^0.5)(4^0.5) = [(9)(4)]^0.5 = [36]^0.5 = 36^0.5   : doing the reverse, you can factor the base of an indicated
                                                                                   power, and each factor will have the same exponent

A SIMPLE EXPRESSION TO A POWER

        The word simple in the title refers to addition and subtraction. First, a very important word of caution: You cannot simply distribute the power to the values being added or subtracted in an expression that is raised to a power.

Ex. (5 + 2)^2   /=/ (does not equal)   ((5^2) + (2^2)) = 25 + 4 = 29

The correct result can only be found (when not using the order of operations directly) by performing the indicated operation. This is sometimes referred to as "extending" or "fully expressing" (the given expression) to the indicated power.

(5 + 2)^2   =   (5 + 2) (5 +2)

To simplify this, especially when algebraic variables are used, requires you knowing about the distributive law of multiplication which will be shown later. Since only numeric values are used here, by using the order of operations, we can get the correct answer by simplifying the original expression which represents a value that could be simplified and-or expressed as a single number that is contained within the grouping symbols.

(5 + 2)^2   =   (7)^2   =   7^2   =   (7)(7) = 49   : basic simplification within the grouping symbols first, or by:

(5 + 2)^2   =   (5+2)^1 (5+2)^1   =   (5 + 2)(5 +2) =   (7)(7) = 49   : "extended method", then simplifying

DIVIDING LIKE VALUES THAT HAVE EXPONENTS

When dividing indicated or expressed powers that have the same base value, or variables as in algebra, simply subtract their exponents so as to have the exponent of the resulting quotient and which will also have that same base value.

Ex. $\dfrac{5^3}{5^1}$ = $\dfrac{5^{(3-1)}}{1}$ = 5^2 = 25   :here, the same base value is 5

checking: $\dfrac{5^3}{5^1}$ = $\dfrac{125}{5}$ = 25   or: $\dfrac{5^3}{5^1}$ = $\dfrac{\overset{1}{(5)(5)(5)}}{\underset{1}{(5)}}$ = 5^2 = 25

Hence, subtracting (to reduce, make less) exponents is essentially equal to canceling some factors common to both the entire numerator and denominator. As another check, the product of the quotient and the divisor (or "divider") should equal the dividend. Using the example above:

   $(5^2)(5^1)$ = $5^{(2+1)}$ = 5^3 : checks

Ex. $\dfrac{7^9}{7^3}$ = $\dfrac{7^{(9-3)}}{1}$ = 7^6 : 7^9 = $7^3 \times 7^{(9-3)}$ = $7^3 \times 7^6$ = $7^{(6+3)}$

Checking: Here, the numerator is factored by using the inverse of the rule (add the exponents) when multiplying like values, or variables as in algebra. It is factored to create like values in both the numerator and denominator which can then be canceled. Since 7^3 is the smallest power of the value, or variable as in algebra, in question, there is no reason in this example to factor it. 7^9 will be factored so that there will be a highest common factor (HCF) of 7^3 for both the numerator and denominator which can then be canceled out:

$\dfrac{7^9}{7^3}$ = $\dfrac{\overset{1}{(7^3)}(7^6)}{\underset{1}{7^3}}$ = $\dfrac{7^6}{1}$ = 7^6 : A way to factor 7^9 is: 7^9 = $7^3 \times 7^{(9-3)}$ = $7^3 \times 7^{(6)}$

What if the exponent of the same base in the denominator is larger? One answer is to subtract the least value exponent from the largest exponent of the like or similar base values, and this ensures a result with a positive valued exponent rather than a negative valued exponent (which requires you knowing about signed values or numbers that will be shown further in this book).

Ex. $\dfrac{5^1}{5^2}$ = $\dfrac{1}{5^{(2-1)}}$ = $\dfrac{1}{5^1}$

checking: $\dfrac{5^1}{5^2}$ = $\dfrac{(5^1)}{(5^1)(5^1)}$ = $\dfrac{1}{5^1}$: after canceling a common factor of (5^1).

also: $\dfrac{5^1}{5^2}$ = $\dfrac{\overset{1}{5}}{\underset{5}{25}}$ = $\dfrac{1}{5}$ = $\dfrac{1}{5^1}$: equivalent fractions, and both equal 0.2

AN EXPONENT OF 0?

Any value, or variable as in algebra, with an exponent of zero (0) is equal to one (1). Besides being logically or mathematically correct, using 0 as an exponent is sometimes (explicitly) expressed or used to help clarify a simplification or a derivation process.

Ex. $10^0 = 1$: This is easily verified when the exponent in the numerator and denominator are equal. Also, any value divided by itself is equal to 1:

$\dfrac{10}{10} = \dfrac{10^1}{10^1} = \dfrac{10^{(1-1)}}{1} = 10^0 = 1$ In general, or symbolically expressed: $number^0 = 1$

(0) indicates that the given base is to be repeatedly multiplied to a value of 1, a total of 0 (ie. none) times. Since there won't be any multiplication performed on this value of 1, the result is that this 1 will remain unchanged. Also note that 10^0 is the weight of the "ones" digit position (in the decimal number system) expressed with exponential notation.

$10^0 = 1$ $= 1$: weight of the "ones" digit position in the decimal number system
$10^1 = 1 \times 10$ $= 10$
$10^2 = 1 \times 10 \times 10$ $= 100$: with 10 as the base, each new value is 10 times more, and therefore, each preceding value is 10 times less (ie. a division by 10)

Also note:

$3^0 = 1$ and this is also the same value as for example:

$5^0 = 1$

Hence: $3^0 = 5^0 = 1$

$4.7^0 + 3.2^0 = 1 + 1 = 2$

If you had $3^2 / 3^2$, the result is 1 since any value divided by itself it 1. Hence both the base (here it's 3) and the exponent (here it's 2) is essentially eliminated and the result is just 1 as it would be for any other base and exponent (including 1 also) divided by itself. Mathematically, shown in an expression form, you can eliminate (set to 0) the exponent by subtracting the exponents. Subtracting exponents is a common form for simplification of the division of like values (or variables) that have the same base:

$3^2 / 3^2 = 1 = 3^{(2-2)} = 3^0 = 1 = n^0$: n = any other number

A FRACTION TO A POWER

A fraction to an indicated power is equal to both the numerator and denominator "raised" to that indicated power. The exponent (ie. the indicated power) of the fraction is essentially "distributed" (applied to, and essentially multiplied) to the numerator and denominator exponents.

Ex. $\left[\dfrac{(\ 5\ )}{(\ 1\ )}\right]$^2 $= \left[\dfrac{(5^1)}{(1^1)}\right]$^2 $= \dfrac{(5^1)^2}{(1^1)^2} = \dfrac{5^{\wedge}(1\times2)}{1^{\wedge}(1\times2)} = \dfrac{5^2}{1^2} = \dfrac{25}{1} = 25$: for this book, large brackets will be utilized, but large parentheses can also be used

Checking this by performing the indicated operation:

$\left[\dfrac{(5)}{(1)}\right]$^2 $= \dfrac{(5)(5)}{(1)(1)} = \dfrac{25}{1} = 25$: this expressed the extending (showing multiplication) of the indicated power

Ex. $\left[\dfrac{(\ 2\ )}{(3^2)}\right]$^3 $= \dfrac{(\ 2\ )^3}{(3^2)^3} = \dfrac{2^3}{3^6}$

Likewise, in a reverse-like manner, the division of two values, or variables as in algebra, having the same exponent or raised to the same indicated power is equal to the fraction (division) of those variables raised to that indicated power.

Ex. $\dfrac{2^2}{3^2} = \left[\dfrac{(\ 2\ )}{(\ 3\ )}\right]$^2

checking: $\dfrac{2^2}{3^2} = \dfrac{4}{9}$ and $\left[\dfrac{(\ 2\ )}{(\ 3\ )}\right]$^2 $= \dfrac{(2)\,(2)}{(3)\,(3)} = \dfrac{4}{9}$

Ex. $\dfrac{5^3}{7^3} = \left[\dfrac{(5)}{(7)}\right]$^3

Ex. $\dfrac{(3^4)(3^3)}{5^7} = \dfrac{3^{\wedge}(4+3)}{5^7} = \dfrac{3^7}{5^7} = \left[\dfrac{(\ 3\ )}{(\ 5\ )}\right]$^7

Ex. $\dfrac{3^{10}}{5^7} = \dfrac{3^{\wedge}(3+7)}{5^7} = \dfrac{(3^3)(3^7)}{5^7} = (3^3)\,\dfrac{(3^7)}{(5^7)} = (3^3)\left[\dfrac{(\ 3\ )}{(\ 5\ )}\right]$^7

ROOTS AND RADICALS

Finding the root of a number is mathematically the opposite or "inverse" of finding a power of a number. The following description may seem complex to the new student, however, after viewing a few examples below, the concept becomes an understandable and necessary part of mathematics.

Here, you are essentially given the power of some number (ie. a base), and the indicated power (the base's exponent), and the base (now called a root for this type of mathematical operation) of that power value is to be found. The symbol for a root operation is the radical symbol ($\sqrt{}$). The value, essentially the power value, underneath the radical symbol is now formally called the radicand. The root to be found is indicated by a value correspondingly called an index. This index means "take or find this specific indicated root of the radicand". The index is essentially the exponent that the root (when considered as a base) will need to be raised to equal that radicand (when considered as a power value or number). Some older ways would refer to a radical as a "surd", and an index as the "root extraction number".

Here is the general "root" expression and formula:

$$\text{root} \ = \ {}_{\text{index}}\sqrt{\text{radicand}} \qquad\qquad : \text{GENERAL ROOT EXPRESSION OR "FORMULA"}$$

This can be rearranged mathematically, to solve for the radicand as:

$$\text{root}\ {}^\wedge\text{index} \ = \ \text{radicand} \qquad\qquad \text{this is mathematically equivalent to a power expression:}$$

$$\text{base}{}^\wedge\text{exponent} \ = \ \text{power} \qquad\qquad \text{hence, comparing this to the general root expression above:}$$

$$\text{base} \ = \ {}_{\text{exponent}}\sqrt{\text{power}} \qquad : \text{showing that the base of a power is found using the}$$
$$\text{root of a power, and therefore base=root}$$

Mathematical operation or expression to solve for the index (the indicated root of a radical, or the indicated (as an exponent number) power of a base value) value is called a logarithm, and it will be discussed more further ahead in this book. A logarithm is basically equivalent to the index of a root and-or exponent value of a base.   As expressed above, to solve for a value of these root and power expressions that have, contain or include these three values will require having or knowing the other two values used. Likewise, to solve for the specific value of the logarithm equation will require the two other specific values (the radicand and corresponding root value, or the base and corresponding power value) of which it is in reference to.

The radical symbol may look somewhat like a division symbol, and the only casual relationship among theses operations and the symbol used is that the result, either a quotient or a root, is usually a smaller value. When the index or "indicated root to be taken" is not explicitly given, it is to be understood as being 2 or the second or "square-root".

It is easy to imagine any of the various or different possible factors of a number, for example:

$$36 \ = \ (12)(3) \ = \ (4)(3)(3) \ = \ (2)(2)(3)(3) \ = \ (18)(2) \ = (9)(4) = (6)(6)$$

Here, the only factors of the number that are all identical, and have a product of 36, are 6. 6 is the only factor that the product of itself (here twice, or 2) several times equals the given value of 36.   The repeated product of 6, twice, is mathematically expressed as 6 x 6 = 6^2 = 36 . The expression 6^2 is called the square (or second (2) indicated power) of 6.   This single factor value of 6 is therefore called the square (or second = 2) root of 36. A very common root of a number to "take" (solve for) is a square root. Squared = second = 2 as a mathematical value.   When solving for the square root of a number, power or product value, you are effectively solving for a single factor (or base) that when it is squared or "raised" (increased in value) to the second (2) indicated power, that it will equal that given value.

For special note of a roots similarity and-or its relationship to division: A root can be thought of as a special instance of

division where if we divide (and sometimes repeatedly) a value by a certain or specific dividend value (called the root), we will have a final result or quotient that is equal to that same dividend or root value used. For example, the square root of 25 is 5. If you were to divide 25 by 5, the result is 5 which is the same value as the divisor. Checking $5^2 = 25$. $5/5=1$ shows a complete and pure division (after 2 divisions) with no remainder. For example, the cube root of 8 is 2. If you were to repeatedly (here 3 times) divide 8, and all the following results by 2 repeatedly, you would have: $8/2 = 4$, $4/2 = 2$ and $2/2=1$. Checking $2^3 = 8$. For example, the fourth root of 16 is 2. $2^4=16$. $16/2 = 8$, and $8/2 = 4$, $4/2 = 2$, and $2/2=1$. Mathematically, since: radicand = root^index , we then have: radicand / (root^index) = 1. Checking: $16 / 2^4 = 16/16 = 1$

$$\text{base}^{\text{exponent}} = \text{power} = 6^2 = 36 = \text{radicand} = \text{root}^{\text{index}}$$

$$\text{root} = \sqrt[\text{index}]{\text{radicand}} = 6 = \sqrt[2]{36} = \text{base} = \sqrt[\text{exponent}]{\text{power}}$$

Given a number or value of 9. The two identical factors of 9 are 3 and 3. $9 = (3)(3) = 3^2$. We see that this has a form of the expression: base^exponent Here the base is 3, and the exponent is 2. The expression also has the form of: root^index , 3 is the root, and 2 is the index. 3 is equivalent to the second (2) root of the radicand of 9. The first root of a value it itself.

$$3^2 = 9 \quad \text{and} \quad 3 = \sqrt[2]{9}$$: "the square root of 9 is 3" and "3 squared is 9"
: "the second root of 9 is 3" and "3 raised to the second power is 9"
or "the second power of 3 is 9"

Ex. What is the square root of 16 ?

$\sqrt[2]{16}$: 16 is the radicand, the index is 2 and is common and understood as the (indicated) root value if it is not explicitly expressed or indicated:

$\sqrt[2]{16} = \sqrt{16} = 4$: since $4 \times 4 = 4^2 = 16$

Ex. $10 = \sqrt{100}$: since $10^2 = 10 \times 10 = 100$

Ex. What is the third (3) or "cube" root of 75 ?

$\sqrt[3]{75} = 5$: since $5^3 = 75$

checking: $5^3 = 5 \times 5 \times 5 = 25 \times 5 = 75$

Extra: $\log_5 75 = 3$, since $5^3=75$: "log" is an abbreviation of the word "logarithm" which is essentially the inverse (or reverse) mathematical operation to that of a power.
"The log or exponent of, or for a base of 5 necessary to produce or be equal to its power value of 75 is 3".

The word cube comes from concepts of the measurement of volume. Volume is a 3-dimensional (length, width, height) measurement of spacial (ie., space area). A planar area is measured with units based on only 2 dimension (length and width). The units of measurement of volume have 3-dimensions (length, width, and height) and the reference unit of volume also has the same unit of length for each dimension, and is therefore a ("geometric") cube shaped unit of reference or measurement. A cube shape is essentially the shape of a square box with equal side lengths. Here is how cube or "cubic" units of measurement are expressed:

units x units x units or simply: unit^3   : a cubic unit , ex. a cubic inch = in^3 , a cubic centimeter = cm^3 = cc

For general mathematical expressions and purposes, such as here for powers and roots, that are not necessarily related to the specific topic of cubic volumes, the value of 3 , instead of the word cube, should be utilized.

Frequently, calculators are utilized for finding roots since they have preprogrammed **algorithms** (steps and procedures to solve a problem) that operate or function very fast. Most basic "home" or "4- function calculators" now also include a generous square root function. These calculators and others are as inexpensive as $1USD as of 2020, and are very important due to their helpfulness. There are also calculator programs and "apps" (application = programs) available for computers and-or phones.

Ex. $2\sqrt{2}$ ~ (about) 1.414213562   : using a 10 digit calculator. Any least significant digits which cannot be
                                                  be entered, processed and-or displayed are truncated (removed), and are
                                                  therefore not available for further (and more exact) calculation. What is
                                                  displayed is sufficient for 99% of all types of people and professions. Some
                                                  values have an endless number of digits, and only a limited number can be
                                                  processed either by man and-or machine.

        The actual value is often expressed as: 1.414213562$\ldots$: $\ldots$ indicates further, and possibly, endless digits

Several methods will be presented in this book for calculating any indicated root. The appendix section of this book includes a method for calculating integer (basically a number without any fractional part) roots. Many methods for calculating roots, particularly square roots, have been developed and the one shown in the appendix is perhaps the most straight-forward and simplest to remember. The appendix also contains many pre-calculated tables.

Probably the oldest method to calculate a square-root of a number is by using an initial good guess at the square-root. If the guess is correct, the product of the root and the root (ie. root^2) as two factors should equal that number (radicand) of which the corresponding square-root is to be found. If the product or "square" of the guessed or assumed root is not equal to the radicand, the guess is typically adjusted by increasing or decreasing a specific digit of it and trying again. Here, a specific digit usually means the current digit in question or the next least significant digit when you are getting "closer" to the true root.

Ex. What is the square-root of 9.61?

Since 3 squared equals 9, and 4 squared equals 16, the square root of 9.61 is somewhere between 3 and 4, and that the square-root of 9.61 appears to be much closer to 3 than 4. Since 9.61 is slightly more than 9, we will adjust the root estimate value of 3 to a slightly larger value:

Trying 3.1 as the new root estimate and then checking it, we get 3.1 x 3.1 = 9.61, hence 3.1 is the square-root of 9.61:

    $3.1 = \sqrt{9.61}$

Occasionally, the root of a number is exact. Sometimes this root would be called a rational, or rationalized, root. For example, the square root of 9 is 3, the square root of 64 is 8. Often, the square root of values, such as 2, and even though it is a constant value (always the same, and never changing) it is irrational or not-resolvable to an exact, specific or completed value since it will have non-ending and non-repeating decimal digits. Only "in theory" or possibly "conceivable in the mind" can a number like the square root of 2 have an exact or specific value, and rather than express or write the approximate answer out to a certain number of decimal places, it is simply expressed or given as just $\sqrt{2}$.

Irrational numbers cannot be expressed as the division of two integers (ie. plain rational numbers). Just because a calculator displays 8, 9 or 10 digits, don't automatically assume that this is the exact square root, it is however a very close approximation for nearly all practical purposes. If your calculator can display 9 digits and if the displayed result is less than 9 digits, only then can you assume this is the exact square root.

Ex.   An area of land is said as being 100 square feet, how long is a side (assuming the land is square shaped, or if it could be represented as simple to conceive square shape)?

$\sqrt{100}$ =   10   :since 10 x 10 = 100   , and "technically correct" by using the given units:

$\sqrt{100 \text{ ft}^2}$ =   10ft^1   :since 10ft x 10ft = (10)(10)(1ft)(1ft) =   100 square feet = 100ft^2

Ex.   A small hand-held telescope is noted by the manufacturer as being 9X ("nine times" or "9 power") magnification, that is, the eye with the aid of the telescope will appear to see an object, or its area, as 9 times larger than the object as seen (ie. 1x or 1 time, no magnification, true size seen) without the telescope. How many times will the length and width of an object appear to be magnified through this telescope? Since area can be thought of and measured as a two dimensional "square" concept, calculated by multiplying the length and width dimensions which have the same numeric values and units of measurement, then doing the inverse by taking the square root of the given area (of the object as seen magnified) will yield the length (= width for that area, =side) dimension of that apparent area magnification.

$\sqrt{9}$ = 3   since: 3 x 3 = 9

Therefore, an object as viewed through the telescope will appear about 3 times longer and wider (not 9). The total (magnified) area appears as (3 x 3) = 9 times more or bigger. Below is a graphical representation of this example , the "1x" = "1 times" = "1 power" indicated square is a representation of what the unaided eye, without a telescope, will see. [FIG 17]
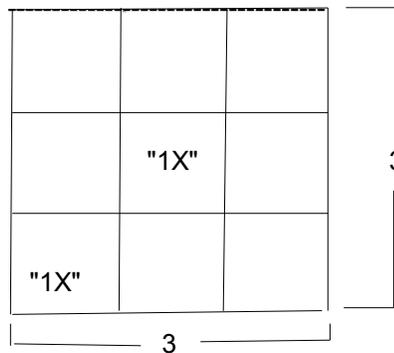
"1X"   3   : 9X = "nine times" = "nine power"
                           Nine of one area.   An area of one
                           increased, magnified, or spread out over
                           to the same size as an area of nine.
"1X"   The "linear magnification" of 9 power is
                           $\sqrt{9}$ = 3
          3

You might have a thought that for an 18 power, "18x", telescope that the length and width, or linear (basically means straight), dimensions seen should double (from 3 to 6) since 18 is twice , or double of 9. Let's check:

$\sqrt{18}$ = 4.243   : $\sqrt{\text{"power"}}$ = $\sqrt{\text{(apparent) area magnification}}$ = (apparent) linear magnification =
                                                                                    (apparent) increases in length and width

The result is less than predicted due to the non-linear nature of powers and roots as mentioned previously in this book. For a "100 power" telescope, the length and width will only appear to be this many times longer;

$\sqrt{100}$ = 10

That is, as the "power of the telescope" (shown as a radicand here), increases or "grows" very large, the dimensions of the object now seen (ie, the root) does not increase or "grow" as fast, but appears to grow less and less. Technically, the rate of change between the linear magnification and the power or area magnification is decreasing or slowing. A large telescope with a large lens or mirror may not have as much magnification as most would think it would.   Today, digital image sensors with a very higher density of individual elements (ie. basically pixels) per unit area are helping to effectively

increase the resolution (ie., "fineness" or "precision") of the image, and which can then even be "digitally magnified", hence effectively magnifying the initial image that the telescope itself can produce.

If an already magnified image of an object or area is then magnified again, the resulting magnification,"power" or "times" = "X", is simply the product of those two magnifications. Ex. If an object is magnified by 100 power, and if that magnified image (perhaps even considered as an initial, 1x, image) is then magnified by a power of 4, the resulting magnification of the object is: (100)(4)power = 400 power.

In a telescope or microscope, the image is linear magnified equally in both the horizontal and vertical (or width) directions. If the linear dimensions of an image or area (A1) are both or equally magnified, say by (n) times longer, the area is magnified or increased by a (magnification or "power") factor of n^2, and results in A2.

A1 = (horizontal length) x (vertical length)   or = (length) x (width) = LW

A2 = (n) horizontal length x (n) vertical length   =   n^2 (LW)   =   n^2 A1

The magnification ratio, factor or "power" is: A2/ A1 = n^2, and the linear magnification is: $\sqrt{n^2}$ = n

As an extra note, you can make a crude, low-power telescope using two magnifying glasses separated by a foot or two, but the image will be "upside down". For the larger objective (initial) lens, a low power lens is preferred because a longer focus distance (ie., focal length) actually increases the magnification. For the eyepiece lens which is used to view the image from the objective lens, the shorter the focal length of it, the greater the magnification of the entire telescope lens system. **Do not look at the Sun with your eyes, or with a telescope, because the concentrated and-or focused light it is too "bright" and can burn and damage your eyes. Remind people.**

When a value less than 1 is raised to a higher power, such as squared, the result is a smaller value since a fraction of a fraction is a smaller fraction. Now, in a reverse type of manner, if an indicated root larger than 1, such as a square root, is then taken on this smaller value, the result is a larger value.

Ex.   0.5^2 = (0.5)(0.5) = 0.25   : A fraction of a value (here, another fraction) is a smaller value.
                                                          Ex. A fraction squared is a smaller value

$\sqrt{0.25}$ = 0.5   : here, the root is larger than the radicand itself.
                                                       The square root of a fraction (a proper fraction, less than 1)
                                                       will be greater than that given fraction.

NEGATIVE POWERS OF 10

10^0 = 1

10^-1 = 1/10 = 0.1   : 10^0 / 10^1 = 1/10^1 = 10^(0-1) = 10^-1   = 0.1 = "a (1) tenth" = "a tenth"

10^-2 = 1/10^2 = 1/100 = 0.01   : 1/100 = "1/one-hundred" = "one/one-hundred"   = "one-hundredth"
                                                    1/100 = 1 / (10)(10) = (1/10)(1/10) = A tenth of a tenth = one-hundredth
                                                                      = (0.1)(0.1) = 0.01
                                                                      = (1/10^-1)(1/10^-1) = 1/10^-2

10^-3 = 1/10^3 = 1/1,000 = 0.001   : one-thousandth
10^-4 = 1/10^4 = 1/10,000 = 0.000,1   : one-ten-thousandth
10^-5 = 1/10^5 = 1/100,000 = 0.000,01   : one-hundredth of one ten-thousandth
10^-6 = 1/10^6 = 1/1,000,000 = 0.000,001 : one-millionth

The square roots of negative powers of 10 will be shown further ahead.

CHART EXAMPLE THAT SHOWS THE SIMILARITIES BETWEEN POWER AND RADICAL EXPRESSIONS

Except for the names used, the operands and results of a given power expression and corresponding root expression are identical in value due to that powers are roots are mathematically the inverse operations of one another.

Operands of a power operation:

Base	Exponent	Power	Power Expression (mathematical notation)
2	1	2	2^1
2	2	4	2^2
2	3	8	2^3
2	4	16	2^4
2	5	32	2^5
etc.			

Operands of a radical operation:

Root	Index	Radicand	Radical Expression (notation)
2	1	2	$1\sqrt{2}$
2	2	4	$2\sqrt{4}$
2	3	8	$3\sqrt{8}$
2	4	16	$4\sqrt{16}$
2	5	32	$5\sqrt{32}$
etc.			

When these values are equal: power = radicand,   base = root, exponent = index   , we have:

power = base^exponent   =   radicand = root^index,   and:

base = $\text{exponent}\sqrt{\text{power}}$ =   root = $\text{index}\sqrt{\text{radicand}}$: a root is essentially the base of a power value

We now know the notation of the expressions to solve for two of the parts (base and power, or root and radicand) of either a power or root expression, and we know the similarities of the power and root expressions. To solve for the exponent or index, a notation called logarithm notation will be used, and this is covered further ahead in this book.

MULTIPLYING RADICALS

Radicals that have the same index (indicated root) can be multiplied. When they are multiplied, the product is a radical with the same index, and the radicand is the product of the radicands. It could also be said that these radicands and radicals that were multiplied together are factors of this new (product) radicand and radical.

Ex. $(2\sqrt{9})(2\sqrt{4}) = 2\sqrt{9}\ 2\sqrt{4} = 2\sqrt{(9)(4)} = 2\sqrt{36} = 6$

Checking by rationalizing the radicals first as indicated in step 1 of the order of operations. Rationalizing a radical expression means ridding or clearing the radical of the radical symbol by solving for the indicated root or expressing it in another form such as with exponential notation which will be discussed further ahead in this book:

$\sqrt{9}\ \sqrt{4} = (3)(2) = 6$

Ex. $3\sqrt{7}\ 3\sqrt{2}\ \sqrt{5}$
$\quad\ 3\sqrt{(7)(2)}\ \sqrt{5}$
$\quad\quad 3\sqrt{14}\ 2\sqrt{5}$

FACTORING RADICALS

Since radicals with the same index can be multiplied, producing a new radical, we can do the reverse and factor a radical for possible simplification. To factor a radical, the radicand value is factored. These factors can then become radicands of radicals which have the same index as the radical from which they were both factored from.

Ex. $\sqrt{16} = 2\sqrt{(4)(4)} = (2\sqrt{4})(2\sqrt{4}) = 2\sqrt{4}\ 2\sqrt{4}$: The factored radical, where each has
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ the same index as the source radical.
$\quad\quad\quad\quad\quad\quad\sqrt{16} = (2)\ (2) = 4$

When factoring the radicand for simplification purposes, it is usually best to find factors of it that are indicated powers whose (indicated) exponent is equivalent to the index of the radicand, such as perfect squares (2), cubes (3), fourths (4) , etc., since these radicals are easily rationalized. The indicated root of a value raised to the same indicated power is simply equal to that given value.

Ex. $2\sqrt{18}$

Since the index is 2, factor the radicand for powers that are perfect squares which are values raised to the second power. Note that 18 has factors of 9 and 2, or 6 and 3. Choose the factors of 9 and 2 since 9 is a perfect square of 3. That is, $3^2 = 9$, and this radicand can be easily rationalized if the indicated root or index is also 2:

$2\sqrt{18} = 2\sqrt{(9)(2)} = 2\sqrt{9}\ 2\sqrt{2}$: factored form. simplifying further, we will factor the radicand
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ of 9 to an indicated (with an exponent) perfect square (2) since the index
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ is 2:

$2\sqrt{3^2}\ 2\sqrt{2}$ In short, for example, the square root of any value squared
$\quad\quad\quad\quad\quad\quad\quad$ (particularly when simply indicated as such) is that value:

$\quad\quad\quad\quad 2\sqrt{3^2} = \sqrt{3^1}\ \sqrt{3^1} = 3$: Using the concept that the square-root of any value
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ multiplied by itself (ie squared) is that value, here 3.

$3\ 2\sqrt{2}$: Since this is a representation for the square root of 18, this value when squared or raised
$\quad\quad\quad\quad\quad$ to the second power should therefore equal 18:

checking: $(3 \cdot 2\sqrt{2})^2 = (3)^2 (2\sqrt{2})^2$: after distributing the exponent

           $= 9 \cdot 2\sqrt{2} \cdot 2\sqrt{2}$: after extending, expressing the indicated square as a multiplication

           $= 9 \cdot 2\sqrt{(2)(2)}$: multiplying the radicands of like radicals with the same index

           $= 9 \cdot 2\sqrt{4}$

           $= 9 (2) = 18$: the original radicand used in this example

Ex. $2\sqrt{147}$

$2\sqrt{147} = 2\sqrt{(49)(3)} = 2\sqrt{(49)} \cdot 2\sqrt{(3)} = 2\sqrt{7^2} \cdot 2\sqrt{3} = 7 \cdot 2\sqrt{3}$

In short, and as an example to this discussion, the square root of any value squared (especially when indicated as being squared with an exponent of 2), is that same value.

Ex. $\sqrt{4^2} = 4$ checking:

    $\sqrt{16} = 4$:It will be shown further ahead in this book that:

           $\sqrt[index]{radicand^{exponent}} = radicand^{(exponent/index)}$: even if the exponent = 1

Ex. $3\sqrt{5^3} = \sqrt{(5)(5)(5)} = 5$ checking:

    $3\sqrt{125} = 5$

       Since a radicand that is a whole number can easily be factored, this leads to another method to calculate the root of a radicand. You can factor a radicand into radicands that you already know the roots of. You can also completely factor a radicand to its prime factors and use a small table for the (previously solved, calculated) roots of prime factors. Here are the approximate roots of the first 16 prime numbers or factors:

$\sqrt{2}$ = 1.414213562 $\sqrt{3}$ = 1.732050808
$\sqrt{5}$ = 2.236067977 $\sqrt{7}$ = 2.645751311
$\sqrt{11}$ = 3.31662479 $\sqrt{13}$ = 3.605551275
$\sqrt{17}$ = 4.123105626 $\sqrt{19}$ = 4.358898944
$\sqrt{23}$ = 4.795831523 $\sqrt{29}$ = 5.385164807
$\sqrt{31}$ = 5.567764363 $\sqrt{37}$ = 6.08276253
$\sqrt{41}$ = 6.403124237 $\sqrt{43}$ = 6.557438524
$\sqrt{47}$ = 6.8556546 $\sqrt{51}$ = 7.141428429

Ex. $\sqrt{105}$ factoring the radicand:

    $\sqrt{(7)(5)(3)}$ factoring the radical:

      $\sqrt{7}$ $\sqrt{5}$ $\sqrt{3}$ using the above table for the square roots of these values:

    $(2.645751311)(2.236067977)(1.732050808) = 10.24695077$

The square roots of negative powers of 10 are also very helpful when the radicand contains a fractional part.

$\sqrt{10^{-1}}$ $= \sqrt{1/10}$ $= \sqrt{10^{-1}} = 0.316227766$ $\quad\quad : \sqrt{0.1}$

$\sqrt{100^{-1}} = \sqrt{1/100}$ $= \sqrt{10^{-2}} = 0.1$ $\quad\quad : \sqrt{0.01} = \sqrt{0.1} \ \sqrt{0.1} = \sqrt{(10^{-1})(10^{-1})}$

$\sqrt{1000^{-1}} = \sqrt{1/1000}$ $= \sqrt{10^{-3}} = 0.0316227766$ $\quad\quad : \sqrt{100^{-2}} \ \sqrt{10^{-1}} = \sqrt{0.001}$

$\sqrt{10000^{-1}} = \sqrt{1/10000}$ $= \sqrt{10^{-4}} = 0.01$ $\quad\quad : \sqrt{0.0001}$

Ex. $\sqrt{1.05}$ $\quad\quad$: compare this to the last example shown above which has the same digits

$\sqrt{1.05}$ $=$ $\sqrt{(105)(10^{-2})}$ $\quad\quad : \sqrt{(105)(1/100)} = \sqrt{(105/100)}$

$\sqrt{1.05}$ $=$ $\sqrt{105} \ \sqrt{10^{-2}}$

$\sqrt{1.05}$ $=$ $(10.24695077)(0.1)$ $\quad\quad : 0.1 = 1/10 =$ division by 10, so simply move the decimal point 1

$\sqrt{1.05}$ $=$ 1.024695077 $\quad\quad\quad\quad\quad$ place leftward so as to make a smaller number

This can also be solved as:

$\sqrt{1.05}$ $= \sqrt{(105)(0.01)}$ $= \sqrt{(5)(21)(0.01)}$ $\quad\quad$: factored to "primes"

$\quad\quad\quad\quad\quad = \sqrt{5} \ \sqrt{21} \ \sqrt{0.01}$ $\quad\quad$: see the table shown previously for these

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ roots, and then multiply them together.

We see that a radical that has a radicand that is not whole value, but a value that contains a whole part and a fractional part, can essentially be changed or reduced to one of solving the product of radicals or roots with radicands that are whole values.

A RADICAND RAISED TO A POWER EQUAL TO THE INDEX

The root of a radicand that is raised to an indicated power equal to the index of the radical is equal to the base of that radicand.

Ex. $2\sqrt{5^2}$ = 5   : checking: $2\sqrt{5^2}$ = $2\sqrt{25}$ = 5

: also, since $5^2 = 5^1 \times 5^1$, $\sqrt{5^2}$ = $\sqrt{5^1 \times 5^1}$ = $\sqrt{5}\sqrt{5}$ = 5

Ex. Given the square root of 10, and multiplying it by the square root of 10, we should have 10:

$\sqrt{100}$ = $2\sqrt{10^2}$ = $\sqrt{(10)(10)}$ = $2\sqrt{10}\ 2\sqrt{10}$ = 10

Ex. $3\sqrt{4^3}$ = 4   : checking: $3\sqrt{4^3}$ = $3\sqrt{64}$ = 4 since (4)(4)(4) = 64 = 4^3

Ex. $4\sqrt{2^4}$ = 2   : checking: $4\sqrt{2^4}$ = $4\sqrt{16}$ = 2

Ex. $3\sqrt{(2^2)^3}$ = (2^2) = 4

checking: $3\sqrt{(2^2)^3}$ = $3\sqrt{2^{(2x3)}}$ = $3\sqrt{2^6}$ = $3\sqrt{64}$ = 4

checking: $3\sqrt{(2^2)^3}$ = $3\sqrt{4^3}$ = 4   since $(4^1)(4^1)(4^1) = 4^3$

By observing the examples given, a basic way to rationalize radicals with a radicand raised to an indicated power equal to the index, is to place (or "convert") the radical into an exponential form. This can be accomplished by simply removing the radical symbol, and divide the radicands exponent by the index of the radical.

Ex. $2\sqrt{5^2}$ = $5^{(2/2)}$ = 5^1 : $5^{(2/2)}$ is the exponential form of $2\sqrt{5^2}$

Ex. $3\sqrt{4^3}$ = $4^{(3/3)}$ = 4^1

Ex. $4\sqrt{2^4}$ = $2^{(4/4)}$ = 2^1

Ex. $3\sqrt{(2^2)^3}$ = $(2^2)^{(3/3)}$ = $(2^2)^1$ = 2^2 = 4

If the radicand's exponent is evenly divisible by the index, then the radicand is a "perfect (integer) power" of the index. For example, given the radicands below:

5^4 is a perfect square (2nd power) of $5^{(4/2)}$ = 5^2, since $(5^2)^2$ = 5^4.

Therefore, the square root of 5^4 should be equal to 5^2:

$2\sqrt{5^4}$ = $5^{(4/2)}$ = 5^2

5^4 is also a perfect fourth (power) of $5^{(4/4)}$ = 5^1 since $(5^1)^4$ = $5^{(1x4)}$ = 5^4

Hence, $4\sqrt{5^4}$ = 5^(4/4) = 5^1 = 5   checking:

$4\sqrt{625}$ = 5

Ex. $3\sqrt{7^6}$ = 7^(6/3) = 7^2 = 49   : 7^6 is a perfect third power or "cube" of 7^2: (7^2)^3 = 7^6

    If the radicand's exponent is not evenly divisible by the index when rationalizing, the radicand can be factored into factors having exponents equal to the index. This factorization is essentially the reverse of multiplying like values, or variables as in algebra, where the product is that value and its exponent is the sum of the exponents. Below are some examples with their intermediate steps.

Ex. $2\sqrt{5^3}$

$2\sqrt{5^3}$ = $2\sqrt{(5^2)(5^1)}$ = $2\sqrt{5^2}\, 2\sqrt{5}$ = 5^(2/2) $2\sqrt{5}$ = 5 $2\sqrt{5}$

Note that even though the indicated power of the radicand is not evenly divisible by a value equal to the index, still, the result can be correctly indicated as such. This is shown below using the last example, and more will be said about this further in this book.

$2\sqrt{5^3}$ = 5^(3/2) or 5^1.5   and as shown above, this = 5 $2\sqrt{5}$, and: (5^1.5)^2 = 5^(1.5 x 2) = 5^3

Ex. $3\sqrt{(7^2)^4}$

$3\sqrt{(7^2)^4}$ = $3\sqrt{(7^2)^3\,(7^2)^1}$ = $3\sqrt{(7^2)^3}\, 3\sqrt{7^2}$ = (7^2)^(3/3) $3\sqrt{7^2}$

= 7^2 $3\sqrt{7^2}$ = 49 $3\sqrt{7^2}$ = 49 $3\sqrt{49}$

OR: $3\sqrt{(7^2)^4}$ = $3\sqrt{7^{(2x4)}}$ = $3\sqrt{7^8}$ = $3\sqrt{(7^3)(7^3)(7^2)}$ =

= $3\sqrt{7^3}\, 3\sqrt{7^3}\, 3\sqrt{7^2}$ = 7^(3/3) 7^(3/3) $3\sqrt{7^2}$ = (7^1)(7^1) $3\sqrt{7^2}$ =

= 7^2 $3\sqrt{7^2}$ = 49 $3\sqrt{7^2}$ = 49 $3\sqrt{49}$

RADICALS WHOSE FACTORS ARE ALL "PERFECT POWERS" OF THE INDEX

The above title is short-hand for saying radicals whose factors are all raised to the same indicated power that is also equal to the index of the radical. This radical can be easily rationalized or simplified by multiplying the base of each power in the radicand's factors.

Ex. $2\sqrt{(4^2)(3^2)}$: here, the index is 2, and each base is raised to the 2nd power.
      (4) (3)
        12

checking: $2\sqrt{(4^2)(3^2)}$
         $2\sqrt{(16)(9)}$: 16 and 9 are "perfect powers", here, perfect squares which are due to values raised
         $2\sqrt{144}$ to the second power. Since the index is also 2, these radicand values, when expressed
          12   as powers of two, can be used to easily rationalize (ie. remove) the radical expression.

You can also perform a check by "distributing" the index to each factor of the radicand when converting them to their equivalent or corresponding exponential form so as to rationalize the radical expressions.

$2\sqrt{(4^2)(3^2)}$
$2\sqrt{4^2}\ \ 2\sqrt{3^2}$ or: $2\sqrt{4 \times 4}$ $2\sqrt{3 \times 3}$
$(4^{(2/2)})\,(3^{(2/2)})$ $\sqrt{4}\ \sqrt{4}$ $\sqrt{3}\ \sqrt{3}$
     (4)   (3)   4   3
       12   12

Another check is this:

$2\sqrt{(4^2)(3^2)}$
  $2\sqrt{(4 \times 3)^2}$
    $2\sqrt{12^2}$ $= \sqrt{144}$
      $12^{(2/2)}$
       12

EXPONENTS THAT ARE LESS THAN ONE OR ARE FRACTIONS

Exponents can be any value, they need not be only integers (essentially the familiar basic counting values). This process of dividing the exponent of the radicand by the index leads us directly to exponents that are fractions. This is also a method of rationalizing radicals such as shown on the right hand side of the following equation:

$$\sqrt[index]{base\textasciicircum exponent} \quad = \quad base\textasciicircum(exponent/index)$$

: a general format for converting between radical and exponential expressions. Note that a power or exponential expression can be a radicand of a radical. Note also that both sides represent a root.

Ex. $\sqrt[2]{5\textasciicircum2}$ = $\sqrt[2]{25}$ = 5^(2/2) = 5^1   Ex. $\sqrt[3]{1000}$ = $\sqrt[3]{10\textasciicircum3}$ = 10^(3/3) = 10^1 = 10

Since a number (ie. base) raised to the first power is still that number, it should be of no surprise that when a number is raised to a value less than one (ie. a fraction), that the resulting (power) value is smaller than that number, hence we find it useful as another way to express some root of a number.

Ex. $\sqrt[2]{5\textasciicircum1}$ = 5^(1/2)   : "five to the one-half (1/2 = 0.5) power is equal to the square root of five"

$$\sqrt[2]{5} \quad = \quad 5\textasciicircum0.5$$

Hence, we have found a way to convert a radical form of a root to it's equivalent exponential form and vice-versa.

Ex. $\sqrt[2]{25\textasciicircum1}$ = 25^(1/2) or= 25^0.5 = 5

Ex. 27^(1/3) = $\sqrt[3]{27\textasciicircum1}$: = 3

Ex. 27^(2/3) = $\sqrt[3]{27\textasciicircum2}$: = 9

Ex. $\sqrt[2]{5\textasciicircum3}$ = 5^(3/2) : = 5^1.5

Ex. 81^(1/4) = $\sqrt[4]{81}$: = 3

Ex. $\sqrt[3]{(5\textasciicircum3)\textasciicircum4}$ = (5^3)^(4/3)   also, by using the "power to a power" rule:

(5^3)^(4/3) = 5^((3/1)(4/3)) = 5^12/3 = 5^4

In general, if the index is less than the indicated power of the radicand, the result is a power of the radicand, and if the index is greater than the indicated power of the radicand, the result is a root of the radicand.

Ex. $\sqrt[0.5]{10\textasciicircum2}$ = 10^(2/0.5) = 10^4 = 10000

$\sqrt[2]{10\textasciicircum2}$ = 10^(2/2) = 10^1 = 10

$\sqrt[3]{10\textasciicircum2}$ = 10^(2/3) = 10^0.66$\underline{7}$ = 4.64$\underline{2}$

$\sqrt[4]{10\textasciicircum2}$ = 10^(2/4) = 10^0.5 = 3.16227766...   : also = 10^(2/4) = 10^(1/2) = $\sqrt[2]{10\textasciicircum1}$

A RADICAL TO A POWER

One way to rationalize a radical (ie. an expression for a root) which is raised to a power, is to first convert the radical to its equivalent exponential form. Then use the "power to a power" rule where the exponents are distributed (ie., multiplied).

Ex. $(2\sqrt{3})$^5

$(2\sqrt{3^1})$^5   converting the radical to exponential form:

$(3^{(1/2)})$^5   "power to a power", "distribute" (multiply) the exponents:

$3^{(\ (1/2)(5/1)\ )}$

$3^{(5/2)}$

Also, converting $3^{(5/2)}$ back to radical form:

$3^{(5/2)}$ = $2\sqrt{3^5}$

Hence, $(2\sqrt{3})$^5   = $2\sqrt{3^5}$ this equivalence leads to this following statement:

A radical and radicand raised to the same power are equivalent in value. The base of the radicands, and the indexes of the radicals must also be the same. The reasoning behind this equivalence is due to the fact that multiplication is commutative. This can be seen in their exponential (power) form before the exponents are multiplied:

$(3^{1/2})^{(5/1)}$ is the same as: $(3^{5/1})^{(1/2)}$, both expressions are equal to: $3^{(5/2)}$

Using the above equivalence, this result can also be seen if the exponents are distributed:

$3^{(\ (1/2)(5/1)\ )}$ = $3^{(\ (5/1)(1/2)\ )}$ = $3^{(5/2)}$

Before leaving this discussion, it must be pointed out that converting an exponent that is an improper fraction to a mixed number can be useful.

Ex.   First: $7^{3/2}$ = $2\sqrt{7^3}$ = $(2\sqrt{7})$^3   = $7^{1.5}$

Now, by converting the exponent to a mixed number:

Since $3/2$ = 1.5 = $1 + 0.5$ = $1 + 1/2$, we have:

$7^{3/2}$ = $7^{(1 + 1/2)}$ = $(7^1)(7^{1/2})$

$7^{3/2}$ = $7\sqrt{7}$ = $7^{1.5}$

A RADICAL RAISED TO A POWER EQUAL TO THE INDEX

The root of a radical that is raised to a power equal to the index is equal to the radicand value of that radical.

Ex. $(2\sqrt{5})^2 = 5$

checking: $(2\sqrt{5})^2$ placing the radical into exponential form :
         $(5^{(1/2)})^2$ with a "power to a power", distribute the exponent :
         $5^{(2/2)}$
         $5^1 = 5$

Or: $(2\sqrt{5})^2$
     $(2\sqrt{5})(2\sqrt{5})$: "extended method" (it should also be obvious here that the square root of the
                               same value, is that value, here 5)

     $2\sqrt{(5)(5)}$: from multiplying "like" radicals (have same index),
                           simply multiply their radicands

     $2\sqrt{5^2} = 2\sqrt{25} = 5$ or:
     $2\sqrt{5^2} = 5^{(2/2)} = 5$

Note that this is very similar to where the root of a radicand in which the radicand is raised to an indicated power equal to the index:

$(2\sqrt{5})^2 = 2\sqrt{5^2} = 5$: Note also for example, that the square (2nd power) of the square-
                                      root of number (here a radicand) is equal to that number. If we
                                      let letter (r) represent or equal the square root of a number (N), then
                                      $(r \times r)$ or r^2 will equal that number: This is verified here
                                      mathematically:

            $r = \sqrt{N}$ then

            $r \times r = \sqrt{N} \ \sqrt{N}$: Multiplying both sides by r or $\sqrt{N}$, or
                                      either (since they are equal in value),
                                      still keeps both sides of the equation in
                                      balance. Note that this is therefore also
                                      the same as simply squaring each side:

            $r^2 = (\sqrt{N})^2$ "extending" the right hand side:

            $r^2 = \sqrt{N} \ \sqrt{N} = N^{1/2} \ N^{1/2} = N^{(1/2 + 1/2)} = N^{2/2}$

                                      or $= N^{(0.5 + 0.5)} = N^1$
            $r^2 = N$

         If r was the cube root of N:

           $r = 3\sqrt{N}$ raising each side to the third power:

          $r^3 = 3\sqrt{N} \ 3\sqrt{N} \ 3\sqrt{N} = (3\sqrt{N})^3 = N^{1/3} \ N^{1/3} \ N^{1/3} = N^{(3/3)}$

          $r^3 = N$

THE ROOT OF A FRACTION

The root of a fraction is equal to the root of the numerator divided by the root of the denominator. The indicated root or index of the roots will also be the same.

Ex. $2\sqrt{\dfrac{36}{4}}$ = $\dfrac{2\sqrt{36}}{2\sqrt{4}}$ = $\dfrac{6}{2}$ = 3

Checking using the order of operations more closely where the value (ie. radicand) underneath the radical is simplified first since the division symbol is an inner grouping symbol for this example, and the radical symbol is an outer grouping symbol:

$2\sqrt{\dfrac{36}{4}}$ = $2\sqrt{9}$ = 3

Here is another check by first converting the radical to exponential notation:

$2\sqrt{\dfrac{36}{4}}$ = $\left[\dfrac{36}{4}\right]^{\wedge}(1/2)$ = $\dfrac{36^{\wedge}(1/2)}{4^{\wedge}(1/2)}$: After using the fraction to a power rule of:
     : (Distributing the exponent to both the num. and den.)

= $\dfrac{\sqrt{36}}{\sqrt{4}}$ = $\dfrac{6}{2}$ = 3

RATIONALIZING FRACTIONS THAT CONTAIN RADICALS

There are two basic methods to rationalize (to make sense of) or rid fractions that contain radicals: The equivalent fraction method, and the equivalent radicand method. They are somewhat similar, but different names are used to distinguish them. This discussion also shows how to rationalize a radicand that is a fraction. The main goal in this process is to create a "perfect (indicated) power" of the index , which can then be easily rationalized.

Equivalent Fraction Method:

First, convert both the numerator and denominator to radicals. This creates a fraction of two radicals. Then an equivalent fraction is made by multiplying the fractions numerator and denominator by the denominator raised to a power such that the resulting radical in the denominator will be raised to an indicated power that is some multiple of the index. This denominator, when simplified, is equal to the radicand.

Ex. $3\sqrt{\dfrac{2}{3}}$ = $\dfrac{3\sqrt{2}}{3\sqrt{3}}$ = $\dfrac{3\sqrt{2}}{(3\sqrt{3})^{\wedge}1}$ Multiplying the numerator and denominator by $(3\sqrt{3})^{\wedge}2$:

$\dfrac{2\sqrt{2}\ (3\sqrt{3})^{\wedge}2}{(3\sqrt{3})^{\wedge}1\ (3\sqrt{3})^{\wedge}2}$ = $\dfrac{3\sqrt{2}\ (3\sqrt{3})^{\wedge}2}{(3\sqrt{3})^{\wedge}3}$ After converting the radical in the denominator to its equivalent exponential form and simplifying. $(3\sqrt{3})^{\wedge}3 = 3^{\wedge}(1/3)^{\wedge}3 = 3^{\wedge}(3/3) = 3^{\wedge}1$:

$\dfrac{3\sqrt{2}\ (3\sqrt{3})\ (3\sqrt{3})}{3^{\wedge}(3/3)}$ = $\dfrac{3\sqrt{18}}{3}$: In the numerator, the radicands were multiplied since the index (indicated root) is the same for each radical.

Equivalent Radicand Method:

$3\sqrt{\dfrac{2}{3}}$ = $3\sqrt{\dfrac{2\ (3^{\wedge}2)}{(3^{\wedge}1)(3^{\wedge}2)}}$ = $3\sqrt{\dfrac{(2)(9)}{3^{\wedge}3}}$ = $\dfrac{3\sqrt{18}}{3\sqrt{3^{\wedge}3}}$ = $\dfrac{3\sqrt{18}}{3^{\wedge}(3/3)}$ = $\dfrac{3\sqrt{18}}{3}$

The above concepts should be taken into general account and especially when rationalizing (essentially simplifying) fractions containing radicals, or rationalizing radicals whose radicand is a fraction.

A CAUTIONARY NOTE WHEN USING POWERS AND ROOTS WITH FRACTIONS

When creating an equivalent fraction, it is incorrect to "raise" the numerator and denominator to the same power, perhaps in an attempt at creating an equivalent fraction.

Ex. $\dfrac{4}{8}$ /=/ $\dfrac{4^2}{8^2}$ or $\left(\dfrac{4}{8}\right)^2$: /=/ or != are symbols meaning "is not equal to" or "not equals"
Often, this symbol is used: $\neq$

Checking by taking the decimal form of both sides:

0.5   /=/   0.25   : This is so since the numerator and denominator are not actually being multiplied by the same identical value when you raise them to the same power. For this example, the numerator was multiplied by 4, and the denominator by 8.

Likewise, when creating an equivalent fraction, it is incorrect to take the same indicated root of both the numerator and denominator.

Ex. $\dfrac{16}{64}$ /=/ $\dfrac{\sqrt{16}}{\sqrt{64}}$ or $\sqrt{\dfrac{16}{64}}$: A fraction, and the root of a fraction are two different values, and not equal. This is similar to above example, but the indicated power that the numerator and denominator values would be raised to would be 0.5 or= 1/2.

Checking by taking the decimal form of both sides:

0.25   /=/   0.5

ADDING RADICALS

Radicals that have the same index and radicand are said to be "like" or similar radicals. Since the radicals represent the same value or quantity, simply add their numerical coefficients (the multiplying factor(s)). If a numerical coefficient is not shown, it is considered as being one (1), and we know that multiplying anything by one is alright since it does not change it's value.

Ex. $\sqrt{9} + \sqrt{9} \quad = \quad (1)\, 2\sqrt{9} + (1)\, 2\sqrt{9}$

$\qquad (1 + 1)\, 2\sqrt{9}$

$\qquad \quad 2 \quad 2\sqrt{9}$: This result should be easily seen since multiplication is really repeated addition of the same value. With further simplification:

$\qquad \quad 2\,(3)$

$\qquad \quad \ 6$

Checking:

$\quad 2\sqrt{9} + 2\sqrt{9}$ Order of operations, step 1 (simplify powers and roots) :

$\quad \ 3 \ + \ 3$ Order of operations, step 4 (additions and subtractions; to combine)

$\qquad 6$

Ex. $3\sqrt{7} + 2\sqrt{7}$

$\quad (3 + 2)\sqrt{7}$

$\qquad 5\sqrt{7}$: in general, $\sqrt{7}$ is to be added to itself (or to an initial sum of 0 if you will) 3 times, and then 2 more times for a total of 5 times, hence multiply by 5 since it is essentially being summed (to a starting reference value of 0) a total of 5 times.

Checking:

$\qquad \qquad 3\sqrt{7} + 2\sqrt{7}$ since multiplication is repeated addition:

$(\sqrt{7} + \sqrt{7} + \sqrt{7}) + (\sqrt{7} + \sqrt{7})$

$\qquad (1 + 1 + 1 + 1 + 1)\sqrt{7}$ After clearing grouping symbols, and expressing

$\qquad \qquad 5\sqrt{7}$ the sum of similar things or values.

MULTIPLYING RADICALS WHICH HAVE UNLIKE INDEXES

It is possible to multiply radicals that have unlike indexes. First, convert and express each radical into an equivalent exponential form, then the exponents can be changed to equivalent fractions with like denominators, and this is essentially creating like indexes for their corresponding radical forms.

Ex.　　$2\sqrt{2}$　$3\sqrt{4}$

$(2^{(1/2)})\ (4^{(1/3)})$　　　　: exponential forms of the radicals

Below, equivalent fractions of the exponents are created in a manner similar to that of adding fractions. A common denominator of 6 was chosen for the fractions and it is also the lowest common denominator (LCD) of the two.

$$\frac{1}{2} = \frac{3}{6} \qquad \text{and} \qquad \frac{1}{3} = \frac{2}{6}$$

Hence, $2^{(1/2)} = 2^{(3/6)}$　and　$4^{(1/3)} = 4^{(2/6)}$　　　: $2^{0.5}$　and　$4^{0.3333}$**...**

Converting each back to its radical form and multiplying:

$(2^{(3/6)})\ \ (4^{(2/6)})$

　$6\sqrt{2^3}$　$6\sqrt{4^2}$　　　: created radicals with like indexes

　$6\sqrt{8}$　$6\sqrt{16}$　$=$　$6\sqrt{(8)(16)}$　$=$　　　: multiplying radicands of radicals which have the same index

　　　$6\sqrt{128}$

THE ROOT OF A ROOT

The root of another indicated root is equal to a radical whose index is the product (ie., multiplication) of the index of each radical.

Ex.　$3\sqrt{2\sqrt{64}}$　　　　: note here that the inner radicand is actually another radical (indicated root to be taken)

$3\sqrt{2\sqrt{64}}$　$=$　$(3)(2)\sqrt{64}$　$=$　$6\sqrt{64}$

This is verified below:

Converting the inner radical to exponential form:

$3\sqrt{(64^{(1/2)})}$

Then this entire radical is converted to exponential form:

$(64^{(1/2)})^{(1/3)}$

By using the "(indicated) power to a power" rule of multiplying exponents, we distribute the exponents:

$(64^{(1/2)})^{(1/3)}$　$=$　$64^{(\ (1/2)(1/3)\ )}$　$=$　$64^{(1/6)}$

Converting this back to radical form, we get:

$64^{\wedge}(1/6) \; = \; 6\sqrt{64}$: checks

Due to multiplication being commutative, the exponents of a "power to a power" can be exchanged.

Ex. $(64^{\wedge}(1/2))^{\wedge}(1/3) \; = \; 64^{\wedge}(\,(1/2)(1/3)\,) \; = \; 64^{\wedge}(1/6)$

   $(64^{\wedge}(1/3))^{\wedge}(1/2) \; = \; 64^{\wedge}(\,(1/3)(1/2)\,) \; = \; 64^{\wedge}(1/6)$

Converting both expressions on the above left sides to radical form, we get:

$$3\sqrt{\,2\sqrt{64}\,} \;\; = \;\; 2\sqrt{\,3\sqrt{64}\,}$$

checking: $3\sqrt{\,2\sqrt{64}\,} \;\; = \;\; 2\sqrt{\,3\sqrt{64}\,}$ first simplifying the "inner radical" radicand:
   $3\sqrt{8} \;\; = \;\; 2\sqrt{4}$
   $2 \;\; = \;\; 2$

This concept can also be used to find some higher integer roots using only square or cube roots. For example, you can find the fourth (4) root of a number (N) by taking the square root of the square root of the number:

$4\sqrt{N} \;\; = \;\; 2\sqrt{\,2\sqrt{N}\,}$: the right hand expression is like a "factored index" form of a radical expression

The APPENDIX section has a similar method of how to calculate odd roots, such as cube (3rd) roots, using square roots.

The following discussion is somewhat of an advanced concept, and perhaps not used much in practice, but for completeness and familiarity, it will be presented since it is based on, and is a natural progression stemming from the previous discussions.

A radical that has an indicated power of a radicand, is equal to a radical having that radicand base value, and the index is divided by that indicated power.   Expressing this using words as symbolic identifiers or variables:

   $\text{index}\sqrt{\text{radicand}^{\wedge}\text{exponent}} \;\; = \;\; (\text{index}/\text{exponent})\sqrt{\text{radicand}}$

Also consider in this discussion:

   $\text{index}\sqrt{\text{radicand}^{\wedge}\text{exponent}} \;\; = \;\; \text{radicand}^{\wedge}(\text{exponent}/\text{index})$: the exponential form of the radical form

Here is the derivation of the first shown equation:

 For simplicity, let: i = index,   r = radicand,   e = exponent ,   z = 1/e , therefore e = 1/z

   $i\sqrt{r^{\wedge}e} \;\; = \;\; i\sqrt{r^{\wedge}(1/z)} \;\; = \;\; i\sqrt{\,z\sqrt{r}\,} \;\; = \;\; iz\sqrt{r} \;\; = \;\; i(1/e)\sqrt{r} \;\; = \;\; (i/e)\sqrt{r}$

Ex. $3\sqrt{10^{\wedge}2} \; = \; (3/2)\sqrt{10} \; = \; 1.5\sqrt{10} \; = \; 10^{\wedge}(2/3) \; = \; 10^{\wedge}0.666\underline{7} \; = \; 3\sqrt{100} \; = \; 4.641\underline{6}$

Checking, and a verification using what we know already:

   $(3/2)\sqrt{10^{\wedge}1} \;\; = \;\; 10^{\wedge}(1\,/\,(3/2)) \;\; = \;\; 10^{\wedge}(2/3) \; = \; 3\sqrt{10^{\wedge}2}$

EXPRESSING RADICANDS LESS THAN ONE WITH INTEGER RADICANDS

      This is an infrequently used method of calculating roots, but if you only have a table of the roots of some integers, it will be very useful. If you plan to calculate roots using other methods, such as those presented in this book, you will at least get another good lesson in mathematical manipulations here. First, represent the radicand in an equivalent proper fraction form, then represent this radical form as the division of two radicals. Placing a value less than one into a (proper) fractional form is discussed ahead in the topic of scientific notation (SN) if you need help doing so.

Ex. $\sqrt{0.5} = \sqrt{5(10^{-1})} = \sqrt{\dfrac{5}{10}} = \dfrac{\sqrt{5}}{\sqrt{10}}$: calculate the square root of 0.5 using the square roots of 5 and 10

Or: $\sqrt{0.5} = \sqrt{\dfrac{50}{100}} = \dfrac{\sqrt{50}}{\sqrt{100}} = \dfrac{\sqrt{50}}{10} = \dfrac{\sqrt{5}\,\sqrt{10}}{10}$

Here is another method to calculate this:

$\sqrt{0.5} = \sqrt{\dfrac{0.5^2}{0.5^1}} = \dfrac{\sqrt{0.5^2}}{\sqrt{0.5}} = \dfrac{0.5}{\sqrt{0.5}} = \dfrac{0.5\sqrt{1}}{\sqrt{0.5}} = 0.5\sqrt{\dfrac{1}{0.5}} = 0.5\sqrt{2}$

By observing the procedure for the last method above, a formula can be written. Letting N represent (be equivalent to) the radicand in question:

$\sqrt{N} = N\sqrt{\dfrac{1}{N}}$:The square root of a value is equal to that value times the square root of its reciprocal.
                                From this we also have: $1/N = (1/N)(N/N) = N/N^2$ = the reciprocal of a number

A check on this formula can be made by converting the radicals to their equivalent exponential form:

$\sqrt{N} = N^{0.5}$ and $N\sqrt{\dfrac{1}{N}} = N\dfrac{\sqrt{1}}{\sqrt{N}} = \dfrac{N}{N^{0.5}} = \dfrac{N^{1-0.5}}{1} = N^{0.5} = N^{(1/2)} = \sqrt{N}$

Here is a formula, that is somewhat similar to the above formula, that can be used to calculate roots. It is a good mathematical lesson, but is perhaps somewhat advanced or excessive, and the reader may simply skip over it.

$\sqrt{N} = \dfrac{1}{\sqrt{\dfrac{1}{N}}}$: The square root of N is equal to the reciprocal,
                                of the square root of the reciprocal of N.

The derivation of this is as follows: The reciprocal of N is 1/N. To do the "reverse", we will express the reciprocal of 1/N to have N:

$\dfrac{\dfrac{1}{1}}{\dfrac{1}{N}} = N$: The reciprocal of the reciprocal of N is equal to N. After taking
                        the square root of both sides, we have the equation above.

The same should be true for any value, including square root values. The reciprocal of: $(1/\sqrt{N})$, or the reciprocal of the reciprocal of the square root of N is:

$$\cfrac{\cfrac{1}{1}}{\cfrac{1}{\sqrt{N}}} \quad = \quad \cfrac{\cfrac{1}{1}}{\cfrac{\sqrt{1}}{\sqrt{N}}} \quad = \quad \cfrac{\cfrac{1}{1}}{\sqrt{\cfrac{1}{N}}} \quad = \quad \sqrt{N}$$

: this can also of been solved by inverting the denominator, and creating an equivalent fraction.

Ex. $\sqrt{5} \quad = \quad \dfrac{1}{\sqrt{0.2}} \quad \sim \quad 2.23607$

: the "wavy" equals symbol: $\sim$ or $\sim=$ means "approximately" or "about" equal to. The reciprocal of 5 is 0.2

RADICANDS THAT VARY BY A CONSTANT FACTOR

This is a method of evaluating radicals where the radicands differ by a common factor. The result is basically the same as that of where a radical is equal to a product of radicals whose radicands are factors of the original radicand.   If the many radicands have a common factor, this common factor only needs to be evaluated once. This concept can be of great use if you are creating a table of roots.

Ex.   Here, a constant factor between the radicands is 10, and there are others such as 2 and 5:

Since $\sqrt{10} \ = \ 3.16227766$ approximately.   Now observe this radicand of 100 which is 10 times greater than 10:

$\sqrt{100} \ = \ \sqrt{(10)(10)} \ = \ \sqrt{10}\,\sqrt{10} \ = \ 3.16227766\,\sqrt{10} \ = \ (3.16227766)(3.16227766) \ = \ 10$

$\sqrt{90} \ = \ \sqrt{(10)\,(9)} \ = \ \sqrt{10}\,\sqrt{9} \ = \ 3.16227766\,\sqrt{9} \ = \ (3,16227766)(3) \ = \ 9.48683298$

$\sqrt{20} \ = \ \sqrt{(10)(2)} \ = \ \sqrt{10}\,\sqrt{2} \ = \ 3.16227766\,\sqrt{2} \ = \ (3.16227766)(1.414213562) \ = \ 4.472135955$

A general "formula" for this discussion can be written or expressed mathematically. Below, R is used to represent the initial radicand, N and F represent factors of the radicand, where F is also a common factor between several radicands that you already know the root of:

$\sqrt{R} \ = \ \sqrt{FN} \ = \ \sqrt{F}\,\sqrt{N}$: for the above examples, F = 10 and its root is 3.16227766

R, F and N are examples of what are called "variables" in algebra, as you can see, each is simply a (symbolic) mathematical representation or placeholder for an actual number value. The actual symbol(s), letters, or words used to represent a variable is sometimes called the variable identifier or variable name. A variable is a general representation of many possible numeric values and-or a particular value being solved for.

LOGARITHMS (LOGS)

For a logarithm (or log) mathematical operation, you are given in the expression a base value and a number (N) equal to some power of that base, and are to find the (indicated) exponent of the base that would make that base equal to this power value or "Number". **Try to think of a log as simply the operation to find an exponent, and-or that a log is equivalent to an exponent.** The word "logarithm" was from that table values or "entries", such as for exponents and power values of a given base that were often logged (written or entered) into a (quick "lookup", find) reference table (list). Instead of using a fancy symbol such as like how a square root symbol is, the symbol chosen for a logarithm operation is simply the world "log".

exponent = Log $_{base}$ Number   : GENERAL LOG EXPRESSION, EQUATION OR "FORMULA"

Note, in the expression above on the right side, number is not the exponent of base, but a power of the base, that is:

base^exponent = Number = the actual power value of the base.
         The left side of the equation is the indicated or expressed power value.

A simplified notation for a log expression, using simple variable identifiers or names, can be something like this:

$e = \log_b N$: b=base (of the power value), N=number or power value, e=exponent

If no base value is given, it is understood as being 10. Logarithms of base 10 are called "common logarithms" due to the commonly used decimal system that is based on 10.

If you want to view all the basic algebraic relationship between powers, roots, and exponents or logarithms, then see the paragraph called: Algebraically Expressing The More Advanced Mathematical Operations, that is further ahead in this book

Ex. Since 10^3 = 1000, if given just the base and the number (a, or the, power of the base), the notation used to solve for the necessary exponent, of that base so as it would equal that given power value, is the logarithm notation (expression):

exponent = $\log_{10} 1000$ or: exponent = log 1000 : when a base value is not indicated, it is
                    understood or considered as base 10, the
                    common log base.

exponent = 3 = $\log_{10} 1000$: since 10^3 = 1000   : 3 is the logarithm or exponent.
                     "The log of 1000, using a base of 10, is 3".

Frequently, a calculator with a logarithm function is used to find logs. Generally, "scientific" calculators have function keys for finding only common and natural logarithms. If the base is not 10 or (e = about 2.72) which is the base for "natural" logarithms what will be discussed in this book, these calculator functions cannot be immediately utilized. Still, most logarithms to be found in common everyday home and scientific usage have either a base of 10 or (e).

Ex. Solve $\log_2 8$

To solve this is to find what would be the exponent of 2 such that the resulting power value of it is equal to 8:

exponent = $\log_2 8$: here the base of the logarithm or power is 2, and the number or actual power value is 8

$3 = \log_2 8 \qquad$ since 2^3 = 8

The examples given were very simple and could have been done simply by inspection. More information on finding exponents will be given ahead in the discussion of: Finding Exponents With Logarithms.

The **Number in a logarithm expression must be a positive value greater than 0**. A logarithm (ie., the exponent value) will be a positive value greater than 0 when the number is greater than 1.   First, a logarithm is always zero when the number is 1: log 1 = 0 : here n=1, consider: 5^0 = 1 , ex. 7^0=1.   Now consider when (n) is greater than 1: Ex. log 100 = 2   : here n=100 is greater than 1, and which can be expressed as n>1. Now consider when (n) is less than 1 but still greater than 0 as required: Ex. log 0.5 = about -0.30103 : here n=0.5 is less than 1, and which can be expressed as n<1 or (n<1), and the logarithm value is therefore less than 0 and is negative in value.

Considering the above statement, some will say for example that given (-2)^3 = -8 , and therefore: $\log_{(-2)} -8 = 3 = \log_2 8$

, and the problem with this is the both the base and Number ambiguity (ie., uncertainty) of the signs of the base and number.   Also consider the base ambiguity of this example: $\log_{-2} 4 = 2 = \log_2 4$

Without the aid of a modern **electronic calculator**, logarithm tables or mathematical calculation, a mechanical device or simple mechanical calculator or computer (that which computes, performs or solves computations) called a **slide-rule** can (only) give a good approximation (typically only a couple of decimal places) of a logarithm or anti-logarithm (finding the number or power value of the base). The slide rule gets its name from that it is similar to two rulers constructed to be adjustable by sliding one next to (ie., parallel) or against the other. Basically, a result is indicated by the corresponding value directly opposite and indicated on the other (logarithm) ruler scale.   The slide rule is also capable of other mathematical calculations. The longer the slide rule is, the more precision capable in the result since more subdivisions (smaller fractional parts) can be drawn or fitted on the indicated scales.

SOLVING FOR THE BASE OF A POWER

Given a power value of a base value, and it's indicated exponent, what is the corresponding base? An easy way to solve for the base is to place the values into exponential form.

$4 = \log_{base} 16$: logarithmic equation form , exponent = $\log_{base}$ power : power = number

base^4 = 16 : exponential equation form , base^exponent = power : power = number

Since the base is being raised to the fourth power, by taking its root with an index equal to the indicated power (the exponent), the base can be isolated or solved for. This is clearly seen once the radical is placed into exponential form. This operation must be done to the entire values or expressions on both sides of the equal sign of an equation so as to keep the equivalence or balance of both sides. More will be said about this during the discussion of equations. Since a base is effectively the "origin" or "root (of a power)" that the power value (ie., Number) is created or based upon, it should be reasonably obvious that a root operation on this power value will be needed to solve for the corresponding base.

Base^4 = 16 expressing the taking of the 4th root of both sides:

$\sqrt[4]{base^4} = \sqrt[4]{16}$ expressing or converting the left side to its equivalent exponential form

base^(4/4) = $4\sqrt{16}$: exponential form (a rationalized form)

      base = 2

hence,   4 = log 16   or   2^4 = 16
                 2

FINDING THE NUMBER (POWER VALUE OF THE BASE)

Ex. Find number or power value given: 4 = log number

Here, the base value is understood as the common base of 10. Expressing this in exponential form:

   10^4 = number

Solving for the fourth power of 10 using repeated multiplication, or "extending" the indicated power:

10^4 = (10)(10)(10)(10)
10^4 = 100(10)(10)
10^4 = 1000(10)
10^4 = 10000

number = 10,000

Expressing the indicated power 10^4 in log form:

4 = log 10,000   or = log 10^4
        10   10

Ex. Find number given: log   number = _1_
                       81   4

Again, placing this information into exponential form is the best way to start:

   81^(1/4)   = number   or: 81^0.25 = number   : 1/4 = 0.25

Since the indicated power (the exponent) is not an integer, number (the power) cannot be solved for by repeated multiplication. Converting this "fractional power" to its equivalent radical notation:

   81^(1/4)   = $4\sqrt{81^1}$ = $4\sqrt{81}$ taking the fourth root of 84:

$4\sqrt{81}$ = 3 = number   : root form   or by switching sides (formally known as the Symmetric Law):

   number   = 3

Hence,   log 3 = _1_   = 0.25   :log form   and   81^0.25 = 3   : exponential form
              81   4

THE LOG OF 1

The log (exponent) of any base to equal a number (the power value of the base) with a value of 1, is zero (0). This can be shown from the fact that a base with an exponent of zero is equal to one.

Ex. since $10^0 = 1$: extra, 10^0 is also the weight of the ones (1's) column or digit position

Converting and expressing this exponential form to its equivalent logarithmic form, we get:

    $0 = \log_{10} 1$: using 0 or zero is necessary since if $10^1 = 10$, 10^0 is needed to express 1
                                   When the number is less than 1, specifically between 0 and 1, the log of it will be negative, hence less than 0.

AN EXAMPLE OF LOGS IN EVERYDAY LIFE

The human ear is said to hear or respond to a sound's intensity or energy level it receives in a logarithmic type of manner, and not a linear or "direct" equivalent manner. Light intensity seen is another example of a logarithmic-like sensing or quantification by the body of outside stimulus or input. For example, the sound energy from someone talking may have hundreds of times more sound energy than someone whispering, however, the human ear will only hear or represent this as only several times louder. Placing some examples in logarithmic form where the number (a power of the base) represents the amount of sound energy (actual physical power) or intensity, and the exponent (or logarithm) numerically represents what energy or volume level that the human ear will apparently or effectively hears, senses or perceives.

Apparent volume or energy heard expressed   Number represents actual amount of energy.
as the log of the actual sound energy

               (Logarithm) ,   (Number)   : logarithm = exponent of the base, and here, base = 10

           pin drop: 1 = log 10
            whisper: 2 = log 100
            talking: 4 = log 10,000
            yelling: 6 = log 1,000,000

Clearly it is much easier to express the large range of number values (that change greatly or exponentially) into a highly "compressed", linear-like, and more manageable form such as the logarithmic (ie. exponent) values and representation presented. For example, note that the number of 100 is 10 times (magnification) more than 10, or an increase of 90 from a value of 10, but the log value (or exponent) only increased only by 1 to represent this increase. It could be said that logarithms are "slow (and get slower and slower) growing", and powers are "fast (and get faster and faster) growing".

Extra: In electronic and energy studies, vast increases, gains or comparing two values can be easily expressed and understood as logarithmic or "compressed" values.   The common formula used was initially developed for use with the Alexander Graham Bell's telephone system:

A gain, increase or magnification factor can be expressed as: Gain = (output / input), but as we know, gains can be very large and unwieldy, so Bell made a way to compress these values to be more manageable.

Gain in Bel units: log (output / input) Bels   : initial Bel unit , and this after an accepted, standardized revision by others, this becomes is a tenth of a Bel , hence a deciBel. **1 Bel =10 deciBells** = 10 (0.1Bel) = 1Bel

The deciBel unit then redefined as: log (output / input) deciBels = dB , and the Bel unit is then ten times this amount:

   1 Bel = 10 (1 dB) = 10 log (output / input) dB : this is equivalent to: log (100)(output/input)) dB , the value of 100
                                                         effectively magnifies the gain before the log is applied, and this is
                                                         good for small gains less than 1.

The Bel unit is also then defined such that a real gain of 10 would be expressed as a gain of 1 Bel or= 10dB.
A real gain of 2 is expressed as a gain of 0.3Bel = 3dB

For example: If the output power doubled (ratio=2):   10 Log 2 = 10 (0.30103) = 3.0103 dB =~ 3db in gain
                 If the output power increased by one-million: 10 log (1000000) = 10 (6) = 60 db in gain

To find the actual gain or gain ratio given an amount of dB gain:

   Gain = 10^(gain in dB / 10)   : (gain in dB / 10) then corresponds to the number of 0's following 1 in the result

Other popular use for logarithms are: earthquake intensity measurement, measuring the relative brightness or
"magnitude" of stars, and some types of electronic power measurement. Logarithms can also be used as an excellent
mathematical tool in any field of study where calculations need to be made, and especially for solving for an exponent.
The word "magnitude" essentially means greatness or the level of , and is related to the word magnification which means
an increase, bigger or larger.

FACTORING THE NUMBER IN A LOG

The log of a number can be expressed as a sum of the logs of the factors of that given number. This can be remembered
as the "(log) product rule" or "(log) factor rule". The log factors must also have the same base as the source logarithm.

   log number   = log factor_1 + log factor_2 + . . .   : . . . = "and so on" since there may be
                                                                              more than two factors

Ex.   log 100 = log 10 + log 10   : using the log factor or log product rule
             10   10   10

checking: 2   =   1   +   1

Also: log 100 = log (10 x 10) = log 10 + log 10 = 2   : shows 100 in factored form, or something like:

           log 100 = log (50 x 2) = log (25 x 4)   : showing some other factors of 100 for the Number

           log 100 = log 50 + log 2 = log 25 + log 4 = 2

The product rule in a reverse manner can be remembered as the "(log) sum rule". The sum of two logs having the same
base is equal to a log with the same base, and a number equal to the product of the numbers.

Ex.   log 8 + log 4 = log (8 x 4) = log 32   : log sum rule
             2   2   2   2

checking: 3   +   2   =   5   : since 2^3 = 8,   2^2 = 4 , and 2^5 = 32

THE LOG OF A FRACTION

The log of a fraction is mathematically equal to the log of the denominator subtracted from the log of the numerator. This can be remembered as the "log quotient rule", or "log of a fraction rule":

log (_numerator_)   =   log numerator - log denominator   : log quotient rule
    (denominator)

Ex.   log (1000) =   log 1000 - log 10   = 3 - 1 = 2
              (10)   10   10

checking: log 100   =   3   -   1   : simplifying first by considering the order of operations
              10   step 3; perform multiplications and divisions

              2   =   2

Ex.   log 1 = log _5_   : this is a particular and special example featuring the log of 1.
                      5

    log 1 = log 5 - log 5   combining like values or terms of log 5:
    log 1 = 0   : Clearly, whatever the value of an expression such as the log 5 is, if you
                                     subtracted it from itself, the difference is always 0. The log (regardless of the
                                     base used) of 1 is always equal to 0. This is the reverse of the exponential
                                     rule that the power of any base with an exponent of 0 is always equal to 1.

                    0 = log   1   and   base^0 = 1   : using any base for the log, and any
                            base   base for the indicated power

A cautionary note about the log quotient rule: This rule can only be used if the number in question is a fraction.   If the numerator and denominator are both logarithm expressions, the log quotient rule cannot be used. It could, perhaps, be used separately on the numerator and denominator.

Ex.   The log rule cannot be used on this expression:

    log 1000   /=/   log 1000 - log 10   =   3 - 1   =   2   : (incorrect method)
    log 10

    This is also incorrect:

    log numerator   /=/   log (log numerator) - log (log denominator)   : (incorrect method)
    log denominator   Since this value is actually the logarithm of
                                                                                the result, raising the base used to this
                                                                                indicated power, as for finding an anti-
                                                                                logarithm (or the Number), will yield the
                                                                                result.

This correct result is found if the order of operations is followed more closely where the numerator and denominator are simplified separately:

    log 1000   =   _3_   =   3   : (correct method)
    log 10   1

Mathization   Edition 1.0

The reverse of the quotient rule can be remembered as the "log difference rule". The difference of two logs having the same base is equal to a log with the same base and whose number is the quotient of the minuend number divided by the subtrahend number.

Ex. $\log_3 27 - \log_3 9 = \log_3 \left(\dfrac{27}{9}\right) = \log_3 (3) = 1$: log difference rule

checking: $3 - 2 = \log_3 (3) = 1$

THE LOG OF A POWER

The log of a number raised to a (indicated, with an exponent) power is equal to the log of that number times the exponent of that number. This can be remembered as the "log exponent rule" or "log of a power rule". The result is that the indicated (with an exponent) power of a number (N), of which the log is to be found, becomes a numerical coefficient (ie. a multiple, or multiplying factor) of the log of just the base value of that indicated power.

$\log \text{Number}^{\wedge}\text{exponent} = (\log \text{Number})(\text{exponent})$ or $= (\text{exponent})(\log \text{Number})$, expressed symbolically:

$\log N^{\wedge}e = e \log N$: LOG (INDICATED) POWER RULE, or the
                                                LOG EXPONENT RULE

Ex. $\log_2 4^{\wedge}3 = 3 \log_2 4$: log (indicated) power rule, or the log exponent rule

checking: $\log_2 64 = 3(2)$

$6 = 6$

Here is a verification of this rule:

log 5^3	"extending" the indicated power:
log ((5)(5)(5))	using the log product rule:
log 5 + log 5 + log 5	Since multiplication is repeated addition, adding
	their numerical coefficients (here, 1, for each) we have:
(1) log 5 + (1) log 5 + (1) log 5	: like values, add their numerical coefficients:
(1 + 1 + 1) log 5 = 3 log 5	: the left side is also a "factored" form of the previous line, and
	the common factor to all the values summed is (log 5)

There is a simple rule when dividing two log expressions when the base of the logs are the same, and both bases of the numbers (N), expressed as indicated (with an exponent) powers, are also identical. Below, (x) and (y) represent any possible value for the indicated exponents of both numbers that are expressed, indicated powers and which have the same (power) base (here indicated as A) of that indicated power:

$\dfrac{\log A^{\wedge}x}{\log A^{\wedge}y} = \dfrac{x}{y}$ this is easily verified using the log exponent rule:

$\dfrac{\log A^{\wedge}x}{\log A^{\wedge}y} = \dfrac{x \log A}{y \log A} = \dfrac{x}{y}$: a value divided by itself is 1, or after canceling out the common factors (here, log A) to both the numerator and denominator.

Even if the base of the indicated power of N is not identical, if you can convert and express N as a power of the other base, the above method will work. In the following example, the base of the expressed power of 2 is converted to its equivalent value having a base of 4:

Ex. $\dfrac{\log 4^3}{\log 2^3}$ = $\dfrac{\log 4^3}{\log (4^{0.5})^3}$ = $\dfrac{\log 4^3}{\log 4^{1.5}}$ = $\dfrac{3}{1.5}$ =   2

Note that $4^{0.5}$ = $\sqrt{4}$ = 2

checking:

$\dfrac{\log 4^3}{\log 2^3}$ = $\dfrac{\log 64}{\log 8}$ = $\dfrac{1.806179974}{0.903089987}$ =   2

Another method to simply the original equations is with the exponential log rule:

$\dfrac{\log 4^3}{\log 4^{1.5}}$ = $\dfrac{3 \log 4}{1.5 \log 4}$ = $\dfrac{3}{1.5}$ = 2

Actually, since the numerator and denominator are both composed of a logarithm expression only, and with no indicated logarithm base, any logarithm base can then be consistently utilized in the equation for the correct solution. If you happen to use a base of 8, this can be solved easily since both Numbers (here 64 and 8) are a power of 8. There are a few other instances shown in this book where any base can be used. Below is a verification of this "any log base" concept:

Given any value, say expressed or represented as N1 (for Number 1), the division of two logarithms of it which always have the same two and different logarithm bases, always results in the same (constant value) quotient, say F.

Ex. $\dfrac{\log_2 64}{\log_4 64}$ = $\dfrac{6}{3}$ = 2   : N = 64, b1=2, b2=4

Ex. $\dfrac{\log_2 16}{\log_4 16}$ = $\dfrac{4}{2}$ = 2   : N = 16, b1=2, b2=4
                                                         Though the numbers differ in both equations, notice that
                                                         the result is always 2 when the same two and different bases
                                                         bases are used consistently.

Expressing this into a formula:

$\dfrac{\log_{b1} N1}{\log_{b2} N1}$ =   F   F is a constant when the Number is changed and when log bases are used consistently (ie., the same). Mathematically or algebraically:

$\log_{b1} N1$ = (F) $\log_{b2} N1$: We see that given N, the logarithms with different bases, will only vary by a constant factor of F (or 1/F) when one is divided by the other.

Given another value, say N2, and using the same bases, the quotient will still be F as shown in the last two examples above. This can easily be verified when N2 is some indicated power of N1 and using the log of an indicated power rule and canceling the common factors:

Let N2 = N1^x   since (x) can have any possible value, N2 can be any possible value, then:

$$\frac{\log_{b1} N2}{\log_{b2} N2} = \frac{\log_{b1} N1^x}{\log_{b2} N1^x} = \frac{x \log_{b1} N1}{x \log_{b2} N1} = \frac{\log_{b1} N1}{\log_{b2} N1} = F$$

We see that regardless of the Number used, given any two bases, there is a common or constant factor (ie. here a ratio value) between those bases.   Mathematically, in the algebraic sense (for a formula) we have:

$$\log_{b1} N2 = (F) \log_{b2} N2$$ dividing the two results, we have another interesting result:

$$\frac{\log_{b1} N1}{\log_{b1} N2} = \frac{(F) \log_{b2} N1}{(F) \log_{b2} N2} = \frac{\log_{b2} N1}{\log_{b2} N2}$$: Same two numbers (N1 and N2) but using another log base for both, will be equal.

Ex. $\frac{\log 25}{\log 5} = \frac{1.397940009}{0.698970004} = 2$: common base of 10,   and:

$\frac{\log_e 25}{\log_e 5} = \frac{3.218875825}{1.609437912} = 2$: base of (e), the "natural log base" (about 2.71)

This is further verification that different bases (when used consistently within a problem) can still be utilized to achieve the correct result. Sometimes a log with a base of 10 is converted into one having a log base of (e), and vice-versa. Here, the value of the Number is adjusted so as the actual proper or true result is maintained.

THE LOG OF A POWER OF A PRODUCT

        The logarithm of an indicated (with an exponent) power of an expressed product of two or more factors can be expressed as equal to the sum of logarithms where each number (factor of the original expressed Number or product) of the log is raised to the same indicated power as the expressed product of those factors. This rule is similar to the log product rule.

  log Number^exponent   =   log ((factor_1) (factor_2) . . .)^exponent   =   log factor_1 + log factor_2 + . . .

Ex. log (1000 x 10)^2   = log (1000^2 x 10^2) = log 1000^2 + log 10^2
                                                log(1,000,000) + log (100)
                                                        6   +   2
                                                             8

checking: log (1000 x 10)^2   OR: log (1000 x 10)^2   OR= 2 log (1000 x 10)
                 log (10,000)^2   log (10,000)^2   2 (log 1000 + log 10)
                 log 100,000,000   2 log 10,000   2 (3 + 1)
                         8   2 (4)   2 (4)
                                                           8   8   : checks

If you were to write a "formula" for the above procedure it would look something like the expressions on both sides of the

equality ("equals") sign below where A and B represent the factors of the Number, and x represents the exponent or indicated power.

$$\log (AB)^x = \log(A^x\, B^x) = \log A^x + \log B^x$$

Using the log exponential rule on both of the equivalent expressions (an equation) above, we can arrive at other useful "log formulas".

$$\log (A^x\, B^x) = \log (AB)^x = x \log (AB) = \log A^x + \log B^x = x \log A + x \log B = x (\log A + \log B)$$

Ex.   5 log a + 7 log b   considering the log exponent rule, this can be expressed as:
        log a^5 + log b^7   considering the log factor and log product rule, this can be expressed as:
         log (a^5 b^7)

"SPECIAL" LOG RULE

When the base of the logarithm, and the base of the number are the same, the result or logarithm (ie., the exponent) is equal to the number's exponent.

Ex. $\log_{10} 10$ = $\log_{10} 10^1$ = 1   : special log rule

checking: 10^1 = 10

Ex. $\log_{10} 10^3$ = 3

checking: $\log_{10} 1000$ = 3

        3 = 3

A check can also be made using the (log) exponential rule:

  $\log_{10} 10^3$ = $3 \log_{10} 10$ = 3 (1) = 3

If we were to express this concept symbolically or algebraically, so as to be non-specific in values or showing a more "generalized" formula for any possible values used, we would have something like:

  $\log_N N^{exponent}$ = exponent   : here, the log base, and the base of the number are the same value of N

Knowing all of the above, it is easy to reason that when the Number is less than the base, that the exponent or log is less than one. When the Number is more than the base, the exponent or log is greater than one:

$\log_b N$ = 1   : when N = b , "When the Number (or power of the base) equals the base of the logarithm,
               the log equals 1".

Ex. log 10 = 1
        10

log N > 1   : when N > b , "when N is greater than b", the log is greater than 1
    b

Ex. log 100 = 2
        10

log N < 1   : when N < b , "when N is less than b, the log is less than 1".
    b

Ex. log 5 = 0.69897
        10

log 1 = 0   : when N = 1   You can now imagine that when the number is less than 1, (N<1), but still
    b   greater than zero, (N>0), since 0 is not a power value of any base value, that
                                     the log will be less than 0. This requires knowing about "negative or signed
                                     numbers" that are discussed further ahead in this book.

                        Ex. log 0.99 = - 0.00436481   : When (0<N<1), the log is negative or
                                                              less than 0. Here, since N is almost or
                                                              close to 1, the log is close in value to 0.

When the Number is greater than 0, and less than 1, (ie. between 0 and 1), this can be symbolically expressed as :

    log (0<N<1) = (<0)   : a value less than 0 is called a negative value, and is indicated
        b   with a negative sign, (-) preceding it, as shown in the example above.
                                   This will account for the logarithms of all the (fractional) values between 0 and 1.
                                   Negative values are used quite a bit when the temperature is below zero degrees
                                   = less than 0 degrees = < 0 degrees = < 0°. Negative value < 0. Ex. -1 < 0

The fundamental log rules described up to now will be verified in the ALGEBRA section of this book.

FINDING EXPONENTS WITH LOGARITHMS

Expressing and finding exponents with logarithms is perhaps the greatest use of logarithms.

Ex. Given $10^x = 1000$, find the exponent x.

First, take the logarithm (using any consistent base) of each side of the equation:

```
   10^x    =    1000
log 10^x   =    log 1000          using the "log exponent" rule:
x log 10   =    log 1000          now isolate (solve for) x:
```

Since log 10 is a factor of the left hand side (of the equation), remove it by dividing it by itself, effectively canceling or ridding it from that side and thereby isolating x so as it can be solved for. Of course, to keep both sides of the equation in equivalence or balance, this process must be done to both sides.

$$\frac{x \cancel{\log 10}}{\cancel{\log 10}} = \frac{\log 1000}{\log 10} = \frac{3}{1} = 3$$

: Note, you cannot use: log 1000 - log 10 = 3 - 1 = 2 , since both the numerator and denominator are logarithms.

Extra note: **The log of a negative numbers (less than 0) are undefined.**

If you were to write a generalized representative or algebraic formula for this procedure, it would look something like the following:

```
given      b^x   =   N          : Here (b) represents the base of the indicated power,  (x) is the indicated exponent
                                   of the base, and N represents the number or power value of that base.
       log b^x   =   log N      : after taking logs of both sides
        x log b  =   log N      : using exponential rule,  solving for (x):

         x   =   log N          : GENERAL FORMULA FOR SOLVING FOR AN EXPONENT
                 log b            ( given:  N  =  b^x )
```

Ex. Solve for x given:

$$x = \log_3 9$$

$$x = \frac{\log N}{\log b} = \frac{\log 9}{\log 3} = 2$$

: Use any base, but the result of 2 can be found by inspection when you consider that both numbers have a factor of 3 in them, and then using a common or consistent base of 3. Note also that:
log 9 = log (3x3) = log 3 + log 3 = log 3^2 = 2 log 3

FINDING POWERS WITH LOGARITHMS

Both of the processes for finding powers and roots with the aid of logs essentially results in the need for evaluating an anti-logarithm which is also known as an inverse logarithm. For anti-logarithms, you are to find the number (ie., power value) given the base and the exponent (ie. the logarithm). You can "think of" an anti-logarithm as the reverse of finding the logarithm of a number.

Ex. Evaluate: $N = 10^3$

Note that this example given can easily be solved by repeated multiplication, but this is not the case when the exponent is not an integer. This simple example was chosen to make the procedure clear.

$$N = 10^3$$ taking logs of each side, here, using a base of 10; the common log base:
$$\log N = \log 10^3$$ this can be expressed as:
$$\log N = 3 \log 10$$

Equating (3 log 10) equal to the required exponent of the base (here 10) to equal N (a power of that base):

$$N = 10^{(3 \log 10)} = 10^{(3 \times 1)} = 10^3 = 1000$$

In general, if you were to write a (algebraic) formula for this procedure, it would look something like:

$$N = b^x$$: N = power, b = base, x = exponent , taking the log of both sides of this equation:
$$\log N = \log b^x$$ or:
$$\log N = x \log b$$ therefore, according to the definition of logarithms:

$$N = 10^{(\log N)}$$ and:

$$N = 10^{(x \log b)} \quad or = \quad 10^{(\log b^x)}$$: A GENERAL FORMULA FOR FINDING POWERS
(where $N = b^x$)

$N = 10^{(\log N)}$ shown above is the general notation for finding antilogarithms (the number) of common (base 10) logarithms. Another, simpler derivation of this notation is from the log definition itself:

$$x = \log_{10} N$$: an expression to find the logarithm of a number using a base of 10, therefore:

$$N = 10^x$$ since $x = \log_{10} N$, and using substitution:

$$N = 10^{(\log_{10} N)}$$: = 10^x, and if any other log base (b) is considered, this formula would be expressed as:
$$N = b^{(\log_b N)} = b^x$$

Below are some more examples of evaluating powers with logarithms. Included in the examples is more terminology associated with the concepts of logarithms.

Evaluate 10^4 using logs.

$N = 10^4$ taking the log of both sides, we can use any log base, here we will use the common log base of 10:
$\log N = \log 10^4$
$\log N = 4 \log 10$
$\log N = 4 (1)$
$\log N = 4.0$

The 4, or the whole portion of the log of (10^4) is technically called the **characteristic** of the logarithm. The fractional portion of the logarithm (here it's 0) is technically called the **mantissa**. Both the characteristic and mantissa are part of the logarithm or exponent. Expressing this in a pseudo (or algebraic) mathematical statement:

log N = exponent = characteristic + mantissa   or: characteristic part of the log + mantissa part of the log

= characteristic.mantissa   : expressed symbolically as decimal mixed number

Therefore, using the inverse log, or anti-log, expression to find the number::

N = 10^logarithm = 10^exponent   : since   logarithm = log N , b^logarithm = base^exponent = N
                                                                                      b

N = 10^(characteristic + mantissa)   OR:

N = (10^characteristic)(10^mantissa)   therefore:

N = (antilog characteristic)(antilog mantissa)

Substituting values for the example just given:

N = (antilog mantissa)(antilog characteristic)
N = (antilog 0)(antilog 4)
N = (10^0)   (10^4)   or =   10^(0+4) = 10^4
N =   (1)   (10,000)
N = 10,000

Here is a small table to assist in understanding the characteristic and mantissa values (here, using a base of 10):

    10^0.0 = 1   : characteristic = 0   mantissa = 0.0 :   log 1   = 0.0

    10^0.5 = 3.16228   : characteristic = 0   mantissa = 0.5 :   log 3.1622$\underline{8}$ = 0.5

    10^1.0 = 10   : characteristic = 1   mantissa = 0.0 :   log 10   = 1.0

    10^1.5 = 31.622$\underline{8}$: characteristic = 1   mantissa = 0.5 :   log 31.622$\underline{8}$ = 1.5

    10^2.0 = 100   : characteristic = 2   mantissa = 0.0 :   log 100   = 2.0

    All the values from 10 up to 99.999... are said to have a characteristic of 1 when base 10 is
    being considered. That is, these values are essentially powers of 10 and the characteristic
    is the integer (whole number) of the indicated power. For all the values from 10 up to
    100, where the characteristic of these values is always 1. A log value having an integer
    value of 1 is characteristic or common of Numbers between, or in, the range from 10 to 100,
    and the mantissa value of their logarithms will increase from 0.0 to 0.999999... in this range.
    In short, the characteristic value of the logarithms remains the same (here 1) for this range of
    values, but the mantissa value of that logarithm increases from 0 to 1 as like a fraction of a
    unit does. The (common, base 10) logarithms of values in the range from 100 to 1000 have a
    characteristic of 2, and in the same manner, the mantissa will increase from 0.0 to
    0.999999... in this range.

    When you see a logarithm (ie. exponent) that has a characteristic of 1, you will know that the
    anti-logarithm or Number (the power value of the base raised to that indicated power) has

a value between 10^1 to 10^2 = 10 and 100. When you see a logarithm that has a characteristic of 2, you will know that the anti-logarithm or Number has a value between 10^2 to 10^3 =100 to 1000, and so on.

Ex. Find the common (base 10) antilog of 2.17

Here, 2.17 is a logarithm of some number, hence it's an exponent value. We see that the characteristic of this logarithm (or exponent) is 2 and the mantissa is 0.17. Since 2 is the characteristic, the corresponding Number is between 10^2 to 10^3 = 100 to 1000.

According to the definition of logs:

2.17 = Log N   and therefore:

N = 10^2.17   : N is the number or antilog.   Factoring this indicated power value:

N = (10^2)(10^0.17)
N = (100) (1.479108388)
N = 147.9108388

When solving for the log of a number, the characteristic portion of the log can easily be found by observation or by expressing the number in standard scientific notation (discussed further ahead in this book under the topic of SCIENTIFIC NOTATION which is a form of exponential notation with a base of 10) where the characteristic is equivalent to the whole part of the indicated (as in the exponent) power of 10, that is, it's equivalent to the log of this power of 10, and the mantissa is equivalent to the log of the other factor to the power of 10. Since this other factor (to the power of 10) is less than 10 in value, and the log of (10^1) is 1, its log, and therefore the mantissa, is always less than 1. Another similar method is to express the number using a slightly modified scientific notation where the number is not between 0 and 10, but always less than 1. This second method eventually requires that you know about adding signed numbers as shown below.

Ex.   Log of 50 can be found by:

Log 50 =   Log (5)(10^1)   or =   Log (0.5)(10^2)
Log 50 =   Log 5 + Log 10^1   or =   Log 0.5 + Log 10^2
Log 50 =   0.698970004 + 1   or =   - 0.301029995 + 2
Log 50 =   1.698970004   =   1.698970004

Again, whenever you see (log N), think of it as being an exponent value:
log N = logarithm = exponent = characteristic part + mantissa part = whole part + fractional part
For this example, on its left side, 1 is the characteristic, and 0.698970004 is the mantissa.

Notice that when a number is factored to any form of scientific notation that the digits remain the same, and only the power of 10 changes. If numbers of various values with the same digits in them are all factored to standard scientific notation, then all of them will have the same mantissa portion in their common logarithm. In respect to the above example:

Log 500 = Log (5)(10^2)
Log 500 = Log 5 + Log 10^2
Log 500 = 0.698970004 + 2
Log 500 = 2.698970004

For comparison, if the digits are the same, the mantissas of the logs are identical:

(antilog) N	(integer power of 10) characteristic	(fractional power of 10) mantissa
5	0	0.698970004
50	1	0.698970004
500	2	0.698970004
5000	3	0.698970004

For illustrative purposes, this same sort of "similar digit results" will occur when multiplying (or dividing) any two numbers with the same digits, for example:

$$\begin{array}{r} 123 \\ \times\quad 57 \\ \hline 7011 \end{array} \qquad \begin{array}{r} 1.23 \\ \times\quad 5.7 \\ \hline 7.011 \end{array} \qquad \text{: both products have the same digits}$$

In this book, you can find more about this subject in the topic of LOGARITHMS OF SIMILAR DIGITS

Ex.   Evaluate 2^1.5 using common logarithms.

This example will show that the indicated power value is converted to an equivalent power value, which has a base of 10 and a correctly adjusted exponent, that can be evaluated using common logarithm and antilogarithm tables, etc. For much of this book, the concepts of the characteristic and mantissa of a logarithm are not used much, but you should have an idea of what they are.

N = 2^1.53

log N = log 2^1.53	: after taking the common (base 10) log of both sides
log N = 1.53 log 2	: log power or exponent rule
log N = (1.53)(0.301029995)	
log N = 0.460575893	: characteristic = 0, and mantissa = 0.460575893

Since the characteristic is 0, the antilogarithm or the value of the number must be between 1 and 10 since 10^0 = 1 and 10^1 = 10. That is, 0 is a common "characteristic" for the exponents of the values being represented (with a power of a base of 10) between this range. It can also be stated as that the integer portion of the logarithm in question has this characteristic value. If the characteristic was 1, the number would be within the range of 10 and 100, but not including 100, or higher, since 10^2 = 100, and the characteristic would actually be 2.

N = (antilog mantissa)(antilog characteristic)	using common antilogs or powers of 10:
N = (10^0.460575<u>9</u>)(10^0)	multiplying like values or variables, add their exponents:
N = 10^0.460575<u>9</u> (1)	The underlined 9 means rounded to this position.
N = 10^0.460575<u>9</u>	

Usually, to save space and cost, common antilogarithm (base 10) tables range from 0 to 1 for the exponent, hence the antilog of the characteristic must be performed by inspection. Since the antilog of the characteristic is an integer power of 10, simply adjust (ie., move) the decimal point in the antilog of the mantissa in accordance with the indicated exponent in the antilog of the characteristic. It should also be pointed out that log tables are for, or thought as for, the log of numbers whose values are within the range of 0 to 1, or 0 to 10 (when the number is placed in standard scientific notation). It is also possible to use log tables in a sort of reverse type of process to evaluate antilogs (finding the Number value).

After evaluating 10^0.460575<u>9</u> :

N = 2^1.53 = 2.887858435   : correct to 6 decimal places due to the previous rounding above

Ex. Evaluate 250^2.5

     N = 250^2.5
log N = log 250^2.5
log N = 2.5 log 250

Let us try to express or place the Number into a value that's within a log table that goes from log 0 to log 1:

log N = 2.5 log (1000 x 0.250)
log N = 2.5 (log 1000 + log 0.250)
log N = 2.5 (3 + log 0.250)   After solving (or from a table) the log of 0.250:
log N = 2.5 (3 + (- 0.602059991))

Note, logarithms of values between 0 and 1 are always negative in sign. Negative in sign simply means that the value is less than zero. More will be said about negative numbers later. Since log 1 = +0, it is easy to reason that the log of a number less than 1, and between 0 and 1, is less than 0 and is indicated as being negative.   Continuing the above example:

log N = 2.5 (2.397940009)   : after combining (formally, it means joining, addition, summing) values in the
                                       second factor, and the result here was essentially due to subtraction.
log N = 5.994850022   : N = antilog (log N) = antilog 5.994850022 = 10^5.994850022

We see that the characteristic is 5 and the mantissa is 0.994850022. Using the basic concepts of logarithms, expressing the antilogarithm of both sides to solve for the number (N):

N = 10^5.994850022
N = 10^(5 + 0.994850022) = (10^5)(10^0.994850022)

Using a table to find the common antilog of 0.994850022, the result is: 9.882117688.   This value is to be multiplied by a power of 10 (here, indicated as the 5th power), and the easiest way to do that is to simply move the decimal point a number of places equal to the indicated exponent. Here, the decimal point will be moved 5 places rightward, and this is the same as multiplying by 10^5 = 100,000.

N = (10^5)(9.882117688)
N = (9.882117688)(100,000)
N = 988,211.7688

Ex. Evaluate 1.74^52

To find the value of this indicated power would be very tedious if performed by hand using repeated multiplication.

N   =   10^(x log b)   : basic formula, previously derived, for finding powers given the base and exponent.
N   =   10^(52 log 1.74)   factoring 1.74 to have a number less than 1 for solving with a log table:
N   =   10^(52 (log (10)(0.174)))
N   =   10^(52 (log 10 + log 0.174))
N   =   10^((52)(1 + (- 0.759450751))   =   10^((52)(1 - 0.759450751))
N   =   10^((52)(0.240549248))
N   =   10^ (12.50856091)
N   =   (10^0.50856091)(10^12)   using a scientific calculator or common anti-logarithm table:

N = (3.225231628)(10^12)   : this form itself would be typical of an acceptable answer
N ~ 3,225,231,628,000   : approximate, since the last three least significant digits were
                                    dropped or omitted and unused during the calculation using
                                    a 10 digit calculator, or by visual inspection moving the
                                    decimal point.

Note, as mentioned previously, that any base of a logarithm can be used to evaluate a power as long as it is consistent in the calculations. Below is an example of the concept using a base of (e) instead of 10. (e) is well defined and roughly equal to 2.718282, and is used as the base for "natural" logarithms which are often used to mathematically represent things that happen in nature and expressed scientifically or mathematically. Evaluating powers of 10 usually requires a table or calculator to solve, however, all powers of (e) can be solved directly from a formula which will be presented further ahead in this book.

From: N = b^x   taking logs (with any base) of both sides:

      log N = log b^x   and if the log taken had a base of (e) instead of 10,
                                this must be explicitly expressed as:

      ln N = ln b^x   ln indicates a natural logarithm, and is usually substituted for the word log when
       e   e   the base is understood as being (e), the natural logarithm base. This can be
                                also be simply expressed as:
      ln N = ln b^x   Solving for N considering the concept of antilogs:

       N = e^(ln b^x)   or from the "log of a power rule":

       N = e^(x ln b)   : Finding a power of a value using natural logarithms (given N = b^x).

The appendix contains a general method for finding any power.

FINDING ROOTS WITH LOGARITHMS

The process of finding roots with the aid of logarithms is similar to that of evaluating powers with logarithms.

Ex.   Evaluate $N = 3\sqrt{1000}$: here, N is the root value. You can use another variable identifier for N, like R.

First, let's convert this radical form to its equivalent exponential form.

   N   = $3\sqrt{1000}$
   N   =   1000^(1/3)   taking logs of both sides:

   log N   =   log 1000^(1/3)   or:

   log N   = $\frac{1}{3}$ log 1000   = $\frac{\log 1000}{3}$

Finding N by taking the antilogarithm:

   N   =   10^((log 1000) / 3)   =   10^(3/3)   =   10^1   =   10

If you were to write an algebraic formula for the above procedure, it would look something like this:

   N   = $x\sqrt{b}$: N = root, x = index, b = radicand
   N   =   b^(1/x)   : exponential form
   log N   =   log b^(1/x)   : taking logs of both sides

or: log N   = $\frac{1}{x}$ log b   = $\frac{\log b}{x}$

N = $x\sqrt{b}$ =   10^log N   =   10^((log b)/x)   **: A GENERAL FORMULA FOR FINDING ROOTS**
                                                                  using logarithms and antilogarithms

An important observation can be made here. The only difference between the (antilogarithm) notation for solving powers and roots is the placement of the variable(x). For powers, (x), the exponent of the indicated power, becomes a multiplier to the exponent of 10 (or any proper base used). For roots, (x), the root-index of the radical, becomes a divisor to the exponent of 10 (or any proper base used).

For powers: N   =   10^(x log b)

Ex. 5^2 = 10^(x log b) = 10^(2 log 5) = 10^((2)(0.698970004)) = 10^1.397940009 = 25

For roots, (x) (the index or the indicated root) becomes a divisor to the exponent of 10 (or the proper base used).

Roots: N   =   10^ ((log b)/x)

Ex. $\sqrt{25}$ = 10^(log b)/x = 10^((log 25)/2) = 10^(1.397940009/2) = 10^0.698970004 = 5

As like evaluating powers with any base, roots can be evaluated using any base as long as the base is used consistently.

Ex.. **N = $x\sqrt{b}$ = e^((ln b)/x)**   **: Formula for finding a root using a power base of (e), and the**
                                    natural logarithm base of (e).   b = radicand, x = index

(This space for book edits.)   Please make a helpful donation to the author of this book.

SIGNED NUMBERS

Signed numbers are numbers preceded by a positive (+) or negative (-) mathematical symbol. Numbers preceded by the negative symbol (-) are said to be negative numbers and have a value less than 0 or some other reference value.. Numbers preceded by the positive symbol (+) are said to be positive numbers and have a value greater than or equal to 0 or some other reference value. If a sign does not precede a number, or the sign is not explicitly indicated, the number is to be considered as a positive number. For example:

   5 = +5 = "positive 5"   and   -5 = "negative 5"

How can a value be less than 0? A value can be less than 0 when that value is, or is considered as, a negative change, decrease, loss, or reduction from another given value which is considered as the reference value that change value is in reference to. That is, the change is in reference to some other value. So that you can get a good feel or understanding of signed numbers, their are several discussions below.

A very common use of both positive and negative numbers is with temperatures. It is common in cold climate areas to have a temperature that is "less than 0 degrees" with either the Fahrenheit or Celsius temperature scales. For example, the temperature may be "negative (or minus) ten degrees", and is indicated as -10°. Since no other reference value was given, this -10 value is in reference to 0 degrees which is the reference value. A negative sign indicates that a value is less than the reference value being considered. Here is a helpful "number-line" that is a graphical or visual reference to consider about signed numbers: [FIG 18]

opposite direction
of measurement

one direction
of measurement

less than reference value   reference value   greater than reference value

The reference value can be any value, either positive or negative in sign. Usually when a reference or "starting" value is not given of which a number is in reference to, that number is understood to be in reference to a value of 0.

distance or length from
reference value

distance or length from
reference value

7 6 5 4 3 2 1 0 1 2 3 4 5 6 7

negative values

positive values

- 7 - 6 -5 - 4 - 3 - 2 - 1 0 +1 + 2 + 3 + 4 +5 +6 +7

The length or distance between any two values is calculated as the difference between those two values. For example: The difference between +3 and +1 is: (3 - 1) = 2. The difference between 1 and 0 is: (1 - 0) = 1. The difference between 0 and (-1) is: (0 - (-1) = 1. The difference between +1 and -1 is: (1 - (-1)) = 1 + 1 = 2. The difference between +3 and -1 is: (+3 - (-1)) = +3 + 1 = 4

- 7 - 6 -5 - 4 - 3 - 2 - 1 0 +1 + 2 + 3 + 4 +5 +6 +7
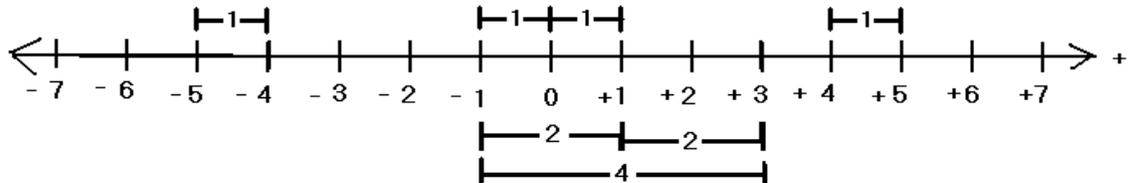
As you can see in the above drawing, the sign of a value also indicates its physical direction and-or location, and in a mathematical sense, this direction and-or location sign is either: "less than" (ie., - = negative in sign), or "greater than" (ie., + = positive in sign) a reference value.

If you had a starting or reference value of 5 things, and got 2 more things, this value of 2 can be expressed as :
   The change of, in, or applied to the value of 5 was an increase of 2 = positive 2 = +2

   If you had a starting or reference value of 5 things, and gave away 2 things, this value of 2 can be expressed as :
   The change of, in, or applied to the value of 5 was a decrease of 2 = negative 2 = -2

As we can observe above, signed numbers, in particular when they have a mathematical sign (+, -) symbol, give or assign a "direction" or further meaning to that number. This meaning could be that the number represents an increase or gain from an existing or reference value (including 0), and therefore it would be noted with a positive (+) sign or symbol. If the number represents a loss or decrease from any existing or reference value (including 0, or even a negative value), it would be noted with a negative (-) sign. In short, signed numbers are in reference to some existing or reference value, and not necessarily in reference to 0 only.

Since signed numbers represent gains or losses with respect to an existing reference value (0 if none was specified, or if the signed value is a final result of a computation), signed numbers can simply be considered as a "change-value" or simply as the change (as a verb: to modify, or have an affect, and as a noun: the amount of the modification or affect, the numeric change value or value of the change) from the reference value. If a value changed to 5 less, this change would be noted as -5 numerically. Of course, to find 5 less of a value, or a value reduced by 5, this would normally be expressed and processed as a subtraction operation: value - 5   Below, the right hand side of the equations represents expressions for the signed number concepts:

   value - 5 = reference_value - 5   : with signed numbers, value becomes a reference value
                                                  for the change or change-value to be applied to, and have
                                                  an affect on it

              = reference_value with the specific change applied to, added to, or combined (+) with it
              = reference _value + change_value

   value - 5 = (reference_value) + (-5)   : the sign of the change value is still attached to
                                                  and associated with it

              = new_value   : the resulting or result value due to a change

Considering this example, note that when signed numbers are considered, expressions such as:

   reference_value - 5 ,   are to be now considered or understood as actually indicating or meaning:

   (reference_value) + (-5)   =   reference_value (+) -5   : reference_value combined (+) with a decrease value

This is so, since when a value changes to a lower value, that value is said to have decreased, or the change in that value is said to be a decrease, and the specific amount of this change or the change value is mathematically said, expressed or represented as negative 5 = -5 , rather than positive 5 = +5 such as if the change was an increase. You can consider -5 as expressing a change or decrease of 5.

What looks like a subtraction expression, is now to be considered or understood to the be combining (as noted with the addition or combine symbol of: +) of signed numbers or values rather than a subtraction expression. The values being combined are called "signed (numeric) values" or "terms".   Note that in the original expression above, and in relation or reference to signed numbers, that the combine symbol (+) can be considered as "hidden"

or "understood", just as when the positive symbol (+) is considered "understood" when not explicitly indicated for a value greater than 0.

What if you added or combined two decreases or losses? The answer is that the net or total decreases or losses actually gets larger or greater in value when they are combined (essentially signed number addition, which is the addition of signed numbers). The decrease or loss would be said to be greater, a greater loss, hence it would be expressed or represented as a larger negative value.   In a notation-like form:

decrease + decrease = larger decrease   : the sum of decreases, so as to find the net or total decrease
  (-5)   +   (-5)   : add the losses together and keep the (common) sign that
                                               they are decreases or losses:
       -5 - 5   : here, shown without the "hidden" addition or combine symbol

   -(5 + 5) = -(10) = -1(10) = -10   : - (5 + 5) = -1(+5 + 5) = (-1)(5) + (-1)(5) = -5 - 5 = -10
                                            This can also be thought of as:

  - (total sum of [signless] decrease values)   :And it is also a factored form where (-1) was factored
                                             from each term of:
                                             (-5 + -5) = (-1)(5 + 5) = -1(5 + 5) = - (5 + 5) = -(10) = -10

Any positive number has an equivalent negative number counterpart (an (opposite) correspondence or resemblance), and vice-versa. Though the signs are different, they both have the same magnitude (essentially the "plain (signless)" distance or length), value, position or distance from 0. Below is a graphical representation of signed numbers using a "signed number line" graph. From the reference value, here 0, numbers in one direction are assigned (signed) as being positive, and for the opposite direction, they are assigned negative signs and are therefore called negative values, and which could simple be stated or "thought of" as numbers, lengths, distances, changes, etc., in the opposite direction:

```
  <--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------->
  - ∞ . . . -5      -4      -3      -2      -1       0      +1      +2      +3      +4      +5 . . . +∞
```

∞ is the "infinity" symbol which indicates that there is no limit to the possible values, hence "infinity" has no specific, constant or actual value, and is more of a concept for an unending process. Consider something that is "infinitely big" or "infinitely small". Outer-space is considered as infinitely big or unending.

A change can be either an increase or a decrease.   The opposite of an increase is a decrease. The opposite of a decrease is an increase. Increases and decreases are counterparts (or the "negative") of each other. If the change is an increase, the value of that change is noted or expressed with a positive sign: +value. If the change is a decrease, the value of that change is noted or expressed with a negative sign: -value.

You might now ask, how do you mathematically express the opposite of a positive value, and that is to take the negative, or opposite of it:

The opposite or negative of a positive value is: - (positive value) = negative value = -value

Ex. - (increase value) = - (+5) = -5.   , a decrease value   : this is the same as multiplying the initial value by
                                               negative 1 = (-1) = -1, as will be shown later.
                                               Consider that if you multiply a value by 1 that the result
                                               is still that same (unchanged) value:
                                               (value)(+1) = (value) ,and if you multiply that value by
                                               (-1), the result is: (-value), or the opposite, whether
                                               value itself was actually either positive or negative in
                                             sign as a signed value.

Likewise to mathematically express the opposite of a negative value, you also take the negative or opposite of it, and the result is a positive value:

The opposite or the negative, of a negative value is: - (negative value) = positive value = +value

Ex.   - (decrease value) = -(-5) = +5   , an increase value   : again, this is also the same as multiplying the value by
                                                                            negative 1 = (-1) = -1, as will be shown later

      Choosing any signed number value such as graphically expressed on the signed number line, even if it's a negative number, all values rightward of any (reference) value are said to be "more positive" or greater (larger) in value. Likewise, all values leftward, of any value on the line, are said to be "more negative" or less in value (since those values are headed closer to, or towards that direction on the signed number line).

Here is a signed number, or "change value", line positioned or aligned to be in reference to 7 on the common whole number line: [FIG 19]

whole number line, reference values

0   1   2   3   4   5   6   7   8   9   10   11 . . . +∞

- ∞ . . . -5   -4   -3   -2   -1   0   +1   +2   +3   +4   +5 . . . +∞

signed number line, change values
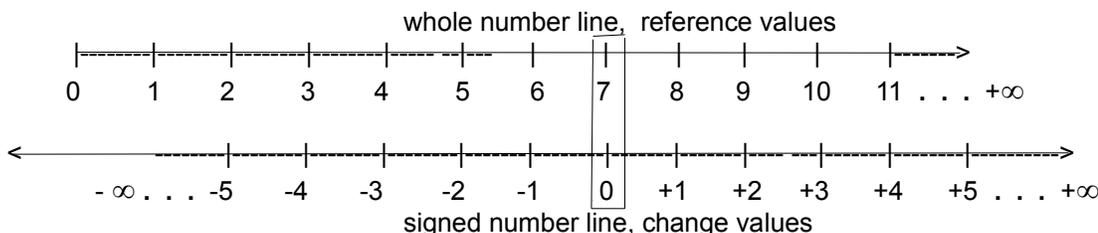
On the signed number line, or change number line, 0 means no change (yet) applied to the reference value chosen, and here, it's set or indicated as 7. To the left means a decrease or loss. To the right means an increase or gain.

Given a reference value of 7, if the change to 7 is a loss or decrease of 1, move to the left of 7 by 1 position for the result. This process can be indicated and automatically calculated mechanically, by first aligning the 0 change value with the reference value of 7. then move or slide the (amount of) change indicator ("pointer" or "cursor") to the (-1) position.   The result is indicated directly across from it on the other ruler or number line, and it is 6.
7 combined with a change of (-1) is 6.   7 + (-1) = 7 -1 = 6 :   [FIG 20]

whole number line, reference values

0   1   2   3   4   5   6   7   8   9   10   11 . . . +∞

- ∞ . . . -5   -4   -3   -2   -1   0   +1   +2   +3   +4   +5 . . . +∞

signed number line, change values
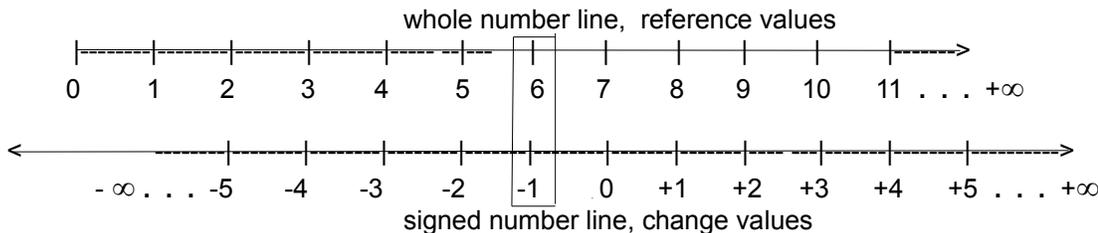
The above concept is also an example of a simple automatic mechanical calculator than can perform addition or subtraction and display the result visually. This construction is a type of "slide rule" system that you can construct with two rulers aligned to be in parallel. To the right of 0 on the signed number or change in values line, you could place a word such as "INCREASE", and to the left of 0, you could place a word such as "DECREASE". To uses this calculator, move or position the zero (or "no change") mark of the signed or changed number line directly across from the reference number in question. Though it may not be apparent, the difference between

any two corresponding values on these two number lines as set and shown above, has a value equal to the value in reference (the reference value), and for this example, it is 7:

reference value + change = sum   : the change can be either positive or negative.
      7   + (-1)   = 6   mathematically, if change is subtracted from both sides:

sum - change = reference value   : again, the change can be either positive or negative

 6 - (-1)   =   7   : this should be interpreted as: (+6) + (-(-1)) = (+6) + (+1) = 7

We see that when combining a positive value and a negative value, it results in the difference (or mathematical amount of separation, such as on the number line) between those two values. Considering 7 as the reference value, the difference between 7 and 6 is (7 - 6) = +1 , hence it can be said that 7 is 1more than 6, and-or 6 is 1 less than the reference value of 7. Considering 6 as the reference value, the (signed) difference between it (6) and 7 is (6 - 7) = -1, hence it can be said that 6 is 1 less than 7.   The negative value indicates the result is less than the reference value, and this would be so, even if that reference value was also negative in sign.

The common whole number line can also be extended (leftward) to include the negative whole numbers leftward of 0, and so as to make a more functional (mechanical) calculator that can also work with negative (reference, or starting) values.

Where are the sign symbols from?   They are a direct result from the addition, adding, or combination (combine, combining, "and", "mixed with" or "joined to or with") symbol of: (+), and the subtraction (remove, take-away, reduce, lessen) symbol (-).   However, with signed numbers where the subtraction symbol appears to be utilized, it is rather a negative symbol (-) associated with a value, and is not a mathematical operation symbol.

Ex. If you had zero and added one to it, the expression would be:

   0 + 1 = 1   :adding a number to 0, is essentially not adding anything to that number: 1 + 0 = 1, and the result is
                      that same number

Therefore, if you had 1 and subtracted 1 from it, you would have:

   1 - 1 = 0   : First, subtracting is applying a decrease or loss. This decrease can be noted mathematically as a
                 negative change, and the actual value of this change will be assigned a negative sign to indicate
                 or express this.   Here. the decrease or change is 1, and it is expressed as a signed number as:
                 (-1) or = -1. This change will be applied to the value of 1. 1 combined, added to, or applied with
                 the change = 1 + change = 1 + (-1). Hence: 1 - 1 is to be understood as = 1 + (-1).

                 1 - 1 = 0, this the same as   (+1) - (+1) = (+1) combined with, or applied to (-1) =   +1 (+) -1 = 0
                 Just the same, since the order of addition or combining does not matter, and the sum remains
                 the same, this can be considered a: (-1) combined or applied to or with (+1):
                 +1 (+) -1   =   -1 (+) +1 = 0

                 Hence when you see: 1 - 1 this can be considered as equal to the sum of two signed values of:
                 (+1) and (-1)   =   (+1) + (-1)   =   +1 -1   = 1 - 1 = (-1) + (+1) = -1 +1 = 0

If you then take this 0 and subtract 1, as in "1 less than 0 (as the reference value)", you would have:

   0 - 1 = -1   As a check to this, consider adding 1 to each side, and the equation will still be in balance.
                 Here is where the subtracting operator, operation or symbol becomes the negative symbol
                 or sign, and continuing this concept of notation: 0 + 1 = +1, we have the positive sign.

0 - 1 + 1 = -1 + 1   : each side can also be expressed as just :   +1 - 1 or = -1 +1 since 0 adds nothing.
      0 = 0   : The combining of an increase and decrease, both having the same value, results as if
                    no (net, total, or final) change has taken place:
                    (increase) + (decrease) = 0 = (+value) + (-value) = +value - value = 0
                    Ex. 7 was increased by 3, and then decreased by 3. This can be expressed as:
                    (7 + 3) -3 = 10 - 3 = 7 = 7 + 3 - 3 = 7 + 0 = 7
                    Mathematically: 7 + 3 - 3 = 7 - 3 + 3   = 4 + 3 = 7 = -3 + 3 + 7 = 7

OR, it can be said that -1 = "negative one" added to nothing is still that same value of -1:
 (-1) + 0 = 0 + (-1) = -1 = 0 (+) (-1) = 0 (+) -1 = 0 -1, since when 0 is added to any value, the result is that same value,
or that the value is unchanged. So, this is where signs and terms (numbers preceded by a sign) started into use.   Rather
than say -1 means "subtract one", which is a mathematical operation, people now can even use negative-one (-1) to show
that this is a result of a mathematical operation, rather than a mathematical operation that needs to be performed. Writing
an expression for the addition of -1 and +1, both being 1 more or less from 0, their combined (+) sum should be 0:

    (+1) + (-1)   =   0   : a value summed or combined to its opposite value is 0, or no value.
                    A value combined or added to 0 is that same value:

(0 + 1) + (0 - 1) = (1) (+) (- 1) = 0   or:

-1 (+) +1 = -1 + 1 = 0   : you can either add -1 or +1 to both sides to see that the equation is still in balance.

   Rearranging the elements, operands, or addends to be summed or combined , their sum should still be the same,
   no matter what order they are summed or combined together:

+1 (+) -1 = 0   On the number-line, if given 1, you would go rightward from 0 to +1, then if you were to
                    take away 1, this is indicated as -1, which means to move along the number-line in the
                    opposite direction 1 position leftward. Doing this, we arrive back at 0. We see that signed
                    numbers indicate the new value or change to be added in or combined into the running
                    sum is to reduce (-), or increase (+) the current total of the sum.

   The add or combine symbol (+) can be removed if we consider that the sign goes with each operand that is to
   be combined, and then this combine symbol (+) can be eliminated or hidden, and understood that the signed
   number operands, or terms, are being added or combined.

   Without signed numbers or terms, it would be difficult to solve equations by transposing (essentially eliminating
   a term, and moving its signed number counterpart) values from one side of the equation to the other so as to keep
   the equations in balance.

   In short, a simple explanation of signed numbers goes something like this:

      +5 is five more than something, and -5 is five less than something.   or:

      +5 is a change that is an increase, and -5 is a change that is a decrease.

   If something was 80 degrees and cooled off to 70 degrees, the change in temperature was (80 - 70) = 10
   degrees.   Since the change in temperature was a decrease in temperature, this would be noted with a
   negative sign as -10. If the change in temperature was ten degrees higher, this would be noted as +10.
   -10 means a decrease of 10, or 10 less than something (here its 80 degrees, and for another day it could
   perhaps mean 10 less than 50 degrees). See, negative numbers don't have to mean less than 0
   always, when they are considered as a decrease or a (negative) change from any another value (ie. the
   reference, initial or starting value, and not necessarily 0).

Ex.   Consider we have a "running" (continuously updated) current total or sum, that is, we are to add in all new values as they come along or happen, to the current or accumulated value of the sum. Perhaps we are putting apples into a basket as we pick them. Sometimes there will be a decrease in the number of apples because someone took some apples out of this basket. This essentially causes the sum of apples to change to a lesser or lower value. This decrease or change can be mathematically indicated as a negative sign preceding its (decrease) value. If someone put some apples back into the basket later, this change can be mathematically indicated as a positive sign preceding its (increase) value. With the concepts of signed numbers (numbers that mathematically indicate or represent an increase or decrease), we can mathematically express this running total or sum as a single equation, of running addends or operands, rather than in multiple steps such as something like:

Equation 1:   first_sum = number1 + number2   : perhaps various amounts of apples put in the basket
Equation 2:   second sum = first_sum - number3   : here, number3 is negative in sign since it is the number
                                                                                     of apples that were taken from the basket. This value
                                                                                     or term is being summed or combined to: first_sum

These multiple equations can be mathematically expressed (more simply) as:

sum = number1 + number2 - number3 +/- **. . .**

This expression above is actually a method that would be employed without even knowing the concepts of signed numbers. Since all changes are to be added into, or applied to, the running sum, and if we do not yet know if a change will be an increase of a decrease, we will simply write or indicate this expression as the combining (over time, as changes take place and we know the values of those changes) of known, or yet to be known values.

sum = number1 + number2 + number3 + **. . .**   : here, the word "number " is essentially a formal
                                                                                     placeholder (ie. a variable) for any possible value,
                                                                                     including its sign, since the value could be either
                                                                                     positive or negative,

Using this form of the expression eliminates what appeared to be a subtraction operation in a running summation (sum) and then assuming that the other numbers or operands are always positive in value. With this new notation, it is understood that all terms (a number preceded by its sign) or operands (actual values, constants or variable[any, or many possible values]) or are always considered as signed values that are to be added or combined using the rules (which will be discussed) for signed numbers.

ABSOLUTE VALUE

| | is the absolute value (mathematical) operation symbol. It is used to transform a signed number into its general distance or "magnitude" from zero. The absolute value of a signed number is that number without its sign (hence mathematically equivalent to the positive value).

Ex. $|+5|$ = 5   : "The absolute value of positive five is five".

Ex. $|-5|$ = 5   : "The absolute value of negative five is five".

Both of the signed numbers above have an absolute value of 5 since each of the signed numbers has a distance of 5 from 0. This also shows that +5 and -5 are signed number counterparts. Signed number counterparts obviously will sum to 0: value + (-value) = value -value = 0

Consider the difference between 6 and 7.   In reference to 6, it could be said that 7 is an increase of 1 and this could be mathematically expressed , noted or indicated as: +1. In reference to 7, it could be said that 6 is a decrease of 1, and this could be mathematically expressed, noted or indicated as: -1.   Still, they (6 and 7) both differ from each other by an absolute (or real) value of 1:

  $|+1|$ = $|-1|$ = 1   : if this was a length or distance, it would have to be positive or signless in value
                          since there is no actual physical length or distance that has a negative value.

WHAT ARE THE SIGNS USED FOR ?

        Values that represent decreases or losses are noted using a negative sign (-) preceding it, and values that represent increases or gains are noted using a positive sign (+) preceding it.

What are terms?

Together, a number and its sign is called a **term**.   Terms are the operands of signed number addition or combining. A single term by itself is sometimes called a monomial or single term expression. The word "mono" means "one" (= 1). To solve or "simplify" a multiple (sum of) terms expression, often called a multinomial, is to possibly combine (essentially a summation by signed number addition) the terms into a single term (essentially the sum).

Ex. If the temperature increases by some value, the value of the actual change in temperature would be noted using a positive sign since the temperature increased. If the temperature decreases by some value, this change in temperature would be noted using a negative symbol. Hence, the signs help give meaning to values they precede. In this example, the signs indicate the change in temperature in reference or with respect to an initial temperature of which could also be either positive (pos.) or negative (neg.) in value.

A general expression, statement or formula for the new temperature, in relation to the old temperature, can be:

(old temperature)   +   (change in temperature)   =   (new temperature)   : each could be either pos. or neg. in sign

For example, if the old temperature was +80° ("eighty degrees") and the change in temperature was 10° cooler, this 10° drop or decrease would be noted as: -10°. The new temperature using the formula above is:

      (+80°)   +   (-10°)   =   +70°

Usually, the addition or combine (to add terms mathematically, considering their signs) symbol would be omitted or "hidden", but it still should be understood that these terms are being combined, and the expression would be the familiar

looking:

+85° -10° = +70°   showing the hidden or understood combine symbol between terms: +85 (+) -10 = +70

Here, from now on, it will be understood that the terms (+85°) and (-10°), "positive eighty-five, and negative ten", are being added or combined. That is, do not think of the (-) symbol as a subtraction symbol, but rather think of it as the negative symbol that goes with the value of 10, making that term of (-10°). Hence, in a way, there is "no real (formal) subtraction operation" with signed numbers, and there is really just the addition or combining of signed numbers or terms. Differences (or the amount of separation) in values may still need to be found, and then a subtraction operation is used on the signed numbers or terms.

To advance you're mathematical capabilities, it is perhaps better to consider all values as terms or signed values. That is, don't think of signed values or terms as a "special unique feature", but think of them as more like the "main feature", particularly for simplifying and solving equations.   With some occasional practice, you will naturally gain more familiarity and confidence using these concepts.

SUBTRACTING A VALUE THAT IS LARGER

With the concepts of signed numbers, it is now mathematically possible to subtract a larger value from a smaller value. As a graphical illustration, observe the drawing below of the number-line: [FIG 21]

```
   <--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------->
   - ∞  . . . -5      -4      -3      -2      -1       0      +1      +2      +3      +4      +5  . . . +∞
```
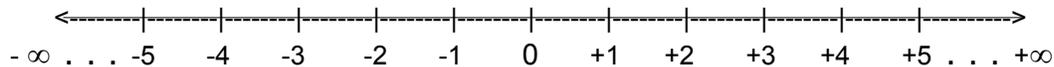
If you subtract 1 from 5, you have 4. This can be accomplished using the drawing above and then by moving 1 position leftward (to the lesser values) from the (reference, or "starting") position marked 5. Likewise, if you were to subtract 7 from 5, the result can be found by moving 7 positions leftward from 5. The result is: -2. The mathematical method to get the same results will be shown in the discussions to follow.

ADDING SIGNED NUMBERS

We know that adding two "common or plain" unsigned or positive numbers equals a larger unsigned or positive value. Symbolically:

(positive value or term) + (another positive value or term) = a larger positive value or term

Ex. 5 + 2 = 7   which really means when considering the signs as associated with signed numbers (ie. terms, of an expression):

(+5) + (+2) = +7   : Here, the "hidden" addition operation or combine symbol is explicitly shown between the two terms that are shown parenthesis. The two terms being summed or combined are: positive five: (+5), and positive two: (+2). The result of this expression of a sum of terms is a single term having a value of positive seven, (+7). This may also be expressed sometimes as:

+5 (+) +2 = +7   : here, the implied and understood combine (add) symbol is explicitly indicated or expressed, also:

(+5) (+) (+2) = (+7)   : is a possible expression of the above equations

(+5) (+2)   =   : if the above expressions are expressed this way, it is an error since it now indicates the product of the two values, rather than the addition of the two values, and the result it actually +10, and not +7.

Adding or combining a negative value to another negative value is effectively adding or combining a decrease to another decrease, hence making a larger total or net decrease, and the sign of the sum is therefore negative. Symbolically:

(negative value or term) + (another negative value or term) = a larger negative value or term

Ex. -5 - 2 = -7   which really means:

 -5 (+) -2 = -7   : Again, the hidden addition operation or combine symbol is shown here for the understanding that these terms, or a series of terms, are to be mathematically combined (added) to produce a net result or sum term.

If a negative value (or decrease) is added or combined with a positive value, or vice-versa, the resulting (signed) sum is equal to the largest difference of the absolute values of the two, and the sign of this sum is from the operand (here a term) with the largest absolute value. In other words, subtract (take difference) the lowest absolute value from the highest absolute value, and keep the sign of the highest absolute value. On the number-line, the resulting sum will be someplace between those two given values.

 (positive value or term) + (negative value or term) = | largest term | - | smallest term | = sign of largest term, difference

Even though the expression is for the sum of two terms, the result is actually found by a difference operation when the signs of those terms differ (are not the same).   The result will neither be a larger positive value, or a larger negative value, but a less positive value and-or a less of a negative value.

Ex.   +5 -2 = +3   :"positive five added (or combined) to negative two is positive three"
OR.= -2 +5 = +3   : since addition is commutative (commutative law), you can rearrange the terms or (signed) operands being combined, added or summed.

Ex.   -5 +2 = -3   : | -5 | - | +2 | = 5 - 2 = 3 applied with the sign of the value that has largest absolute value,
OR.   +2 -5 = -3   and since 5 is greater than 3, the proper sign will be from its corresponding term. Here, the
                                  term of which 5 was from is: (-5), so therefore, the sign to apply to the resulting sum of the
                                  combining of those two terms will be the negative sign. This is so, since although the
                                  result is less negative, there is still some of its negative value still remaining after the
                                  combine operation, and it must therefore be indicated or expressed as part of the result.
                                  Here, the result is negative three = (-3).

If you want, you can easily verify all these results on the number lines shown previously. It is interesting to note that the
difference or separation between -5 and +2 on the number line is actually +7, and not -3, and this can be shown by
subtracting those two values from each other:

   -5 - (+2) = -5 -2 = -7 = 7 in absolute value   , or:
   +2 - (-5) = +2 +5 = +7 = 7 in absolute, signless value

Again, if given a negative value and you then add a positive value to it, the sum may still be negative, as when the largest
of the two signless values was negative, but the sum or result can be said as being "less negative" in reference to its
previous and larger negative value. For example, when +2 is added to -5, the sum is -3. -3 is said to be less negative
than -5.

Ex. If the temperature outside was -20 and then it "warmed up" to -3, this change is usually said to be better or positive
even though it is still below zero (ie. negative) degrees outside. The temperature is headed or getting closer to being a
positive value.

     If you are performing addition of signed numbers with a standard "home" calculator which does not have a "sign"
or "change of sign" key, below is a somewhat "wordy", but simple example of how to add or combine signed numbers in
such a situation. First, some calculator that do not have a "sign" key actually allow you to enter negative numbers by first
pressing the "minus" or "subtract" key before entering the number (which is essentially the same as entering: 0 - number =
-number, and then press = or the next operation key, or by pressing: number - and then = , where the calculator assumes
the first operand is 0. For a simple check to see if your calculator can display negative numbers, subtract a larger positive
(or signless) value from any smaller positive value, and the result should be negative in sign (indicated before, or
sometimes after the result). For example, enter: 2 - 3, and the result displayed should look something like -1.   If your
calculator does not work with signed values (particularly negative values) you must establish and record the sign of each
operand and result mentally or manually on paper.

   Add positive five and negative three using a simple "home" calculator.

       (+5) + (-3)

 Since subtraction of a number is really equal to the addition (combining to make a sum) of the negative, "opposite" or
  "reverse" of that of that number, we can transform this addition problem back into a common subtraction problem by
   subtracting the negative, "opposite" or "reverse" of that number.

       (+5) + (-1)(-3)   : Multiplying by (-1) to have the reverse or counterpart of (-3).

       (+5) + (+3)   We must change the sign of the operation since we are now
                                  adding two positive numbers for an increase, however we
                                  want to combine (ie. add) a decrease to (+5) as shown in
                                  the original problem, so we must therefore use the common
                                  subtraction operation instead of the addition operation:

(+5) - (+3)   : as mentioned, this subtraction of a value is equal to the addition of that negative
                             of that value.   (+5) - (+3) = (+5) (+) -(+3) = +5 (+) (-3) as first given.
                             And this will be performed on the calculator as:

  5 (-) 3   : five minus three
     +2

Summarizing the results: +5 (+) -3   in common, non-signed number form   =   +5 (-) +3

Again, the conceived subtracting a number is really adding or combining the negative of that number when working with signed numbers.

If your calculator does have a (change or enter) sign-key or button (often indicate as: [+/-]), to enter a negative number, first enter the absolute (signless) value of that negative number and then press the "sign key" to display the negative symbol with it.   If you press the sign-key again, the value displayed should change back to a positive value.

As an extra note which could be helpful to understanding the difference or separation of absolute values: If the signs of two values being added or combined are exchanged, the result is opposite in sign, hence both sums have the same absolute value:

First note that: $|-7| + |+2| = 7 + 2 = 9$,   and $|-7| - |+2| = 7 - 2 = 5$

 Now consider addition with the operands signs reversed:

$+7 -2 = +5$ and $-7 +2 = -5$, Hence: $|+7 - 2| = |+5| = 5$ and $|-7 +2| = |-5| = 5$: A difference of 5 for both.

GETTING LESS AFTER ADDITION

Here is a very understandable descriptive analogy of how something can be less after adding or combining.

There is an empty container, its volume or space is first filled halfway with hot water, then the container is filled (ie. combined, or added, (+)) to the top with cold water. Surely, the amount of water has increased, but this is not the case for the temperature. The net (final total or sum) result of the water temperature is that it is not hotter (higher in temperature), but is now only a warm temperature, and that the temperature of the water is now less and is said as being "cooler", "lower" or "less" in value as compared to the original, starting, or reference temperature.. This is commonly thought of as:

       "hot plus cold, equals warm"   : you can view warm as "less hot", or "not as hot" in temperature, and that
  or   "cold added to hot, equals warm"   is similar to saying a value is less positive than before

Note that the cold water temperature is really in reference to the warm water temperature, and vice-versa. For example, cool 34 degree water can still be called cooler water than 35 degree water, even though its barely noticeable, and when both are actually cold to most people. To account for all possible temperatures, and all possible changes in temperature, a (more generalized) equation such as this is written:

       temperature + change in temperature   =   new or resulting temperature.

Ex. Without using any specific units of measurement such as pounds or kilograms, let's say that a force of 10 is being applied to an object in one direction, and another force of 3 is being applied to the object in the opposite direction. What is the net , effective, or resulting force?

The force in the opposite direction will be assigned a negative sign. The net or effective sum of the forces is:

       force_1 + force_2 = new or resulting force   or:
       force + change in force = new or resulting force

       +10 + (-3) = +10 -3 = +7   : The right hand side of the equation shows the expression without the
                                             hidden combine symbol since it should be understood by now that the
                                             individual terms (+10, and -3) of this expression are being combined.

If the force being applied in the opposite direction had a value of 10, the resulting force would be:

       +10 + (-10) = +10 -10 = 0   : Objects usually move when a force is applied to it, and here, the object
                                             would not be moving since the net or total force acting upon it is 0.

Ex. Suppose we give an algebraic sign of (+) for the distance an object goes upwards, and a (-) sign for the distance an object goes in the opposite or downward direction. A ball is thrown straight upwards for 50 units of distance, and due to gravity, it will fall downwards 50 units of distance. How far (as in the "signed number" or "algebraic" change in distance from its starting or reference position) did the ball travel?

Total Distance   =   Sum Of Distances

              +50 units (+) -50 units   : shown with the implied or hidden combine symbol
              +50 units - 50 units   combining terms (to simplify):
                  0 units   :the balls ending location is the same as its home, initial or start location

This result might appear to be very odd. You could argue that it went 50 units up, and then turned around and went another 50 units as it fell back, and hence a total of 100 units of distance. This is of course absolutely true if the distances are considered as undirected (essentially the signless or absolute values) or non-signed, but because of the directional

mathematics with signed values, the net distance moved is 0 units, as can be clearly seen since the ball is in the same position or location as where it started from. The change in distance of it's ending position and its starting position is essentially 0, whether the ball moved or not.

Although much of this has already been discussed here and there previously in this book, here is another helpful discussion and an example that shows that the sign of a number indicates the difference or change from another (reference) value in such a manor so as indicate (ie. with a sign) if the change was an increase or decrease from the starting or reference value. If no reference value is indicated or known, it is understood that the signed number value is a simple result of an equation, or a basic measurement (which has a reference value of and for that measurement, usually considered or understood as 0, such as on a ruler (a reference scale to measure distance or length) or a weighing scale).

If the value is not considered as a difference, or in reference to some value other than 0, but is a final or a resulting value of some expression, the value is understood at being in reference to a value of 0:

  Ex. +10 , "positive ten", without any reference value, is understood as being 10 greater than 0
       -10 , "negative ten", without any reference value, is understood as being 10 less than 0

If the value is in fact a difference of two values, it indicates the amount of change in the first value so as to be equal to the second value. This difference can be expressed in two ways:

  1. A signless or "magnitude, absolute or plain common numeric value":   Ex. 10 , "ten"

  2. A signed numeric value: Ex. +10 = "positive ten" , or -10 = "negative ten"

Now consider two values, let v1 = value_1 , and v2 = value_2

 Ex. v1 = +10 and v2 = +7   : or if just signless values were given: v1 = 10 and v2 = 7

Lets consider v1 was the initial value and v2 is its final, current, new or resulting value after some change in or of v1. Since these values are different (not the same or identical), a change has occurred to the first, starting or initial value (here, considered as v1). This change (as a measure of the specific or actual difference (a measure of the separation of two values)) or differences in values can be calculated with a subtraction (or "difference operation", since the result of a subtraction operation is a difference) operation.

  v1 + change in v1 = v2   or simply:
  v1 + change = v2   solving for change:

  change = v2 - v1   since the change (in v1) is actually expressed here as a difference (d) value:

  change = d = v2 - v1 = ending value - starting value   Note, for an unsigned number system, you must use:
                                          d = highest absolute value - lowest absolute value

          d = v2 - v1
          d = +7 - (+10)
          d = +7 - 10
          d = -3   : the difference or change was negative, or "negative in value", indicating
                     the new value is a decrease or less than (<) the starting value. +7 < +10

Knowing there is a difference and its value, it is used to indicate a "numerical comparison or relationship" of one value with respect (in reference) to another. It will indicate if a value is greater than or less than another, and by how much.

  If the specific value of d is positive in sign, it indicates an increase or gain, and that the new or ending value is greater (>) than the starting value: new value > starting value.

If the specific value of d is negative in sign, it indicates a decrease, lessening, reduction or loss, and that the new or ending value is less-than (<) the starting value: new value < starting value

For the example, variable (d), the difference or change value is -3.

ending value < starting value   : likewise, mathematically: starting value > ending value
        +7   <   +10   +10   >   +7

Ex.   If v1 = +7   and   v2 = +10

      From: change = d = v2 - v1

                change = d = (+10) - (+7)
                change = d = +10 -7
                change = d = +3   : a positive change value or difference indicates a gain or increase
                                                                   has taken place or occurred.
                                                             here,   ending value > starting value
                                                                              +10   >   +7

If you were to apply a change of (-3) to a value, such as (+10), you would express this operation as an addition or combining operation of two or more signed values. Here the combining operation is applied to or upon the starting value of (+10):

      starting value + change = ending value   , or by switching sides:

      ending value = starting value + change
             +7   =   +10   +   (-3)
             +7   =   +10   -3   this is the same as: +7 = +10 (+) -3
             +7   =   +7   : checks

We see how useful and important the sign is with its corresponding value, as in a signed number system. If the change was only indicated as 3 and without the negative sign or an indication of being "3 less", the ending value would commonly (in an unsigned number system) be calculated to be: +10 + 3 = 13 which would be the wrong result and which has a difference of +6 (ie, 6 higher, off or away) from the true result of (+7). In a sign number system, the sign is always kept with the value, and generally not to be discarded, so as it will be used in any further calculations so as to achieve the correct results.   The sign is now part of the number or value. (+7) and (-7) are two different numbers. One value, (+7) is 7 more than 0, and the other, (-7) is 7 less than 0 (or possibly some other reference value). (+7) and (-7) are quite different, and expressing this difference mathematically or quantitatively, the difference between them is a relatively huge value of: (+7) - (-7) = +7 +7 = 14. The value or number of (-7) is just as important and useful as (+7) = 7.

MATHEMATICALLY EXPRESSING SIGNED NUMBER COUNTERPARTS

      The mathematical counterpart (corresponding, and "opposite") of a signed value or number is equal to that same value, but with a sign change. Mathematical counterparts of signed values or numbers have the same (signless, or unsigned) absolute value, "magnitude" or distance from 0. To change the sign of a value, simply subtract it from 0, or just take the negative of it by multiplying it by (-1).

The negative (-), "reverse", mathematical or number-line counterpart of a positive value is a negative value. Expressing this mathematically:

Ex. -(+5) = -5   : the negative of positive five is negative five. This is a single term and expression.
                If you add 0 to this term (to create a two term expression and to help understand the result),
                it wont change its value: value + 0 = 0 + value = value
                -(+5) + 0   =   0 (+) (-(+5)) = 0 (+) -(+5) = 0 -5
                0 - 5 = (0) + (-5) = -5   since 0, or no change, added or applied to any value is equal to that
                same value. Likewise any value added or applied to 0 is still equal to that same value.

The negative (-), "reverse", or counterpart, of a negative value is a positive value: Expressing this mathematically:

Ex. -(-5) = +5 , or=   -(-5) = -1(-5) = +5 = (-1)(-5) = +5   : the negative of negative five is positive five

The expressions on both sides of the equations in the above examples will be mathematically verified in the discussions to follow, such as the multiplication of signed numbers. Technically, the set of whole (common "counting") numbers and their corresponding negative counterparts are classified as the "integer" numbers.

MULTIPLICATION OF SIGNED NUMBERS

      The product of two positive terms is a positive term, as expected since it is much like the multiplication of two unsigned values. The product of a negative term and a positive term is a negative term. The product of a negative term and a negative term is a positive term. All these rules can be verified using repeated addition.

Ex. (+2)(+3) = +6   Essentially, this is an example of "repeated increments (addition)". Here two will be added to
                zero (think of the starting point, a sum that is nothing =0) a total of three times. In other words,
                three, twos will be summed, or that two will be summed three times.
                Note also, (+2)(+3) = (+3)(+2) since the order of multiplying does not matter.

since +2+2+2   or=   (+2) + (+2) + (+2)   : shown on the right side with the "hidden"
    +6   addition or combine symbols.

Ex. (-2)(+1) = (-2)(1) = -2   : any value times one is still that value.   If you were to start at the
                  0 or "reference" position of a number line and reduce this by 2, by
                  going in the leftward or the negative direction (essentially a subtraction),
                  you will arrive at -2.   (+0)+(-2) = +0 -2 = -2

Ex. (-2)(+3) = -6   :This is an example of "repeated decrements (subtraction)", so the result
                should then be less, or more negative.   Negative two will be added to 0 a total of
                three times. This might be more easily seen when rewritten as: (+3)(-2) using the
                commutative law. Since the second factor is negative, and not positive, the result
                cannot be positive. Instead of repeatedly adding (+3) a total of (2) times, this is
                effectively repeatedly subtracting (+3), a total of (2) times, from 0.

since -2-2-2   or   (-2)+(-2)+(-2) = (-4) + (-2) = -6

Ex. (-2)(-3) = +6   This is verified below:

First, it can be shown that:

(-3) = (-1)(+3) = the negative or counterpart of +3 is -3.   : adding -1 three times is (-1)+(-1)+(-1) = -1 -1 -1 = -3

That is, (-1)(+3) = -(+3) = -3   : or = (+3)(-1)

Then:

(-2)(-3) = (-2) (-1)(+3) = (-1)(-2)(+3) =   - (-2)(+3)

Essentially then, the negative or counterpart of (-2) will be used as a multiplicand , and for the repeated addition:

  -(-2) = +2   : the negative or counterpart of (-2) is -(-2) = +2
                         "the negative of negative two, is positive two"
  -(-2)(+3)
  (+2)(+3)
  +2+2+2
    +6

An alternate verification is:

(-2)(-3)
(-2)(-1)(+3)   using the commutative law:
(-1)(-2)(+3)
(-1) (-6)   = -(-6)   and the negative or counterpart of (-6) is:
   +6   = +6

A negative value times a positive or signless value results in a larger negative value, and is due to the repeated addition of that negative value. It is not too difficult to then understand that a negative value times a negative value should be of the opposite sign, and cannot have a negative value. The opposite of a negative value is a positive value. Summarizing the signs of the results or products of signed number multiplication:

(positive value) x (positive value) = positive value
(positive value) x (negative value) = negative value
(negative value) x (negative value) = positive value   : (negative) (negative value) = - (negative value)

DIVISION OF SIGNED NUMBERS

If the signs of the dividend and divisor are the same, either both positive or both negative, the sign of the quotient will be positive, otherwise, it will be negative. All quotients can be verified using the signed number multiplication rules when multiplying the divisor and quotient together and comparing their product with the dividend.

Ex. $\dfrac{+10}{+5} = +2$ since $(+5)(+2) = +10$

Ex. $\dfrac{-10}{-5} = +2$ since $(-5)(+2) = -10$

Ex. $\dfrac{-10}{+5} = -2$ since $(+5)(-2) = -10$

Ex. $\dfrac{+10}{-5} = -2$ since $(-5)(-2) = +10 = (-1)(5)(-1)(2)$

$$(-1)(-1)(5)(2)$$
$$- (-1)(10)$$
$$+1(10)$$
$$+10$$

Summarizing the signs of the quotient of signed number division:

$\dfrac{positive\ value}{positive\ value}$ = positive value   **:** division of values having the same sign

$\dfrac{negative\ value}{negative\ value}$ = positive value

$\dfrac{positive\ value}{negative\ value}$ = negative value   **:** division of values having opposite signs

$\dfrac{negative\ value}{positive\ value}$ = negative value

OBSERVATION OF SIGNED NUMBER COUNTERPARTS

The negative (-) or counterpart of a positive value is a negative value.

Ex. -(+5) = -5

Note that the negative symbol preceding (+5) can be interpreted as meaning;

(-1)(+5) = -5

The negative or counter part of a negative value is a positive value.

Ex. -(-5) = +5

Note again that this negative symbol can then be interpreted as being multiplied by (-1).

(-1)(-5) = +5

Also note, -(-5) is actually a single term containing the negative sign that precedes(-5).

Written in this form: (-1)(-5), it is also a single term but it is understood as having a positive sign:

(-1)(-5) = +(-1)(-5)   or: +((-1)(-5))
                  +(+5)
                   +5

So, the apparent subtraction of a term is really adding the negative of that term. Though differences sometimes need to be found, there is "no real subtraction" operation performed with signed numbers (terms), and there is only their summing or combining.

Ex. +5 -(-2)

This expression is the addition of two terms: (+5) and -(-2) or (-(-2)). The (+5) term is being combined (added) to the negative of the expression inside the grouping symbol which is the term (-2).

That is,   +5 -(-2)
          (+5) + (-(-2))   : shown with the "hidden" addition (combine) symbol
   OR,   (+5) + ((-1)(-2))
             +5 + (+2)
               +5 +2   : now shown without the addition symbol
                 +7

After practice, many of the intermediate steps will not have to be performed. For the above example:

   +5 -(-2)
    +5 +2
      +7

Noting the above discussion, the same can also be said about the positive symbol. These rules are very helpful when clearing or ridding grouping symbols when simplifying expressions.

+(+5) = +1(+5) = (+1)(+5) = +5
+(-5) = +1(-5) = (+1)(-5) = -5

Subtracting a negative number is essentially removing a negativity or loss from the sum, because the result is a positive value to be added into the sum.   Therefore, the result is more positive.   Since we are working with signed numbers, it is best to think of subtracting a negative number as the addition of a positive number:

Ex. First: 2-5   : combining with a negative term
            2 (+) -5
                -3

    Now with the combining of the negative of a negative term which can be thought of as subtracting a negative number, and we know this is essentially the addition of a positive number:

     2 - (- 5)   showing the "hidden" addition symbol (+) between the terms:

    2 (+) -(-5)   : - (-5) = (-1)(-5) = +5 :
        2 +5   : the expression resolved to the addition of a positive number
          7

Checking, the difference added to the subtrahend (when the original expression is considered as an expression of subtraction) should equal the minuend:

     7 + (-5)   removing or hiding the addition symbol:
      7 - 5
         2   : checks

Ex.   -2 - (-5)
       -2 +5
        +3

Checking: +3 + (-5)
               +3 -5
                 -2   : checks

WATCH THOSE SIGNS

Here is a common example where by not observing the signs, an error could result.

Simplify: $-3 - \dfrac{-20 + 7}{5}$: The term -3 can be understood to have a denominator of 1

This is a two term expression, it is a sum of terms.

The first term is: -3   The second term is: $-\dfrac{-20 + 7}{5}$

The lowest common denominator of both terms is (1)(5) = 5. After creating an equivalent fraction of the first term:

$\dfrac{(5)(-3)}{5} - \dfrac{(-20 + 7)}{5}$ combine the numerators over the like denominators:

$\dfrac{-15 - (-20 + 7)}{5}$ distributing (-1) to clear grouping symbols, this will essentially reverse the sign of each term in the grouping symbol. You can think of this as multiplying the entirety of what is in the grouping symbols by (-1), and here, each term in the grouping symbols. -(-20 + 7) = -(-13) = +13 = -(-20 + 7) = (-1)(-20) + (-1)(+7) = (+20) + (-7) = +13

$\dfrac{-15 + 20 - 7}{5}$ combining the negative terms in the numerator:

$\dfrac{-22 + 20}{5}$

$\dfrac{-2}{5}$ an alternate method of displaying this is:

$\dfrac{-2}{5} = \dfrac{(-1)(2)}{(1)(5)} = \dfrac{(-1)}{(1)}\dfrac{(2)}{(5)} = -1\dfrac{(2)}{(5)} = -\left(\dfrac{2}{5}\right) = -\dfrac{2}{5}$

Checking: $-3 - \dfrac{-20 + 7}{5}$

          $-3 - \dfrac{-13}{5}$ showing with a "hidden" addition symbol:

          $-3 \;(+)\; -\dfrac{(-13)}{(5)}$ $=$ $-3 \;(+)\; \dfrac{(-1)(-13)}{(+1)(5)}$

          $-3 + \left(\dfrac{+13}{5}\right)$ $=$ $-3 + (+2.6)$ $=$ $-3 + 2.6$ $=$ -0.4

          $-0.4 = \dfrac{-4}{10} = -\dfrac{2}{5}$: made an equivalent fraction (divided both the num. and den. by 2)

Checking: - 3 -   - 20 + 7
                          5

        - 3 - (- 13)   =   -3 - (- 2.6)   =   -3 (+) (-1)(-2.6) =
                (5)

        -3 + (+2.6)   =   -3 + 2.6   =   - 0.4

        - 0.4   =   _ _2_   : -0.4 is the decimal number equivalent of the fraction on the right hand side
                        5

"MINUS" OR "NEGATIVE"?

        People frequently use the words minus and negative as interchangeable or meaning the same thing. but as a good rule, think of minus or the minus symbol (-) as the operation to find a difference (subtraction) of two values, and think of negative, which happens to have the same sign (-), for values (operands) or results of an operations(s). For example, if you see (-5) by itself, then it is best to not say that it is "minus five" since if this is perhaps the result of some operation(s), what are you subtracting (minus is the symbol for a subtraction operation, and not its result) it from if this is the answer or result you are looking for? It is perhaps more correct to say one of the following when you see something like (-5) by itself: "five less than 0", or simply: "negative five".

NEGATIVE EXPONENTS

Negative exponents can be created when you are canceling or reducing fractions using the exponent rules when dividing like values, or variables as in algebra. Typically, a negative exponent is the result when a larger value exponent in the denominator is subtracted from a smaller value exponent in the numerator.

Ex. $\dfrac{5^1}{5^2}$ = $\dfrac{5^{(+1(-)+2)}}{1}$ = $\dfrac{5^{(+1-2)}}{1}$ = 5^{-1}

Checking by multiplying the divisor and quotient to have a product equal to the dividend:

$(5^2)(5^{-1})$ = $5^{(2+(-1))}$ = 5^1 : checks

Note: $\dfrac{5^1}{5^2}$ = $\dfrac{\overset{1}{(5^1)}}{\underset{1}{(5^1)(5^1)}}$ = $\dfrac{1}{5^1}$: if canceling is used.   Also : $(5^1) / (5^2)$ = $5/25$ = $1/5$

Since $5^0 = 1$, we have:

$\dfrac{1}{5^1}$ = $\dfrac{5^0}{5^1}$ = $\dfrac{5^{(0-1)}}{1}$ = 5^{-1}

Hence, $\dfrac{5^{-1}}{1}$ = $\dfrac{1}{5^{+1}}$: We see that a value with a negative exponent really means, or is equal to, a division by that value with a positive exponent. The inverse or "negative" of multiplication (repeated addition) is division (repeated subtraction). As an extra note, the inverse or "negative" of a power (which includes both the base and the exponent; either expressed or simplified to a value) is a root.

Also note that the exponents can also be subtracted in the denominator instead of the numerator:

$\dfrac{5^1}{5^2}$ = $\dfrac{1}{5^{(2-1)}}$ = $\dfrac{1}{5^1}$

Hence, a value with a negative exponent in the numerator is really a division by that value with a positive exponent in the denominator, or in simpler words, multiplying by a negative power is essentially division by its corresponding positive power.

Ex. Simplify $+100(5^{-2})$

$\dfrac{+100(5^{-2})}{1(\quad 1 \quad)}$ = $\dfrac{+100(\quad 1 \quad)}{1 \ (5^2)}$ = $\dfrac{+100}{5^2}$ = $\dfrac{\overset{+4}{+100}}{\underset{+1}{25}}$ = $\dfrac{+4}{1}$ = $+4$

Negative exponent form is sometimes desired when simplifying the product of multiplying like values (or variables in algebra), where to simplify is to add the values exponents, and keeping the same value.

Ex. $(3^3)\left(\dfrac{1}{3^1} \right)$ without using direct multiplication to simplify:

$(3^3)(3^{-1})$ multiplying like values or variables, add their exponents to simplify, keeping that base:
$3^{(3-1)}$ and this can be simplified to:

3^2

Or by representing 1 as = 3^0, like values in both the numerator and denominator can be shown:

$$\frac{(3^3)(1)}{(3^1)} = \frac{(3^3)(3^0)}{(1)(3^1)} = \frac{(3^3)(3^{(0-1)})}{1} = (3^3)(3^{-1}) = (3^{(3-1)})$$ combining the exponents:

$3^2 = 9$ checking: $3^3 / 3^1 = 27 / 3 = 9$

Another method to convert a power which has a negative exponent to an equivalent value which has the same exponent which is positive in value comes from the concept of reciprocals, a fraction to a power, and the previous discussion on negative exponents.

Ex. $3^{-4} = 1/3^4 = 1^4 / 3^4 = [1/3]^4 = 0.333333333^4$ These steps are also verified more below:

$1/3^4 = 1 / 81 = 0.012345679 = 0.3333333333...^4 = 0.012345679$

$$3^{-4} = \frac{1}{3^4} = \frac{1^4}{3^4} = \left(\frac{1}{3}\right)^4$$

In general, if you were to write a (algebraic) formula for this, you would create something like:

$$N^{-x} = \frac{1}{N^x} = (1/N)^x$$: (1/N) is the reciprocal of N   and   (1/N)^x is the reciprocal of N^x ,
By observation, it could be said that the reciprocal of an expressed power (such as N^x) is equal to that same indicated power (such as x) of the reciprocal of the base of that power.

A reciprocal of some value, say N is sometimes expressed as N^(-1), and this is verified here with several intermediate steps which can be omitted during simplification of expressions:

The reciprocal of N is usually expressed as: $\frac{1}{N} = \frac{(1)(1)}{N} = \frac{(1)(N^0)}{N^1} = (1) N^{(0-1)} = (1) N^{(-1)} = N^{-1} = \frac{1}{N}$

Some formula express reciprocal values using an exponent of (-1), and it may be odd for example to see something such as (pi) to the negative 1 power:

$$(pi)^{-1} = \frac{1}{(pi)} = (3.14...)^{-1} = \frac{1}{3.14...}$$

THE ROOT OF A PERFECT SQUARE

The square root of a positive perfect square (of a value) is both positive and negative in sign, that is, a positive perfect square has two roots, one is positive, and one is negative.

Ex.   What is the square root of 25?

$\sqrt{+25}$ =   ±5   : ± means both positive and negative

That is, $\sqrt{+25}$ is +5 and -5, but not both.   25 is said to be a perfect square (value) of both of its square roots.

Checking by squaring (raising to the second power) both roots:

    (+5)^2 = (+5)(+5) = +25
    (-5)^2 = (-5)(-5)   = +25 , symbolically or algebraically expressed: (-n)^2 = n^2
                          checking: (-n)^2 = (-n)(-n) = (-1)(n)(-1)(n) = (-1)(-1)(n)(n) = +1n^2 = n^2

Ex. $\sqrt{20^2}$ = ±20^(2/2) = ±20^1 = ±20   : to put this radical in exponential form, and at the same time, solving
                                for the square root, divide the exponent of the radicand by 2

Ex. $\sqrt{4^8}$ =   ±4^(8/2) =   ±4^4

Ex. $\sqrt{50.9796}$ = ±7.14   : usually, a calculator would be used to find this root

Generally, the "principle" (main and positive) roots have more practical or useful importance, and therefore it is usually the intended and formally defined root of any radical. Negative roots are usually meaningless, such as negative lengths or distances, but don't completely ignore the possibilities of negative roots, for example, they are very important when drawing curves that are symmetrical to either an axis of that curve, or to an axis-line of a coordinate system - where the points [ie. a tiny part of a line or curve] have corresponding or counterpart positive and negative value coordinates (corresponding locations, or simply the location or "address", of a point [drawn as a dot on paper]).

Here is a note on the difference between a negative valued base being squared, and the negative of a value squared.

Consider this: (-5)^2   /=/   - 5^2

This above is read as: "Negative five square is not equal to the negative of five square.":

    (-5)^2   /=/   - 5^2
   (-5)(-5) /=/   -(5^2)
     +25   /=/   -(+25)
     +25   /=/   -25

For -5^2, due to powers and roots having a higher precedence than combining of terms (signed values, and-or partial or intermediate sums of the sum), this expression really indicates and is to be evaluated as: -(5^2). To indicate or "force" the square of negative five, parentheses must be used as shown above as: (-5)^2. You could also say: "the square of negative five is not equal to the negative of the square of five".

Also consider these possibilities: -(-5^2)   /=/   - (-5)^2   or = - ((-5)^2)
                                          -(-25)   /=/   -(+25)
                                             25   /=/   -25

THE DISTRIBUTIVE PROPERTY , DISTRIBUTION

The distributive property means that the product of two factors, in which one or both is a sum of terms (ie., an expression), is equal to the sum of all possible products of each of the terms of one factor, with all of the terms of the other factor. This should be easily seen, as presented below, since multiplication is really repeated addition. Distribution was also briefly mentioned previously in this book. The word distributive comes from the fact that the terms of one factor are "distributed", as in a distribution of something, to be used (here multiplied) with all the terms of the other factor. Frequently, the words "distribute" and "multiplication" are exchanged or used in place of each other due to their similarity or process.

Ex. +5(+1+2)   So simplify this expression, distribute the +5 factor to each member of the other factor
                         in the grouping symbols:

   +5(+1) (+) +5(+2)   : an intermediate step frequently omitted, showing the
                           "hidden" addition or combine symbol shown here as (+).
    +5(+1) +5(+2)   distribute or multiply to clear the grouping symbols in each term:
      +5   +10   add (combine terms):
        +15

Checking, using the order of operations more closely where the expression within the grouping symbols is simplified first:

+5(+1+2) = +5(+3) = +15

Ex.   2(10) = 10(2)   expressing multiplication as repeated addition on the left side, and a
                                   factor composed of, or reduced to, partial sums (here of 2) on the right:
      10 + 10 = 10(1 + 1)
   10(1) + 10(1) = 10(1 + 1)   : this expression is of the basic format of the distributive property, and the
                                   expression on the right side is essentially a factored form of the left side
                                   where 10 is a common factor to each term, and is then factored out of each
                                   term and expressed as a product on the right side.

   For more verification of the distributive property, here is a slightly different way of expressing the above example:

                2(10) = 10(2)   expressing each side using repeated addition:
   2+2+2+2+2+2+2+2+2+2 = 10 + 10   expressing 2 as partial sums:
   (1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1) = (1)(10) + (1)(10)
                10(1+1) = 10(1) + 10(1)   : this expression is of the basic format of the distributive property

   You may also consider this:

   10 + 10 = 20   dividing both sides by 10 will keep the equation in balance, that is, both expressions will
                          still be equal:

   $\dfrac{10 + 10}{10} = \dfrac{20}{10}$ the fraction on the left hand side can be considered as the sum of two "like-fractions" that
                          have the same denominator of 10:

   $\dfrac{10}{10} + \dfrac{10}{10} = 2$

    1 + 1 = 2

Since the original equation was divided by 10, we can multiply both sides of this new equation by 10 to undo what was taken out. A multiplication by a number, and then a division by that same number, or vice-versa, will still keep things in balance since that whole process is essentially multiplying by 1 which does not change a value:

number / number = 1,   or , =   (number)(1/number) = 1.   : a number divided by itself equals 1 which equals the
                                                      number times its reciprocal.
10(1 + 1) = 2(10)
10(1 + 1) = 20   hence, in relation to original equation, we can see another verification of the distributive law:

10(1 + 1) = 10(1) + 10(1) = 10 + 10 = 20

If you were to create a minimal formula to represent this distributive property concept, letting A, B and C to represent (or be "placeholders for") any values (not yet specifically known), it would be:

A (B + C) = AB + AC   **: a symbolic or algebraic formula for the distributive law**
                                       Note that AB = (A)(B) = A x B = "A times B",   and AC = (A)(C), etc.
                                       Since letter "x" used as the symbol for multiplication might be considered as
                                       a variable, it's use is therefore problematic, and parenthesis or brackets are
                                       recommended where needed, such as to clarify an expression.

This expresses that A times the quantity of B + C, is equal to A times B, plus A times C.   The right side of the equation is known as a simpler form of the left side. This is so since it's a simple sum of terms rather than an indicated multiplication (which itself represents another mathematical operation). The left side of the equation is also known as the factored form of the right hand equation (where both terms contain a (common) factor of A), and more about this will be said ahead.

A single term is called a monomial, and a sum of two terms is called a binomial. The expressed product of: A(B+C) is technically a single term product which consists of a monomial (A) times a binomial (B+C). After distributing and multiplying the factor of A to each term in its binomial factor, the simplified or "simpler" result is a binomial expression that is a sum of terms.

   A(B + C) = AB + AC   : distributed form, showing all possible products of the two factors
   AB + AC = A(B + C)   : factored form , each term on the left side has a factor of (A) in it

The above is just the basic representation of distribution, and the factors A or (B + C) could actually be composed of more terms and the basic concepts can still be applied to those just the same.

For additional verification of the distributive property, consider this where it is expressed like a common (vertical) multiplication problem:

A (B + C) = (B + C) A   : due to the commutative law, the expressed order can be switched

                B + C
        x   A
          (AB) + (AC)

For example: 2 (10 + 8)   =   10 + 8   =   18   : this right side is the result simplifying the expression
                              x   2   x _2_   in the grouping symbols first, and then multiplying
                           20 + 16   36
                             36

              2(10 + 8)   =   2(10) + 2(8)   : 18 multiples of 2, is the same as 10 multiples of 2, plus 8 multiples of 2
                         20 + 16 = 36

Note also that the second (from the left) vertical multiplication expression above essentially shows the right hand multiplication as a multiplication of its positional sums in the decimal system. It could also be said that 10 and 8 are "partial sums" of 18. In short, each "partial sum" (whatever values they may be) of the multiplicand is to be multiplied by the multiplier once. Below, integer numbers 9 and 10 are expressed as the sum of two integer "partial sums". This will help give a deeper understanding of partial sums that are integers, and of the concepts of distribution such as that distribution can be "thought of" as either one or all factors converted or expressed as a sum of partial sums of it. After some intermediate multiplications, the result is essentially that of repeated addition, as expected since multiplication is essentially repeated addition.   Though in these examples, it is it is easier to simplify the expressions in the grouping symbols first before using distribution, it also helps verify using the concepts of distribution with values, such as variables, in grouping symbols that cannot be simplified first or further   Here in a pseudo expression or formula:

   (factor1) (factor2)
   (factor1) (sum of partial sums of factor2)
   (factor1) (partialsum1 + partialsum2)   Now expressing the distribution of factor1 to each
                                                           partial sum in factor2:
   (factor1) (partialsum1) + (factor1)(partialsum2)

Ex. (5) (2)   : This example verifies the concepts of distribution.
     (5) (1 + 1)   The multiplication of 5 and 2, expressed as a repeated addition of 5:
     (5)(1) + (5)(1)   : shows the addition or sum of the partial products of
       5   +   5   which are also partial sums of the final product
            10

   Or, since the result is the same if the order of the multiplication is changed:

   (2) (5)
   (2) (1 + 1 + 1 + 1 + 1)
   (2)(1) + (2)(1) + (2)(1) + (2)(1) + (2)(1)
     2   + 2   + 2   + 2   + 2   : extra, for example, = (2+2)+2+2+2 = (4+2)+2+2 = (6+2)+2 = 8+2 = 10
     10   A summation by grouping , showing partial or intermediate sums

Here is a verification of distribution

   Since: (5)(2) = (5)(1 + 1)   : a factor expressed as a sum of partial sums
   And: (5)(2) = 5 + 5   : multiplication is repeated addition of that value,   equating the equivalences:

         (5)(1 + 1) = 5 + 5   : which would result after multiplying the first factor to each partial sum of the second factor

Consider these partial sum integers that an integer value can have, and even these partial sums can be expressed as other partial sums.

         9   10

      (1 + 8)   (1 + 9)
      (2 + 7)   (2 + 8)
      (3 + 6)   (3 + 7)
      (4 + 5)   (4 + 6)
                       (5 + 5)   : after this, the values start repeating, for example (6+4) was already expressed as (4+6)

   Clearly, an integer (N) will have the whole portion of (N/2) integer partial sums. 9/2 = 4.5,   and 10/2 = 5
   Note that all the values from 1 to (N-1) will be included in these partial sums. Any value plus 0, is not
   considered as a partial sum. Any number or factor can be expressed as partial sum:

number x 9 = number(9) = number(sum of 9's partial sums) ,   for example:

number x 9 = number(1+8) = number(1) + number(8) = partial sum or product + partial sum or product = product

In short, if given number added once, and given number added eight times, the sum of those two values is equivalent to number being added nine times. Number being added nine times is the same as number times 9.

A factor can also be expressed as the sum of more than two partial sums, even up to the point where that factor is expressed entirely as the sum of values that are 1.

Ex. 7(4) = 7 (+1 + 1 +2)
      28   = +7(+1) +7(+1) +7(+2)   :shown with indicated (ie. expressed) distribution
                                   Now multiplying (it is also distributing, even if just
                                   one term), and it will "clear the grouping symbols".
      28   = +7 +7 +14   Now combining the terms on the right hand side:
      28   = 28

Ex.   +3(2+1)   Using distribution (D):
    +3(+2) +3(+1)   : an expression with indicated distribution and the resulting indicated products
      +6 +3   : after distributing to clear grouping symbols, and here its a single multiplication in each term
       +9

Ex. +2 +(3-7) -(5-7)   this can be rewritten for better clarity as:
    +2 +1(+3 -7) -1(+5 -7)   distribute (+1) and (-1) to clear grouping symbols:
    +2 +1(+3) +1(-7) -1(+5) -1(-7)   : expressing the distribution, now multiplying to clear grouping symbols:
    +2+3-7-5+7   optionally summing up all positive terms, and negative terms separately:
     +12 -12
       0

Here is another generalized formula of the concept of distribution. Variables, or placeholders representing any value, are used rather than specific values:

            (A + B) (C + D)
          A (C + D)   +   B (C + D)   : an intermediate step often not shown. Note that each term now
                                 includes the same factor of (C + D).
           AC + AD + BC + BD

OR :   (A + B) (C + D)
         (A + B)(C) + (A + B)(D)   this can be expressed as:
         (C)(A + B) + (D)(A + B)   : here, each term has a factor of (A + B)
          CA + CB + DA + DB   this can be expressed as:
          AC + BC + AD + BD   which can be expressed as:
          AC + AD + BC + BD   : the same as above

Ex. (+2+3)(+1+4)
   +2(+1) +2(+4) +3(+1) +3(+4)   : showing the sum of the indicated products of distribution
       +2 +8 +3 +12   : after distributing again to clear grouping symbols
         +25   : after combining terms

checking: (+2+3)(+1+4) = (+5)(+5) = +25   : here the expressions in the grouping symbols were simplified first

Ex. -1(+4 -5 +2)   Distributing the -1:
      -1(+4) -1(-5) -1(+2)   Distributing or multiplying to clear grouping symbols in each term:
                                      : Note, this is three terms being combined (summed): -1(+4) and -1(-5) and -1(+2)
                                        since it is understood that these terms are being summed or combined, and that the
                                        combine symbol (+) is hidden or understood as being there
      -4 +5 -2   Combining these terms:
        +5 -6
          -1

checking: -1(+4 -5 +2)
              -1(+6 -5)
              -1(+1)
                -1

Ex. Given a value of 1.5, this value can be expressed as a sum of the whole value and the fractional value:

1.5 = 1.0 + 0.5

If this value of 1.5 is multiplied by 2, then 1.5 and any possible expressed equivalence of 1.5 is effectively multiplied by 2:

2 (1.5) = 2 (1.0 + 0.5)   distributing:
   3   = 2 (1.0) + 2 (0.5)   : expressed or indicated distribution and the sum of the products
   3   =   2   +   1   : the right hand side is the sum of a multiple of 2 + a fraction (part) of 2
   3   =   3

Notice that the specific fractional portion of 0.5 in 1.5 corresponds to a specific fractional portion of the value of 2, that is, the portion will be less than 2, and here, that this portion of 2 is the value of 1. 1 is numerically this portion (ie. fraction) of 2:

$\frac{1}{2}$ = 0.5   : the decimal equivalent of the fractional part (here, it's 1) of 2

Ex. 5.2 (10) = (5 + 0.2) (10) = (10)(5) + (10)(0.2) = 50 + 2 = 52

The distributive property is used widely in algebra, especially when the expression inside a grouping symbol of one factor cannot be simplified further. To continue simplifying, distribution will create a sum of terms which is a simpler expression. The ultimate goal when simplifying expressions is usually to produce a single value or term if possible.

Note: (+2 +3)(+1 +4)   and   +2+3(+1+4)   are actually very different.

In the expression +2+3(+1+4), only the value of +3 is being multiplied to (+1+4), that is, this is a two term expression. The first term from left to right is +2, the second term is +3(+1+4) which represents a product of two factors +3 and (+1+4). Multiplication has a higher precedence of performing than combining. In the expression (+2+3)(+1+4), the total quantity or value inside the first grouping symbol is being multiplied to (+1+4). That is, (+2+3) is an expression of a factor to be multiplied to (+1+4). The whole expression (+2+3)(+1+4) actually represents only one term, and since the sign is not indicated, it is considered positive in sign, however this does not necessarily mean the result after evaluating (simplifying, solving) the expression will be positive in sign.

(+2+3)(+1+4)   /=/   +2+3(+1+4)
+2(+1)+2(+4)+3(+1)+3(+4)   /=/   +2+3(+1)+3(+4) , and:

```
+2+8+3+12                    /=/     +2+3+12
   +25                       /=/     +17
```

Checking by first using simplification within the grouping symbols:

```
(+2+3)(+1+4)     /=/     +2+3(+1+4)
  (+5)(+5)       /=/      +2+3(+5)
    +25          /=/       +2+15
    +25          /=/        +17
```

Here is a generalized or algebraic version or representation of this process where variables represent any valid number:

$(a + b)^2 = (a + b)(a + b) = (a)(a) + (a)(b) + (b)(a) + b^2 = a^2 + 2ab + b^2$

Ex. $(a + 1)^2 = (a + 1)(a + 1) = a^2 + a + a + 1 = a^2 + 2a + 1$: $(a + 1)^2 = (1 + a)^2$

CANCELING TERMS IN FRACTIONS

       Only terms that are factors of the entire numerator and denominator can be canceled. This is a very important concept such as when there are variables being used in algebraic or symbolic expressions.

Ex. $\dfrac{(2+10)}{2}$

Here, the 2 in the denominator cannot be directly canceled with the 2 in the numerator or the factor of 2 in the value of 10 since neither term in the numerator is a factor of the entire numerator. That is, the entire numerator, and not just a term or selected terms of it, is being divided by 2. If canceling were to occur as just described, you would get the incorrect result. The correct result is actually 6. This can be checked by using the order of operations more closely where the grouping symbols in the numerator and denominator are to be cleared first if possible. Actually, the grouping symbols in the numerator are not needed in this example. To clear grouping symbols here, distribute a (+1) factor preceding it:

$$\frac{(2+10)}{2} = \frac{+(+2+10)}{+2} = \frac{+1(+2+10)}{+2} = \frac{+2+10}{+2} = \frac{+12}{+2} = \frac{6}{1} = 6$$

As a check, consider the expression as the result of a sum of fractions with like or common denominators:

$\dfrac{(2 + 10)}{2} = \dfrac{2}{2} + \dfrac{10}{2}$: now, both fractions have a common factor of 2 in their entire numerators and denominators, and can easily be divided out or canceled:

$\dfrac{(2 + 10)}{2} = 1 + 5 = 6$

Another check can be made by factoring the expression or value in the grouping symbols so as a possible common factor to both the entire and numerator can be created. Factoring each of the terms in the numerator we find that each has a common factor of 2, and this is also a common factor to both the entire numerator and denominator:

$$\frac{(2 + 10)}{2} = \frac{((2)(1) + ((2)(5))}{2} = \frac{\cancel{2}(1 + 5)}{\cancel{2}} \quad \text{or} = \frac{2\,(1 + 5)}{2\,(\ 1\ )} = (1)(6) = 6$$

    Notice that the factorization of the terms in the numerator expression is the same as the distributed form
    of those factors.

Ex. $\dfrac{-3(2+6)}{15}$ Here, rather than performing distribution to simplify the numerator, "canceling out" can be used due to that the entire numerator and denominator have a common factor (here, its 3, or -3, and we will use 3) among them:

$$\overset{-1}{\underset{+5}{\cancel{\frac{\cancel{-3}(2+6)}{\cancel{15}}}}} \quad \text{or} = \overset{-1}{\frac{\cancel{-3}(2+6)}{\cancel{3}(5)}} = \frac{-1(2+6)}{5} = \frac{-2-6}{5} = \frac{-8}{5} = -1.6 \qquad : \text{note } 15 = (3)(5) \text{ or} = (-3)(-5)$$

Checking using distribution to clear grouping symbols (ORD., step 2):

$$\frac{-3(+2+6)}{15} = \frac{-6-18}{15} = \frac{-24}{+15} = -1.6$$

Checking using an alternate version (ORD., step 4, combining, simplification within the grouping symbols first):

$$\frac{-3(+2+6)}{15} = \frac{-3(+8)}{15} = \frac{-24}{15} = -1.6$$

Dividing A Several Or Multi-Term Expression

A several or multiple ("multi", meaning several or many) term expression is called a multinomial. For the specific number of terms present or used, for example, a binomial is a 2-term multinomial, and a trinomial is a 3-term multinomial. A single (one or "mono") term expression is called a monomial. To divide a sum of terms by a monomial or single term, divide that term into each and every term of the given sum of terms. To initially express this concept in the example below, the order of operations is not strictly followed in the following example:

Ex. $\frac{(+7+14)}{7} = \frac{7}{7} + \frac{14}{7} = \frac{+21}{7} = +3$: a multinomial divided by a monomial

Expressing or explaining this in word form:

$$\frac{value1 + value2}{value3} = \frac{value1}{value3} + \frac{value2}{value3}$$ even these expressions can be expressed as something like:

$$\frac{v1 + v2}{v3} = \frac{v1}{v3} + \frac{v2}{v3}$$: expressed by essentially using a "shorthand" version, simple indication, notation or identification of the values used.

This concept above should be clearly seen since for the addition of like fractions (have the same denominator), the numerators are combined (added) and placed over that same denominator:

$$\frac{+7}{7} + \frac{14}{7} = \frac{(+7+14)}{+7} = \frac{+21}{+7}$$ Ex. $\frac{25}{5} = \frac{(20+5)}{5} = \frac{20}{5} + \frac{5}{5} = 4 + 1 = 5$

Each term of a multinomial or several term expression can be divided by a multinomial denominator expression as long as that entire multinomial denominator is considered as like one single divisor and-or value within grouping symbols, and it is then essentially representing and acting as one single value, monomial or single term.

Ex. $\frac{+3+9}{+2+1} = \frac{+3+9}{(+2+1)} = \frac{+3}{(+2+1)}$ (+) $\frac{+9}{(+2+1)} = \frac{3}{3} + \frac{9}{3} = \frac{+12}{3}$, or $= 1 + 3 = +4$

checking: $\frac{+3+9}{+2+1} = \frac{+12}{+3} = +4$

Ex. $\frac{(+3+9)+6}{(+2+1)}$

Note that the numerator is indicated or expressed as having two terms: +(+3+9) and +6. Dividing each of these terms by this multinomial denominator:

$\frac{(+3+9)}{(+2+1)} + \frac{+6}{(+2+1)}$: here, the numerator of the first fraction contains two terms: +3 and +9, then we have:

$\frac{+3}{(+2+1)} + \frac{9}{(+2+1)} + \frac{6}{(+2+1)} = \frac{3}{3} + \frac{9}{3} + \frac{6}{3} = \frac{(3+9+6)}{3} = \frac{18}{3} = +6$, or $= 1 + 3 + 2 = +6$

Multinomials are sometimes called polynomials, but more formally and in terms of algebraic variables and advanced mathematics, polynomials are multinomials that are a sum of terms that are algebraically related such as having the same variable but which are raised to different indicated powers - often to the next whole number exponent so as to produce a sum of terms that are ascending integer powers of the variable. A polynomial may then be simply said to be a sum of terms which contain several powers of the same variable.

CDC Is Another Helpful Plan For Simplifying Expressions

C, D and C are the first letters for the words: Cancel, Distribute, and Combine. Taken together, they represent an easy method, steps or plan to remember or follow when expressions are to be simplified. This plan closely follows and can be used along with the standard order of operations plan. To help you learn this plan, you should frequently write these letters, or the words they represent, next to the problems you are solving:

C - Cancel : Try to reduce or possibly cancel (eliminate) fractions, that is, rid the denominators of fractions (make them +1 if possible). One way is to cancel out common factors to both the numerators and denominators.

D - Distribute : Use the distributive property to eliminate complex (multiple term) factors such as expressions within grouping symbols, or simply multiply to produce a product. Basically, to distribute is to clear grouping symbols.

C - Combine : Combine (add) terms using the rules for signed numbers, or variables as in algebra where the terms must be "like terms" that have the same exact variables.

Ex. Simplify $\dfrac{8(2+1)}{4} + \dfrac{5}{1}$ C , cancel common factors in the fraction (here, the +4):

$\dfrac{\cancel{8}(2+1)}{\cancel{4}} + \dfrac{5}{1}$ D , distribute, or multiply (here, the +2) :
$^{+2}$ $_{+1}$

$\dfrac{+4+2}{1} + \dfrac{5}{1}$ C , combine or add terms (here, in the numerator):

$\dfrac{+6}{1} + \dfrac{5}{1}$:   C , combine, add terms (here, that are fractions with the same denominator of 1):

$\dfrac{+11}{1}$ = +11

Ex. Simplify $2(6^2 + \sqrt{16}) + 5$

Note that this is a two term expression (a binomial). The terms are: $+2(6^2 + \sqrt{16})$, and +5. 2 is a factor of this first term. $(6^2 + \sqrt{16})$ is also a factor of the first term, and which itself contains multiple terms, hence it is surrounded or "delimited" (set-apart, bounded, identified) by grouping symbols. It is a factor of the indicated product of 2 and $(6^2 + \sqrt{16})$.

First, there is nothing to "cancel" here since there are no fractions or division in the expression.

$2(6^2 + \sqrt{16}) + 5$ ORD. step 1 (simplify powers and roots), ORD means ORDer Of Operations:

$+2(+36+4) + 5$ ORD. step 4 (combine values or like terms):
$+2(+40) + 5$ D , Distribute ("distribute to clear grouping symbols"), ORD. step 3:
$+80+5$ C , Combine, ORD step 4:
 $+85$

Dividing Or Factoring A Several Term Expression

This procedure stems from, and is effectively the inverse of distribution. Given a several or multiple term expression (a multinomial) as a dividend or numerator of a fraction, we can divide by any value:

Factor or divide this expression by 2:   (value1) + (value2)

$$\frac{(value1) + (value2)}{2} = quotient \qquad \text{mathematically:}$$

$$(value1) + (value2) = 2 \, (quotient) \qquad \text{: a factored from of the multinomial dividend}$$

$$\frac{(value1) + (value2)}{2} = \frac{value1}{2} + \frac{value2}{2} = quotient$$

We see that the left side expression is equal to that of the right side of adding the like fractions with the same denominator, hence this division or factorization process of the numerator is essentially the reverse of adding like fractions together.   It can also be said that a multinomial divided by a monomial or single term expression is equal to each term of that multinomial divided by that monomial.

Ex.   Divide (8 + 6) by 2.

We see that this is a multinomial (8+6) divided by a monominal (2).

$$\frac{(8 + 6)}{2} = \frac{8}{2} + \frac{6}{2} = 4 + 3 = 7 \qquad \text{checking:}$$

$$\frac{(8 + 6)}{2} = \frac{14}{2} = 7 \qquad \text{:checks , multiplying the divisor and quotient together as a check of the dividend:}$$

$$(2)(7) = 14 = (8+6) \qquad \text{: checks}$$

A good way, if possible, to factor a several term expression is to is to divide it all, or each term, by a factor that is common to all the terms of that expression. Usually, when greater simplification is concerned, the largest, greatest, or highest common monomial factor (HCMF) of all the terms is chosen as the divisor.

The result after canceling common factors in both the entire numerator and denominator will be another multinomial. As similar to basic concepts of division, the product of this quotient and divisor (denominator) is equal to the original multinomial numerator or dividend. The indicated or expressed product of the quotient and divisor is formally called a factored form of the multinomial.

Ex. Factor the expression: +10+20

First, checking for any common factors, we find that the highest common factor (HCF) of or within all the terms is 10. Other examples of common factors of both terms are 2 and 5. Since 10 is larger, dividing the entire expression by 10 we have:

$$\frac{+10+20}{+10} = \frac{+10}{+10} + \frac{+20}{+10} \qquad \text{: dividing each term by 10, or "factoring 10 out of each term"}$$
: here 10 is the highest common factor to all the terms in the dividend, and:

```
 +1        +2
+1̶0̶  +  +̶2̶0̶           : Cancel
+1̶0̶       +̶1̶0̶
 +1        +1
```

$$\frac{+1\ +2}{1}$$ Combine (the fractions):

(+1+2)

Hence $\dfrac{+10+20}{+10}$ = (+1+2)

Expressing the product of the divisor and multinomial quotient is a factored form of the multinomial dividend:

+10(+1+2)   : this is also the result after multiplying both sides of the above equation by 10 and canceling.
                   Hence: +10+20 = (+10)(+1+2) in a factored form, and the left side is the equivalent result
                   after using distribution on the right side.

The "golden rule of factorization" is to first factor out any factors common to all the terms. Usually, the HCMF is chosen since the resulting expression will be completely factored.

As already indicated, to factor a multinomial, you don't necessarily have to divide by a factor common (ie common factor) to all the terms, but can perform a division by any value:

 Given +10 + 20   we can factor by any value such as 2, 5, 7, 9, 1.2, -5.3, or any other value needed:

   $\dfrac{+10\ +20}{7}$ = $\dfrac{10}{7}$ + $\dfrac{20}{7}$ = 1.4285$\underline{7}$ + 2.8571$\underline{4}$ = 4.2857$\underline{1}$ hence:

     +10 +20 = 7 (4.2857$\underline{1}$)   : = 29.99999$\underline{7}$ = almost 30.0 which is the correct result. A slight
                                                      error of only (30.0 - 29.999997) = 0.000003 is due to rounding.
                                                      This error value could be indicated with a negative sign since the
                                                      result we calculated of 29.999997 is 0.000003 less than the
                                                      true result. You can also rather use this formula to find the error
                                                      value, and the sign will be automatically placed:
                                                      (calculated value) - (true value) = (error value)
                                                      The sign of the error value will be in reference to the calculated value
                                                      being more (+), or less (-), than the true value.

FACTORING A BINOMIAL OF PERFECT SQUARES

      To factor a binomial (a two term expression) that is composed or consists of terms that are both perfect squares (indicated with an exponent, or not), one, and only one, of the terms must be a negative term. The expression is therefore much like a difference of perfect squares. The result of factoring a binomial of perfect squares is a product of two binomial factors where one binomial factor contains a sum of the terms square roots, and the other binomial factor contains the difference of the terms square roots. This is a "special factorization" that you should try to memorize so as to recognize it and-or quickly solve some math problems when it can be applied, and including the reverse of this procedure. Symbolically, this concept can be expressed as:

(squared value1 - squared value2) =

(square root value1 + square root value2) (square root value1 - square root value2)　　　: factored form of the binomial
　　　of perfect squares

Ex.　Factor　$(3^2 - 9^2)$　　　　: both terms are (indicated or expressed) squares of a value, here 3, and 9
　　　　　　　　　　　　　The second term (-9^2) is the negative term, or it could be said that the
　　　　　　　　　　　　　 entire binomial expression is a difference of perfect squares.

$(3^2 - 9^2)$ = (+3 +9)(+3 -9)　　　　　: the binomial of perfect squares factored, continuing:
$(3^2 - 9^2)$ =　 (+12)　 (-6)　=　-72　　　checking, using distribution:

(+3+9)(+3-9)　　　　　　　　　　expressing the distribution of the factors:
+3(+3) +3(-9) +9(+3) +9(-9)　　　distribute:
+9-27+27-81　　　　　　　　　　combine, note that the "counter part" terms of -27, and +27 cancel out or sum to 0 :
　-72　　　　　　　　　　　　　checking, using the order of operations:

$(3^2 - 9^2)$　　　ORD. step 1, simplify powers and roots:
　(9-81)　　　combine:
　-72

If $(3^2 - 9^2)$ was initially expressed as (9 - 81) you would have the same result, and a general or algebraic representation for the above procedure would be:

　(a - b) = $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$　　　　　: **Formula for factoring a binomial of perfect squares.**
　　　　　　　　　　　　　　　　　　　　　　You can check this by using distribution in the
　　　　　　　　　　　　　　　　　　　　　　right side expression:

　　　　$\sqrt{a}\ \sqrt{a}\ + \ \sqrt{a}\ (-\sqrt{b}) \ + \ \sqrt{b}\ \sqrt{a}\ + \ \sqrt{b}\ (-\sqrt{b})$

　　　　　　a　　- $\sqrt{a}\ \sqrt{b}$ + $\sqrt{a}\ \sqrt{b}$ - b　　　　combining like terms:

　　　　　a - b

In short, a difference in terms could be viewed or thought of as equivalent to the product of the sum of the terms square roots and difference of the terms square roots. Actually, in more simplified words, and perhaps where the above mathematical derivation stems from, it could be said that the product of the sum and difference of two values is equal (=) to the difference of those two values when they are both squared. A general symbolic or algebraic representation of this might look like:

　　　　$(a^2 - b^2)$ = (a + b) (a - b)　　　: **Factored form of a binomial (a sum of two terms) of the difference
　　　　　　　　　　　　　　　　　　　　　between two perfect squares**

Ex. Given (8+2)(8-2) ,   this can be expressed in simpler terms with the values squared as:
Expressing each value as the square of their square roots:

$\quad$ (8^2 - 2^2)
$\quad\quad$ (64 -4) = 60   : As a check, all these expressions equal (8+2)(8-2) = (10)(6) = 60.

Ex. Factor (25-1)

First, there are no common factors here except 1 of course. Each value (term) here is (considered to be, or can be considered as) a perfect square, and therefore, there is the expressed difference (ie. has the - symbol) of perfect squares.

Note, actually the term of: -1 is not a perfect square of 1, it is though a perfect cube of (-1):

(-1)^3 = (-1)(-1)(-1) = (+1)(-1) = -1

Hence, when factoring binomials of perfect squares, the sign of the negative term is meant to be taken as meaning the difference of positive values, or adding the negative of a positive value. For the above example, "think of"
(25-1) as (25-(+1)), which are equivalent.

$\quad$ (25 -1)   with 5 as the square root of 25, and 1 as the square root of 1:
(5^2 - 1^2)   : expressing the binomial of perfect squares, simplifying:
$\quad$ (5+1)(5-1)   : the factored form of the initial binomial. Here, we have the
$\quad\quad\quad\quad$ product of the sum and difference of the square roots of the
$\quad\quad\quad\quad$ terms of the binomial of perfect squares.

$\quad$ (6)(4)
$\quad$ 24   checking:

(+25-1)   combining terms:
$\quad$ +24

or

(+5 +1)(+5 -1)   distribute:
$\quad$ +25 -5 +5 -1   combine:
$\quad\quad$ +25-1
$\quad\quad$ +24

As an extra note or thought, when two binomials (here, as a factored form of the difference of two squares) differ only in sign as shown above, they are often called **conjugate** forms of each other. Do not confuse the above discussion with the square of a binomial which is usually expressed in the form of: (a + b)^2 or (a - b)^2

COMMON FACTORS CONTAINING THE SAME TERMS WITH DIFFERENT SIGNS

Sometimes, common factors to both the entire numerator and denominator of a fraction appear that they cannot be canceled because all the signs of the terms within one factor are not the same as the corresponding terms of the other factor. That is, all the corresponding terms are opposite in sign. A method to overcome this problem will be shown below.

Ex.   Simplify $\dfrac{(+5+3)}{(-3-5)}$ By use of the commutative law of addition, this can be expressed as:

$\dfrac{(+5+3)}{(-5-3)}$ After factoring (-1) out of each term in the denominator, and expressing the factored form or product:

$\dfrac{(+5+3)}{-1(+5+3)}$

$\dfrac{\overset{+1}{\cancel{(+5+3)}}}{\underset{+1}{-1\cancel{(+5+3)}}}$: now we can cancel, using the common factor of (+5+3), and then distributing in the denominator:

$\dfrac{+1}{-1}$:clearly, the result here should be a negative value since the signs of the numerator (or dividend) and denominator (or divisor) are different.

$\dfrac{\overset{-1}{\cancel{+1}}}{\underset{+1}{\cancel{-1}}}$:canceling to clear the fraction of its denominator and the ridding of the fraction. Here, both the numerator and denominator are divided by -1. This will leave a standardized or commonly expected value of +1 in the denominator. In short, any value divided or multiplied by (-1) is the "(number line) inverse" or signed number counterpart of that value. For example: -1 (-1) = +1,   and -1/(-1) = +1

$\dfrac{-1}{+1}$ = -1   : since any value divided by 1 is that value, or: (+1)(-1) = -1 when checking by multiplying the divisor and quotient to produce the dividend.

Checking, here the numerator and denominator expressions will be simplified first:

$\dfrac{(+5+3)}{(-3-5)}$ = $\dfrac{+8}{-8}$ = -1   : checking (-8)(-1) = (-1)(-8) = +8 , "the negative of negative eight, is positive eight"

The initial problem above could have been solved, perhaps more easily, by "factoring out" or dividing all the terms of the numerator by -1. This would be similar to the concept of dividing a multinomial by a monomial (and here it would be -1), and expressing the product of that monomial divisor and multinomial quotient:

(+5 + 3) = -1 (-5 -3)   and therefore

$\dfrac{\overset{+1}{-1\cancel{(-5-3)}}}{\underset{+1}{\cancel{(-5-3)}}}$ = -1   : note that canceling does not mean to eliminate completely or to make it a value of 0, but rather make it a value of 1, which does not change anything when using it as a factor or divisor.

Ex. $\dfrac{(+5-3)}{(-5+3)}$ = $\dfrac{\overset{+1}{\cancel{(+5-3)}}}{-1\cancel{(+5-3)}}$ = $\dfrac{+1}{-1}$ = -1   checking: $\dfrac{(+5-3)}{(-5+3)}$ = $\dfrac{+2}{-2}$ = -1

AVERAGE AND STANDARD DEVIATION

Average, or average value, which has been previously discussed in this book, is a basic requirement for understanding the concepts of both average deviation and standard deviation. Average gives us a single value to represent a set of data, but it does not guarantee that the data values used are even close to that ("central") average value. Are the date values dispersed (set apart or away from, their difference) close in value to the average, or are the data values dispersed far away in value from the average? The standard deviation value was defined to answer this question so as to get a better idea of the actual values of a given data set.

Here is a simple example to observe:

Data Set 1:   48.5, 50.5, 51
Data Set 2:   2,   50,   98

The average of both data sets above is the same value of 50. However the dispersion or deviation of data values from the average value is clearly greater (in difference) in the second data set. 2 and 98 are not very close to 50, as compared to the data values in the first data set. How far are they from the average? The answer is the amount they differ (as in a difference value) or deviate away from the average:

data - average = deviation   : the (signed) deviation that a data value is from the average value,   The sign of
                                           this deviation indicates if the deviation was greater (ie.: +deviation_value) or
                                           less (ie.: - deviation_value) than the average value.

Though the sign of the deviation is not required for the computation of the average and standard deviation values, the sign with its corresponding deviation can be used elsewhere to indicate if that deviation value is less than (- , in sign) or greater than (+ , in sign) the average. The deviation with it's sign is the (pos. or neg.) change or difference from the reference value, and here, it is the average value of all the data. For example, if the average of a given set of data was 10, and one of the data values was 7, that data value has a deviation of:

        data - average = deviation
         7   - 10   = -3   : the negative deviation indicates that this data value is
                                              three less than the average value of the data set. This
                                              value can be said as being "below (the) average (value)".

If you find out the deviation from the average for each member of a set of data, you could sum up all the deviations and divide by the number of deviations (or number of data values of the set) to find the average deviation for that set of data (values, "members" or "elements"). This seems to be a correct method but it will not work (without an adjustment) due to the mathematical fact that the average of the data has created, in which the sum of deviations below (less than) the average, plus the sum of deviations above (greater than) the average is always equal to 0. To overcome this, an adjustment is used, and that is to simply use the absolute (ie. unsigned, or positive) value of each deviation. Mathematically, when a value is placed within two vertical lines (ex. | value |) it means to take the absolute or signless value, whether the value is positive or negative. The average value of a set of data (values) may be either positive or negative in value, but the average deviation is considered as a signless value, or as a generic or basic "distance" value from the average in order to represent all the deviations of the all the data values. Due to using only the absolute value of the deviation, you may optionally use this next formula that still gives the deviation a sign, but it wont be used in the formal average deviation formula shown below: average - data = deviation

A.D. = Average Deviation = $\dfrac{\text{sum of } | \text{deviations} |}{\text{number of data values}}$: **AVERAGE DEVIATION , A.D.**

Ex. For the first data set above, the (signed) deviations are:

data - average = deviation

48.5 - 50 = -1.5,   use for example: | -1.5| = 1.5 when finding the A.D.
50.5 - 50 = +0.5
  51 - 50 = +1.0

The average deviation is:

average deviation = _1.5 + 0.5 + 1_ = _+3_ = 1   : Average, and average deviations are mathematical
                              3   3   concepts of the entire data considered, and it does
                                                                            not necessarily mean that any, or all data value(s)
                                                                            are equal to the average or average deviation value.

Ex. For the second data set above, the deviations are:

  2 - 50 = -48   use: |-48| = 48 when finding A.D.
50 - 50 = + 0
98 - 50 = +48

The average deviation is:

average deviation = _48 + 0 + 48_ = _96_ = 32
                              3   3

Standard deviation (S.D.) is close in value to the average deviation, but the standard deviation is more like an expectation or prediction, rather than an actual mathematical fact like average deviation. Standard deviation is the deviation that most data values could have from the average deviation, hence standard deviation could be called or thought of as something like: "the most likely deviation".

For example, given the data set of: -5, +5

,the average is 0 and it is easy to see that most values (hear, actually all the values) deviate from it by 5.

Given the data set of: -5,   0,   +5,

,the average is 0 again, and most values (the -5 and +5) deviate from it by 5.

Given the data set of: -5, -5, -5, -5, -4, +4, +5, +5, +5, +5

,the average is 0, and most, but not all, values deviate from it by 5. You should expect the standard deviation value to be close to 5. Let's derive the formula for standard deviation:

The derivation of standard deviation is similar, at first, to average deviation:

1.   For each data value, find its deviation from the average.
     average - data = deviation

2.   Rather than take the absolute value of negative signed deviations for their effective positive value, square each deviation since any value squared is positive in sign. This also has the effect of magnifying a larger deviation
(at least >1) much more than a smaller deviation. Small deviations (particularly those less than 1) are therefore very small in value and practically eliminated since any fractional value squared is a smaller value since a "fraction of a fraction"

yields a smaller value.

3.   Find the average of the squared deviations:

$$\text{average of the squared deviations} \quad = \quad \frac{\text{sum of squared deviations}}{\text{number of data items} - 1}$$

Notice 1 is subtracted from the total number of data items. This is due to the fact that one of the data items is predicted to be at least near or equivalent to the average, and that this data value should not be counted (included) in the set since it does not deviate from the average and therefore cannot play much of a role in any expected or predicted deviation. Remember, we are not looking for a true mathematical value as like average deviation, but we are looking for a "most-likely" value. If one is not subtracted, you will have another form of standard deviation that yields a "compromise" value that is a value between the average and standard deviation values of a set.

4.   Since we squared the deviation values in the formula, the result is essentially another squared value, and we need to do the inverse (or reverse if you will) and take its' square root to effectively arrive at the square root value before it was squared:

$$\text{Standard Deviation} \quad = \quad \sqrt{\text{average of the squared deviations}} \qquad : \textbf{STANDARD DEVIATION}$$
(S.D.)

For example, given the set of:

-5.0,   0.1,   +5.0

The average of this set is +0.033333333.

By observation, you could then say that most values have a deviation, from the average, of about 5. and that the standard deviation for this set should be about 5.

The deviations are:

-5.033333333,   +0.066666666,   +4.966666666

The deviations squared are:

25.33444444,   0.004444444,   24.66777777

Using the formula shown above for the average of the squared deviations (using data items -1), we get:

25.00333333

Taking the square root of this we arrive at:

standard deviation   =   5.000333322

Notice that this value is slightly larger than the predicted value of 5. This is due to the "greater magnification" of the larger deviation values after squaring them.

For comparison purposes:

$$\text{average (unsigned) deviation} = \frac{5.033333333 + 0.066666666 + 4.966666666}{3} = 3.355555555$$

For the data set given, most values had a deviation of 5 from the average value, and the standard deviation value yields a better deviation value for most of the data than the average deviation value shown here does.

The "compromise value", as discussed above, is calculated to be: 4.082755061

SCIENTIFIC NOTATION

Scientific Notation (S.N.) is a standard numerical notation to expresses any numbers, and in particular, very large or small numbers that have many digits where its value perhaps can't be easily realized or worked with.   The notation still uses the same digits values of that number, but has only one whole number between, or including, 1 and 9, and with all the fractional (less than one) digits appended on, and a multiplying power of 10 attached to the number. This notation helps with manually processing calculations, or for expressing the results of calculations to make them appear presentable and intuitively manageable. Since numbers expressed in scientific notation contain powers of 10, the rules for multiplying and dividing like values with exponents, such as with powers of 10, will allow for greater agility and speed for calculating or estimating some results.

To convert a number greater than one to scientific notation, move the decimal point to the right of the most significant digit (MSD), and the multiplying power of 10, to this newly created lesser value, will have a positive exponent equivalent to the number of digit positions ("decimal places") it was moved. Any trailing zeros can be omitted.

$$\text{Ex: } 70 = 7.0 \, (10\text{^}1)$$
$$\text{Ex: } -3579.2 = -3.5792 \, (10\text{^}3)$$

To convert a number less than one to scientific notation, move the decimal point to the right of the most significant, non-zero digit (MSD), and the multiplying power of 10 will have a negative exponent equivalent to the number of digit positions the decimal point was moved. Any trailing zeros can be omitted. Note, that since the number was made apparently larger, there must be a value that would decrease or divide it so that it would be equivalent to its original (lesser) value from which it was created or derived from. This value is the negative power of ten which actually means division by an equivalent positive power of ten.

$$\text{Ex: } -0.00123 = -1.23 \, (10\text{^}{-3})$$

Ex. Convert 35,000 to scientific notation (SN)

35000 = 3.5(10^+4)

checking: 3.5(10^+4) = 3.5(10,000) = 35,000

Ex. Convert 50.0250 to scientific notation

50.0250 = 5.0025(10^+1)

checking: 5.0025(10^+1) = 5.0025(10) = 50.025

Ex. Convert   0.0047 to scientific notation

0.0047 = 4.7(10^-3)

checking: 4.7(10^-3) = $\dfrac{4.7}{1}\left(\dfrac{1}{10\text{^}{+3}}\right)$ = $\dfrac{4.7}{1000}$ = 0.0047

Ex. Multiply 20,000 by 3,000

(20000)(3000)
2(10^+4)3(10^+3)   : each factor represented or expressed in scientific notation
(2)(3)(10^+4)(10^+3)   : associative law

6(10^(+4+3))   : when multiplying like values or variables, add their exponents
6(10^7)   : 10^7 effectively places 7 zeros after the value and moves the
          decimal point seven places rightward:

60,000,000 = "sixty-million"

Another method to convert decimal values to fractional values and vice-versa, is to use some of the concepts of scientific notation.

Ex. Convert 0.247 to an equivalent (proper) fractional form

$$0.247 \;=\; 2.47(10^{-1}) \;=\; \frac{2.47(1)}{1\,(10^{+1})} \;=\; \frac{2.47}{10} \qquad : \text{"two point forty-seven, tenths", OR:}$$

$$0.247 \;=\; 24.7(10^{-2}) \;=\; \frac{24.7(1)}{1\,(10^{2})} \;=\; \frac{24.7}{100} \qquad : \text{"twenty-four point seven, hundredths", OR:}$$

$$0.247 \;=\; 247(10^{-3}) \;=\; \frac{247(1)}{1\,(10^{3})} \;=\; \frac{247}{1000} \qquad : \text{"two-hundred and forty-seven thousandths"}$$

Note, as seen above, the first factor in the indicated products need not be between and including 1 and 9, as in true scientific notation, as long as the power of 10 is adjusted correctly and the resulting expressed values are equivalent. This type of (exponential) notation is then a modified form of the standard scientific notation.

Ex. Convert 52.3/100 to scientific notation, and then convert this equivalent scientific notation to its equivalent decimal form.

$$\frac{52.3}{100} \;=\; \frac{52.3}{10^{+2}} \;=\; 52.3(10^{-2}) \qquad : = 0.523$$

$$52.3(10^{-2}) \;=\; 5.23(10^{+1})(10^{-2}) \;=\; 5.23(10^{(+1\,-2)}) \;=\; 5.23(10^{-1}) \qquad : \text{scientific notation}$$

$$5.23(10^{-1}) \;=\; \frac{5.23(1)}{1\,(10^{1})} \;=\; \frac{5.23}{10} \;=\; 0.523 \qquad$$
: decimal notation. Hence the same as the above, but
       without any multiplying factor such as the power of 10.

checking: $\dfrac{52.3}{100}$ = 0.523

   Scientific calculators and computers often represent values in a modified scientific notation format with a symbol of (E), meaning "exponential", instead of indicating or displaying the common base value of 10. The exponent (of 10) will follow the (E) symbol.

Ex. If the calculator display is: 1.234E-12
  This is to be interpreted as: 1.234E-12 = 1.234(10^-12) = 0.000,000,000,001,234 : "1.234 pico"

This notation is very helpful since the calculator may only have 8 or 10 digits. It should also be pointed out that these digits of the calculator display are for only the significant digits of the result of an operation, and any lesser valued. least significant digits not capable of being displayed, stored, processed, or entered by the user, are essentially lost and won't be available for further calculations. If the calculator "truncates" or discards some least significant digits of a value, this

"error" or deviation from the true value can eventually "build up" (increase in value) and may need to be considered when using that value (with some slight, or insignificant error) for any possible further calculations.

MULTIPLYING AND DIVIDING BY NUMBERS TOO LARGE FOR AN ELECTRONIC CALCULATOR

Most hand-held electronic calculators are either the 8 to 10-digit display types, hence they are limited to that maximum amount of digits, and therefore, any value or calculated result with more digits than this maximum amount of digits cannot be fully entered or displayed. The value, as entered, is too large for that specific calculator to use and work with (ie., process). The least significant digits will then be lost, and the result will usually be rounded at the least significant digit that is displayed. A loss of trailing digits and fractional amounts can eventually lead to incorrect accuracy of the result(s) if this "shortened" or approximated value is used as an operand for further operations. In general, multiplying an operand having N1 (some number of) digits by another operand with N2 digits will yield a product with about (N1 + N2) digits, give or take 1 digit, and which may be more than your calculator can display. If the product is too large for your calculator to display, your calculator will either signal an "overflow error" (the value is greater than the maximum value that can possibly be displayed on and-or processed within that calculator), or it may display a "scientific" (exponential, power of 10) form with only the most significant digits and "dropping" (disregarding, truncating or loosing) the least significant, "last" digits. For most cases this wont be a problem, and is acceptable. Below are some basic examples of how to overcome this number of digits limitation if you need to. These examples assume that your calculator is an 8-digit (for internal calculations, and display) type and you want to maintain the least significant digits within a calculation using that calculator. However, as will be shown below, you may need to perform a final calculation by hand.

Ex.   (7)(4321.567895)   : the second factor contains ten digits, and this can be expressed as:
       (7)(4321 + 0.567895)   distributing :
       (7)(4321) + (7)(0.567895)
       30247 + 3.975265   summing by hand (ie. the calculation performed by you using a pen on paper) :
       30250.975265   : an eleven digit result

Ex.   (5) (1234567891)   : the second factor contains ten digits, and this can be expressed as:
       (5) (1234500000 + 67891)
       (5) ((12345)(10^5) + (5) (67891))
       (61725) (10^5) + 339455   multiplying by observation :
       6172500000 + 339455   summing by hand :
       6172839455   :a ten digit result

Ex. Divide 3 by 10,000,000,000   factoring 10,000,000,000 into two (or more) factors:

$$\frac{3}{10{,}000{,}000{,}000} = \frac{3}{(1{,}000{,}000)\,(10{,}000)} = \underline{0.000{,}003} \over 10{,}000 = 0.000{,}000{,}000{,}3 = 3(10^{-10}) = \frac{3}{10^{10}}$$

The example above can be solved in a variety of ways, and specifically determined by the powers of 10 used.

METRIC SYSTEM OF UNITS

The sub-units (ie., fractions of a unit), and multiples (ie., powers of a unit) of a unit of measurement within the metric system are all based on 10 of the specific "base" or fundamental units in use, such as for example, the fundamental unit of length called a meter. Why 10? The answer is that since the decimal system that we count with is based on 10, there should then be a standard system of units that are also based on 10. The word "metric" is based on the words and concepts of "to meter", which means "to measure", and "numeric". The metric system was created to be a practical, world-wide accepted standard system of measurements, and so as to avoid the many, and with some being antiquated (old and-or unused), systems of measurements in use which can cause problems for those not familiar with them and-or the conversions between two or more different measuring systems.

Type of Measurement	Base units in the U.S.A. system of measurements (Many are also now using the metric system.)	Base units in the metric system of measurements.
Distance or Length	mile or feet (mi)	meter (m)
Weight	pound (lb)	Newtons (N) for weight or force, and gram (g) for mass or matter
Volume	gallon (gal)	liter (L)

The Appendix Section of this book contains many conversions and constants.

Within the metric system of measurements, to change from larger to smaller units (technically called a reduction of units within the system, and commonly (although somewhat incorrectly) referred to as a "conversion" of units which is actually for converting between units of two different measuring systems and their units) that are some multiple or fraction of the base, fundamental or standard reference units, simply multiply or divide by a power of 10. The same process of simple multiples or division of units cannot generally be said about the English or British system of units. For example, in reference to the English length measurement units, 1 foot is 12 inches, 1 yard is 3 feet, 1 mile is 5280 feet. Here, the (conversion) factors vary widely: 12, 3 and 5280, and therefore, there is no systematic or logical approach to the relation of the units within that measurement system. One advantage of anything based on 12 units, such as a dozen or 12 inches, is that it can be evenly divided by or into 2 (a half), 3 (a third), 4 (a fourth or quarter), and 6 (a sixth) smaller fractional parts or fractional units. Half of (1/4) will give (1/8), half of the (1/8) will give (1/16), and half of the (1/6) will give (1/12). A system based on 10 units can only be evenly divided by 2 and 5, and with half of 2 being 1 step or unit.
.
In the metric system, the powers of 10 of the derived units are frequently represented as word or alphabetical abbreviation prefixes before the base unit. These prefixes are usually written as abbreviations of the corresponding name given to that power of 10. For example, all the units of measurement for distance or length values are based on, or are in reference to, the unit of measurement called the meter (m). Hence the base units for length are called meters. A centimeter (cm) is defined as one-hundredth of a meter. A centimeter is a sub-unit or fractional unit and part of a meter. The word or unit prefix "centi" (as previously discussed with the concept percentages) numerically means the multiplying factor of: (1/100), "one-hundredth", or= $(1)(10^{-2})$, and its abbreviation is (c). Thus, 1 centimeter = 1cm = $(1)(10^{-2})$m = 0.01m, which is one-hundredth of a meter. There are 100 centimeters in one meter. In other words, 1 meter is equivalent in length as 100 centimeters, and vice-versa. A kilometer (km) is one-thousand meters. The prefix kilo (k) means (1000) = $(1)(10^{+3})$ = 10^3. 2km = $(2)(10^3)$m = (2)(1000)m = 2000m.

Here's an example of a conversion (although it's technically or formally a "reduction") within the metric system:

Since there are 1000 meters per kilometer, (1000m/1km), or one kilometer per 1000 meters, (1km/1000m), to convert meters to kilometers, simply divide the quantity of meters by 1000. In short, when converting from smaller units to a larger units, use division, otherwise, use multiplication when converting from larger units to smaller units since there is many more smaller things within a larger thing such as larger sized units of measurement. Here is a mathematical verification and example:

1000m = 1km   : this is the known, given and basic equivalence between these two units of measurement.
                    Solving for (1) meter by dividing both sides of the equation by 1000:

$$\frac{1000m}{1000} = \frac{1km}{1000}$$ after canceling common factors:

$$1m = \frac{1}{1000} km$$: "one meter is one-thousandth of a kilometer". This can also be expressed as: 1m = 0.001km

Multiplying each side by some number (N) such as some number or multiple of meters:
N (1 meter) = N (1) meters = N meters, and we must do the same to both sides to keep their equivalence:

$$(N)1m = N\left(\frac{1}{1000}\right)km = Nm = \frac{N}{1000} km = (0.001\ N)\ km$$ or: 1000m = 1km
                                                                                      N (1000m) = N (1km)
                                                                                      (1000N) m = N km

Hence. when converting (some number or multiple of) meters to kilometers, divide the magnitude (value) of meters by 1000. For example, to convert 4752 meters to it's equivalent distance with kilometers units:

$$4752\ m = \frac{(4752)}{(1000)} km = 4.752\ km$$

You may also wish to consider this method which considers the ratio of one unit to that of the other to find the proper multiplier to use when converting between those units. With the division of like or similar values rules, the process to find the multiplier is easy, such as by subtracting the exponents.   Consider this example of converting centimeters to kilometers:

$$\frac{1\ cm}{1\ km} = \frac{10^{-2}\ m}{10^{+3}\ m} = \frac{10^{(-2 - (+3))}}{1} = 10^{(-2-3)} = 10^{-5}$$: shown with some intermediate steps.
                                                                                      : the units essentially cancel out or are
                                                                                        eliminated since its a strict numeric, or
                                                                                        unit-less value, here just a ratio.

Now, mathematically, (dividend = quotient x divisor), and from this equation above we have:

$$1cm = 10^{-5}\ (1\ km)$$: the multiplier is: $10^{-5} = 1/100,000 = 0.000,01$
$$1cm = 0.000,01\ km$$

Multiplying both sides by N, as for some given number of centimeters:

$$N\ cm = (0.000,01\ N)\ km$$ mathematically, we also find:

$$N\ km = \frac{Ncm}{0.000,01} = \frac{Ncm}{10^{-5}} = N\ 10^5\ cm$$

Ex. $$1km = \frac{1cm}{0.000,01} = 1(10^5)cm = 100,000\ cm$$

Here are the common metric prefixes for the base-units of measurement:

	tera(T)	= 1(10^+12)	= 1,000,000,000,000	: trillions
	giga (G)	= 1(10^+9)	= 1,000,000,000	: billions
*	mega (M)	= 1(10^+6)	= 1,000,000	: millions
*	kilo (k)	= 1(10^+3)	= 1,000	: thousands Ex. 5km = 5 (1000)m = 5000m
	hecto (hc)	= 1(10^+2)	= 100	: hundreds
	deca (dc)	= 1(10^1)	= 10	: tens (of units)

		1	= 1	: "ones" or "whole", complete or entire units, no unit prefix used, but explicitly indicated
	deci (d)	= 1(10^-1)	= 0.1	: tenths (of a unit)
	centi (c)	= 1(10^-2)	= 0.01	: hundredths Ex. 100cm = 100 (0.01)m= 1m
*	milli (m)	= 1(10^-3)	= 0.001	: thousandths
*	micro (u)	= 1(10^-6)	= 0.000,001	: millionths
	nano (n)	= 1(10^-9)	= 0.000,000,001	: billionths
	pico (p)	= 1(10^-12)	= 0.000,000,000,001	: trillionths

* Note that these (*) indicated base-unit prefixes differ by a factor of 1000, and including the prefix centi, these prefixes are perhaps the most commonly used. A discussion on CHANGING UNITS given further ahead in this book shows how to make conversions between the Metric and English systems of measurement, or within a single particular system.

The base unit of mass (a measure of the amount of actual physical substance [like atoms], matter or material) in the metric system is the gram (g). The word of "gram" is rooted in words of: "grain", "measure", "measurement", and "weight", and originally meant a small weight or piece of something. A gram of water is defined as having the measurement of one cubic centimeter (1 cc, a volume measurement, ie., "space" or "three dimensional spacial area") when its temperature is just above the freezing temperature of water. 1000 grams of matter or a substance is equal to 1 kilogram of that mass: 1000g = (1k)g = 1kg = (10^3)g. 1cc volume of water is defined as equal to 1g of mass of water. 1cc of other substances will be more or less grams of mass since their density (mass/volume = amount of matter per volume) is different than that of water. 1000cc volume of water is defined as having 1000g of mass = 1kg of mass.   A cube shape, 10 centimeters long, 10 centimeters wide, and 10 centimeters high is (10cm x 10cm x 10cm) volume = 1000cm^3 = 1000 cubic centimeters = 1 liter unit of volume size.

For spacial, space, or volume measurements, the liter (L) is the base unit of measure in the metric system. 1liter = 1L = 1000cc. 1 liter volume of water is defined as having 1kg of mass, and has a "proportional, corresponding or associated weight (technically a force)" indicated as 1kg on a weight scale, however, 1 liter volume of some other substance like copper, gold, zinc, sodium, lithium, plastic, foam, etc., will have a "corresponding weight" of more or less than that of water, due to that the (mass) density (= mass/volume) of that particular substance is not the same as the density of water which is 1gram/1cc.

One-thousandths of a liter = 1L/1000 = 1000cc/1000 = 1cc = 1(10^-3)L = 1 milli-liter = 1ml

For conversion purposes:

1 fluid ounce , water = **1fl. oz** , essentially a volume oz measurement, = **29.5735cc** = 29.5735 grams if water (Often in the U.S.A food industry, 1 fl. oz. is set as equal to an even 30 grams of "corresponding scale weight".)

1 pound = 1lb ,   a weight measurement, = 0.453592 kilograms = 453.592 grams
1 ounce = 1oz ,   a weight measurement, = 1lb/16 = (1/16)lb = 0.0625lb = 453.592g/16 = 28.3495g

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SECTION 2: BASIC ALGEBRA

ALGEBRA

Algebra is a form of mathematics where symbols or letters are used to (symbolically) represent numbers.   The English word "algebra" is just a couple of hundred years old and is derived from the older Arab word al-jbar. The basic interpretation of this word is: "the combining of several things, so as to have one thing". Simple algebraic-like methods have also been used since ancient times, primarily by the Egyptians, Greeks, and Hindus. This mathematical "invention" or knowledge could be said as always being someplace within the ability of humankind right from the start of our first basic civilizations, and therefore it was surely destined to happen, grow and "sprout up" like a nurtured seed as time went by, and to this day, it is still growing and being useful.

When a value remains the same and-or cannot change in value, that number is said to have a constant value, and it is therefore called a **constant**.   When a number is not a constant or a specific known value, or that number is an unknown value yet to be known and-or solved for, it is or has a value that can variate (from the words vary (changeable), and various (several)) and which means that it is non-specific, non-constant, and that it can be one of many possible values, and-or change in value. The symbolic names, letters, identifiers, representatives or any other symbolic placeholders for numbers that can vary in value and-or are yet unknown in value are therefore called **variables**. Variables are often found in mathematical expressions and equations. Constants can also be given symbolic identifiers or names. On of the most common natural constants is Pi ("pie") which equals about 3.141592654 and is the result of any circle's circumference length divided by the diameter length of that circle.

If the temperature of something can variate, fluctuate or change in value, we can use something like the word: "Temperature" itself since it makes a good identifier for the value it represents. Simply using the letter T is more compact and usually acceptable, and when it is understood as to what it represents. Temperature or T, or some other identifier, can be used as a symbolic placeholder yet to be set to a specific numeric value and-or for an updated numeric value of the temperature.

Consider this analogy: Within a box, the current temperature is written on a piece of paper. That box is essentially a placeholder for the temperature value, and this box could be identified and labeled with the letter T on it. That box and value it contains within it can also be mathematically represented and expressed with that identifier of T. Since the temperature can fluctuate or vary in its value, the identifier of T is a therefore also called or identified as "variable T". T is now understood to identify,   represent, and mathematically express and be equivalent to a varying and-or unknown value of the temperature.   The value of T can be set (assigned, written), changed (updated, adjusted), or read (viewed, obtained), since it is a placeholder for a value.

> T = this letter or symbol identifies and represents both the temperature and its corresponding
>   numerical value. If the temperature can vary or change in value, T is called a variable, and is
>   specifically known as:"variable T". If the temperature measurement was 70 degrees, T will be
>   set or assigned with this value of 70:   temperature = 70, and since T = temperature, T = 70.

To use identifiers or variables as equal to, and as if they are, actual numbers is very helpful and useful in mathematics. For example, unknown values can therefore be expressed and mathematically worked with as if they were a (literal, actual or specific) numeric value. Another advantage is that variables don't then need to be a specific numeric value so as to be mathematically useful, and it is understood that the variable is a general representation of one of many possible and-or changing values. Note that even though a variable can have a changing value, it is still only equal to one specific value at a time or measurement, and it should be said, indicated or understood that its value is a current (as of now, or at this time) value. Even after saying all this, it is still possible for a variable to have the same value for quite a long time, and therefore it acts or appears as if it were a (unchanging) constant.

Mathematical operations involving variables are performed in a manner very similar to using actual numeric values. It is easy to work with variables when you begin to "think of" or treat them just like actual numbers. Some

examples of variables are: A, B, C, X, Y and Z. Usually, if you are to create a variable's name (symbol(s)), or "identifier" (often used as a general name for any variable as for computer languages), and as previously mentioned, it is best to choose one that will be descriptive to what the value represents. Sometimes, rather than use A or L to represent length, it is better to use a descriptive name like: LENGTH since this will be helpful for your future reference and-or for anyone else who sees it used in a formula or calculations.

Two or more variables written side by side are considered as being multiplied together. For example, AB means "A times B", (A x B) , or (A)(B). Since multiplication is commutative, the order of performing or expressing it does not matter. for example: AB = BA. A value in front of a variable(s) is called it's numerical coefficient (ie. cofactor). A numerical coefficient is a multiplying factor of the indicated product. For example, 2A = (2)(A). Due to multiplication being associative, the numerical coefficient can be, or perhaps was previously, associated with any other factors of that term of which it is in. For example, 2AB = 2(AB) = (2A)(B) = A(2B) = AB2. For conformity, the numerical coefficient is usually placed first. If a variable does not have a numerical coefficient indicated, it is considered to have a numerical coefficient of one (1) which of course does not change its value. Likewise, for example, 2A can be considered as the coefficient of B in the term of: 2AB.   This coefficient then has a numeric part and factor: 2, and a variable part and factor: A.

It is also of importance to understand that a variable, being just a placeholder, can represent any numeric value, positive or negative in sign, and may even represent the resulting value of some other expression. It should be generally understood that the actual numeric value of a variable can be either positive or negative in sign.

Ex. If variable A is equal either to: -10, this can be correctly expressed as: A = -10,
   It is incorrect to express this situation with the equation of: -A = -10, that is: -A is not equal to -10. As a verification
   of this, if you divided both sides by -1, you will have this equation: A = 10 which we know is wrong.
   It is possible to express the correct equation of: A= -10   as: -A = 10   because the equation is still in balance and
   you can check, determine or verify this if you divide both sides by -1, and the result is A = -10.

Ex. Total = reference value + C   : here C represents some change (from and with respect
                                                  to, or in reference to, the given reference value)

      If C = +10,   Total = reference value + (+10)   =   reference value   +10
      If C = -10,   Total = reference value + (-10)   =   reference value   - 10

Notice that with two possible signs for the value of the variable, here C, that only one algebraic formula is needed as originally stated in the example. If the formula was expressed or written as: Total = reference value - C, hence as a difference between the values, the resulting value of the equation would be different, and specifically for this equation, it would result in opposite signed values.

Another thing to be aware of with variables is when a variable is expressed as being raised to an indicated (with an exponent) power. For example:

XY^2

Above, only the Y variable is to be squared, not both (the product of X and Y) of the variables. The exponent (ie. indicated or expressed power) is applied only to the immediate variable and not to a product of variables or an expression unless those are explicitly indicated within grouping symbols as being raised to the indicated power. Perhaps this rule would be more clear if the commutative or associative law was considered, and the expression was re-written as:

From: XY^2 to= Y^2X   to show that only Y is being raised to the second power.   or, by using parentheses:

From: XY^2 to= X(Y^2)   =   (Y^2)X

If the indicated product of X and Y were to be squared, it would be noted with grouping symbols such as parentheses as:
(XY)^2   expressing or "extending" the indicated operation:

$(XY)(XY) = X \, Y \, X \, Y = X \, X \, Y \, Y = X^2Y^2$, hence: $(XY)^2 = X^2Y^2$: a product to a power, then
distributed the exponent

Note that with: XY^2 , it is incorrect to think that the two factors here are just (X) and (Y). The factors of this product are actually: (X) and (Y^2).

     In daily life, we often need to make what are actually, and can be expressed as, simple algebraic equations to solve a problem. Take the information from the problem stated and apply variables and constants, known or unknown, so as to form an equation to mathematically represent the situation. For example, Mike is three years older than Tom, and we can express this as an equation. Actually, this can be expressed three ways, and that each equation can be mathematically derived from each other:

     Let variable M = Mike's age   and
     Let variable T = Tom's age

    1. $M = 3 + T = T + 3$

    2. $M - T = 3$: here the left side expression is a difference of the variables set equal to the constant
                            difference value of 3.

    3. $M - 3 = T$ and this is equivalent to after switching sides: $T = M - 3$

Mathematically (or "algebraically"), all of the equations above can be derived from any one of the other
shown equations. Any initial choice in the equation you do create and use, and if correct mathematically
such as one of these equations above, will still yield the correct result.

LIKE TERMS

     Terms that have the same exact variables are called similar or like terms. The exponents of corresponding variables of two or more like terms must also be the same. That is, the indicated powers of a variables must also be the same so as to still represent the same numeric value. The numerical coefficients or multiplying factors of like terms do not need to be the same.

Examples of like or similar terms:

  AB and BA
A^2B and BA^2
5X^2 and 2X^2   : here, the numerical coefficients are different.
4XYZ and 3YZX   : Often, you will want to arrange these terms so that they appear more visibly
                           as being like terms. Ex., 4XYZ and 3XYZ , or any other possible way.

If A=10, and B=2: $AB = (10)(2) = 20$ and $BA^2 = (2)(10^2) = (2)(100) = 200 = A^2B$

   Clearly AB and BA^2 or $= A^2B$ are not like terms since they represent different values.
   Here, 20 is not equal to 200. AB and A^2B are not like terms.

ADDING LIKE TERMS

"Like" or similar terms can be combined (added, summed). For example, 2X + 3X is a two-term (algebraic) expression. The terms +2X and +3X are like terms (both have the exact same variable(s), here X, including their indicated powers, and therefore can be combined just like being the same numeric value to produce a multiple of that value.

Given +2X and +3X, how do we combine these (like) terms?

First notice that a common factor (in both terms) of X can be divided (ie. factored) out of each term:

2X + 3X = X(2 + 3) = (2 + 3)X = (5)X = 5X

If you had 3 of a certain value, and also 5 of that same value, you then have (3+5) = 8 of that value, and which also means 8 times that value.

Hence, **to combine like terms, combine their coefficients, and keep the same variable(s).**

Here is a check using the concept of multiplication to represent repeated addition:

(2X) + (3X)	"extending" the indicated products and expressing their "partial sums" or terms:
(X + X) + (X + X + X)	distributing (+1) to clear the grouping symbols:
X + X + X + X + X	using multiplication to represent this repeated addition of X, here five times:

X5   or = 5X   : The left hand side is the "common mathematical form" of expressing this
                     result, and it has the form of a number or variable to be multiplied 5 times.
                     The right hand side is the equivalent "algebraic form" where the multiplier
                     is placed first and is called the numerical coefficient of the variable.

                     We see that summing or combining the numerical coefficients of like
                     terms is a shorthand way to repeatedly add like terms (which may be
                     already some existing multiple of the variable(s))..

Ex. 4X + BAX = X(4 + BA)   : here X, the common factor of each term, was factored out (divided out of
                                    each term) and is expressed as a factor to the resulting quotient so as the two
                                    expressions remain in balance.

Another, perhaps simpler, analogy or verification of the addition of like terms stems directly from the basic concept of adding things which are similar or have like units. To use units as variables is non-standard, but it serves as a good illustration. In a strict sense, variables are a symbolic (representative) numeric concept of representing a numeric magnitude or quantity (of the units) that is known or unknown with symbolic letters, text or other symbols. For example, we could let letter, or the symbol, B (or perhaps: *, or notches such as | or 1) represent an object such as a brick or "brick unit", rather than some numeric or variable value. If there are five bricks, this can be algebraically represented as 5B, meaning 5 Bricks = "five bricks". If you divided or separated the bricks into a group of three bricks, and another group having two bricks, the total sum of the bricks is still equal to five bricks:

2 bricks + 3 bricks = (2 + 3) bricks   =   5 bricks
  B, B   +   B, B, B = (B + B) + (B + B + B) = B + B + B + B + B   : optional, for clarity
   2B   +   3B   = (2 + 3)B   =   5B

If we then let B=Bricks, variable X = 2, and Y = 3 for the quantity of that variable or unit, the above could then be expressed as:

```
XB     +    YB     =    (X + Y)B
2 Bricks  +  5 Bricks  =   (2+5)Bricks  = 7Bricks
```

In the standard sense of mathematical understanding, when B represents a numeric value, variable or constant, and not some object identifier or unit of measurement as temporarily used here to illustrate the summing of "like" or similar terms concept and method, the equations and operations as expressed here are still the same, except that then the numerical coefficients (here, X and Y) of variable or value B, are actually to be considered as (multiplying) factors to B. Likewise, B would also be considered a multiplying factor in each term, and since it is in both terms, it is a common factor to and of those terms, and can be factor or divided out from each term and then expressed in the "factored form" (of the original terms). The factored form of the expression is the product of B and a product that is the expressed sum of the remaining terms. Doing the reverse is called the "distributed form" of the expression.

Ex. Simplify   5X -3X

+5X -3X = (+5-3)X = (+2)X = +2X   : After combining the numerical coefficients of like terms and
                                                 keeping the variable(s). You can check this result by first
                                                 factoring variable X out from each term, and that it is indicated
                                                 here in the second equivalence: (+5 -3)X or= X(+5 -3)

The intermediate steps shown above are often considered as "understood" (known) and hence they are usually omitted in practice for the sake of some speed, efficiency and clarity, however, its always good to see what has actually taken place.

Ex. Simplify   -3X +4B +3B -X

The terms -3X and -X are like terms, and the terms +4B and +3B are like terms.

-3X -X = -3X -1X = -4X   , and   +4B+3B = +7B

The sum is therefore:

-4X+7B   or =   +7B -4X   : since combining terms is commutative, so therefore, the order of which
                                term comes first, or is expressed first (perhaps for some clarity), has no
                                affect on the result.

If X=3 and B=2 above, the value of the expression is found by assigning or substituting these values into their corresponding variable numeric placeholders:

 -4X +7B = -4(3) +7(2)   D (distribute, from the CDC method mentioned previously):
                -12 +14   C (combine terms using the laws for the signs):
                   +2

ALGEBRAICALLY VERIFYING DISTRIBUTION

The concept of distribution has already been mentioned, discussed and used several times in this book, and now we are at the point to where it can be algebraically described and verified below:

a + a + a + a + a + . . .   : The mathematically expressed repeated addition, of a variable or value.
                                     an endless or infinite amount of times, hence the sum is not an
                                     expressible value, other than mentioning that it is infinitely large.

If we consider a total, or limited number, say (n) of the above terms by using grouping:

(a + a + a + a + a) = an = na   : Expressing the repeated addition as multiplication.

Now grouping terms of this set of (n) terms into two groups or sets
of terms of variable (a), with one set containing a total of (b) like
terms ,and the other, a total of (c) like terms; (here, (b) will actually
be 2, and (c) will be 3 for this analysis example):

(a + a) + (a + a + a)   expressing these groups of repeated addition, as multiplication:
      ab   +   ac   : ab + ac = ba + ca   is a more common form with the coefficients of (a) first

Since n = (b + c) ,   : n = the total number of repeated additions of the like terms

a +a +a +a +a = an = na = (b + c)a = ba + ca = a(b + c) = ab + ac   : verifies distribution
                                           2a + 3a   = a(2 + 3) = a5 = 5a

Also notice that a(b + c) is the factored form of (ab + ac), where each term has a common factor of (a).

Ex.   (a + b) (c + d)   expressing an important intermediate step of distribution that is often omitted:

    a (c + d) + b (c + d)   : Notice, and as a verification, that each term here has a common factor of
                                     (c + d), and if you were to factor this out of each term, you would then have the
                                     original expression.

    ac + ad + bc + bd   : this is the distributed form of the above expression, and you can verify that this
                                     is correct by factorization, and it is essentially a reverse process - such that
                                     during multiplication, a factor is put into each term, and during factorization
                                     or division, a factor is essentially pulled back out from each term:

    (ac + ad) + (bc + bd)   : grouping like terms, one group has (a), and the other group has (b) for each term.

    a (c + d) + b (c + d)   : factoring out the like variable(s), or values, from each grouping of like terms

    (c+d) (a + b)   : here, (c + d), an expression, is treated like a single variable or value
                                     contained in the grouping symbols, and was factored from each term.
                                     This can be expressed as, using the commutative or order law:

    (a + b) (c + d)   : this is the initial expression shown above. This gives verification to how
                                     distribution was used and expressed starting from the initial expression.

Below, a few generalized examples of expressing and working with algebraic values, equations, and distribution will be shown.

Ex. If you were to add and subtract the same value, say (b), to or from another value, say (a), what is the difference (d) in the resulting two values?

Letting: N1 = a + b,   and   N2 = a - b   : N1 and N2 represent binomials (2 terms), and N2 is a difference of terms.

Let: d =   N1   -   N2

    d   = (a + b) - (a - b)   : subtracting (or adding the negative of a value) to find the difference
                                            Distributing (+1) and then (-1) to clear grouping symbols:

  d = a +b -a +b   combining like terms of variable (a), and variable (b):

  d = 2b   : You can imagine or see this when you take any value (a) on a numbered-line and
                               increase it by (b), and also decrease it by (b). This change in value, or (b), can be
                               thought of as a line segment graphically, and the difference in the values of N1 and
                               N2, is 2b. The specific value of (a), which we can think of as the starting or reference
                               value, which is also centered on the entire length of 2b, does not affect or play any
                               role in the difference value, and is therefore absent from the resulting equation or
                               formula (for d) for this situation. If (b) was called the change of or from the starting
                               value (a), then (d) could then be expressed as: d = 2(change).

                      Ex. If a=0 and b=5: d = (0+5) - (0-5) = 5 - (-5) = 5 + 5   =   10
                                            d =   2(b) = 2(5) = 10

Ex. What is the formula for the difference (d) between $(a + b)^2$ and $(a - b)^2$?   Note that this is similar to
    the above example, but here the terms, or its (equivalent) value, are now squared.

  d = $(a + b)^2$ - $(a - b)^2$: mathematically expressing the difference. Extending the squared terms:

  d = (a + b)(a + b) - [(a - b)(a - b)]   using distribution (D) within the grouping symbols:

  d = [a^2 + ab + ab + b^2]   - [a^2 -ab -ab +b^2]

  d = +1[a^2 +2ab + b^2] - 1[a^2 -2ab +b^2]   : notice in the two grouped terms, that the term 2ab is
                                                                   present and either pos. or neg. Distributing +1 and -1
                                                                   to clear grouping symbols:
  d = a^2 +2ab +b^2 -a^2 +2ab -b^2 combining (C) like terms:

  d = 4ab   : When a=b, $(a-b)^2$ = $(a-a)^2$ = $(b-b)^2$ = 0^2 = 0 , yet this formula is still valid.
                          When (a) and-or (b) is equal to 0, there is no difference possible, and d is always 0.

  If (a) and (b) are always equal, a=b, this difference (d) formula here could be written as (using algebraic substitution):

  d = 4aa   or =   4bb

  d = $4a^2$ or = $4b^2$

Ex. The sum or difference between a variable squared and that variable results in an multiplication expression:

    a^2 + a = a (a + 1) , ex. 5^2 + 5 = 5 (6) = 30   , a^2 - a = a(a - 1) ,   ex. 5^2 - 5 = 5(4) = 20   : check with dist.

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Ex. What is the difference (d) between the sum of two squared values and the difference of two squared values?

$(a^2 + b^2) - (a^2 - b^2) = d$ after clearing each grouping symbol by distributing (+1), and (-1):
$a^2 + b^2 - a^2 + b^2 = d$ after combining like terms, and switching sides:
$\quad\quad d = 2b^2$

Ex. Algebraically show that (a/b) and (b/a) are reciprocals of each other. The easiest way to do this is by knowing the fact that a reciprocal times its own reciprocal is equal to 1:

$$(\text{reciprocal}) \left(\frac{1}{\text{reciprocal}}\right) = \frac{\text{reciprocal}}{\text{reciprocal}} = 1$$

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \frac{ab}{ba} = \frac{ab}{ab} = 1 \quad : \text{ their product of 1 indicates the values are reciprocals of each other}$$

Let (a/b) = n , then   1/n is its reciprocal, = 1/(a/b) = (b/a)   : algebraic verification that (a/b) and (b/a)
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ are reciprocals of each other

MULTIPLYING LIKE VARIABLES

Multiplying like variables is performed just like the multiplication of like values or numbers. To multiply like variables, add their exponents, and multiply their numerical coefficients together.

Ex. $(A)(A) = A^2$

Checking: $(A)(A) = A A = (1)A^1(1)A^1 = (1)(1)A^{(1+1)} = (1)A^2 = A^2$

If A = 5, $(A)(A) = AA = (5)(5) = A^2 = 5^2 = 25$

Ex. $A^2A^3 = A^5$

Checking: $A^2 A^3 = A^{(2+3)} = A^5$
Checking: $A^2 A^3 = (AA)(AAA) = AAAAA = A^5$

Ex. This example shows that when multiplying like variables that have numerical coefficients other than 1, multiply those numerical coefficients, and add the exponents of the variable.

$(2A)(2A) = 4A^2$

Checking: $(2A)(2A) = (2)(A)(2)(A) = (2)(2)(A)(A) = 4A^2$

Ex. $(10A^2)(2AB) = 20A^3B$

Checking: $(10A^2)(2AB) = (10)(2)A^2A^1B = 20A^3B$

Ex. $(2A^3)(B)(-5A^{-4}) = -10BA^{-1}$ Or=: $\dfrac{-10B}{A}$ Or: $\dfrac{\_\ \_10B\_}{A}$

A POWER TO A POWER

To simplify a "power to a power", that is, an indicated power that is itself raised to an another indicated power, simply distribute (multiply) the exponents.

Ex. $(A^2)^3 = A^{(2x3)} = A^6$

Checking by "extending" the indicated operation, and then using the multiplication of like variables rule of adding the exponents:

$(A^2)^3 = (A^2)(A^2)(A^2) = A^{(2+2+2)} = A^6$

If the base of the indicated power contains multiple factors, "distribute" (multiply) the (outer) indicated power to each of the factors exponents.

Ex. $(x^3y^1)^2 = x^{((3)(2))} y^{((1)(2))} = x^6y^2$ checking by extending (using multiplication) the indicated power:

$(x^3y^1)(x^3y^1) = x^3y^1x^3y^1 = x^3x^3y^1y^1 = x^{(3+3)}y^{(1+1)} = x^6y^2$

MULTIPLYING SIMILAR (INDICATED) POWERS

To simplify the product of like or similar indicated powers (have the same indicated exponent), raise both the product of the numerical coefficients and then the product of the variables to that indicated power.

Ex.   (2X)^2 (3XZ)^2 = ((2)(3))^2 ((X)(X)(Z))^2 = (6)^2 (X^2 Z)^2 = 36 X^4 Z^2   checking:

(2^2 X^2)(3^2 X^2 Z^2) = (4X^2)(9X^2 Z^2) = 36X^4 Z^2   or:

(2X)(2X) (3XZ)(3XZ) = (2)(2)(3)(3)(X)(X)(X)(X)(Z)(Z) = 36X^4Z^2

Another way to simplify this is to consider that the two (same) indicated powers have been previously factored out from a single power of:

(6X^2Z)^2   : this is the result after multiplying the bases of the same indicated powers
                       together and indicating that same power. This can be expressed as:

(6XXZ)^2
((2)(3)(X)(X)(Z))^2   which can be expressed as:
((2X) (3XZ))^2   distributing the exponent to each of the indicated factors:
(2X)^2 (3XZ)^2   : this is the original example given

Now checking by distributing the exponent of the single (or unfactored base product) power:

(6X^2Z)^2   we have:
6^2 X^(2(2)) Z^(1(2))
36 X^4 Z^2   :checks

Hence the bases of similar (indicated) powers can first be multiplied together, and then the exponent can be distributed. Also, as indicated here, you can factor the base of an indicated power, and each factor will have that same indicated power or exponent: (ab)^n = a^n b^n , which is the same result after initially distributing the exponent to each factor.

DIVIDING LIKE VARIABLES

As with numerical values, just take the difference (or add the negative of) of the exponents when dividing like variables.

Ex. $\dfrac{A^5}{A^2}$ = $A^{(+5-2)}$ = A^{+3} checking: $(A^2)(A^3)$ = $A^{(2+3)}$ = A^5

Ex. $\dfrac{A^2}{A^5}$ = $\dfrac{A^{(+2-5)}}{1}$ = A^{-3} : with a negative exponent

OR: $\dfrac{A^2}{A^5}$ = $\dfrac{1}{A^{(5-2)}}$ = $\dfrac{1}{A^3}$: with a positive exponent

checking: $\dfrac{A^2}{A^5}$ = $\dfrac{\overset{+1}{\cancel{A^2}}}{\underset{+1}{(A^3)\cancel{(A^2)}}}$ = $\dfrac{1}{A^3}$ = $\dfrac{A^{-3}}{1}$ or: checking: $(A^5)(A^{-3})$ = $A^{(5+(-3))}$ = $A^{(5-3)}$ = A^2

OR: $\dfrac{A^2}{A^5}$ = $\dfrac{A^2 A^{-5}}{1}$ = $A^{(2+(-5))}$ = A^{-3} : using the multiplication of like values or variables rule to add or combine their exponents and keep that base value

Dividing A Several Term Expression By A Single Term Expression

    To divide a several or multiple term expression (often called a multinomial) by a single term expression (often called a monomial), divide the denominator term into each term of the numerator expression. As discussed previously using only numeric values, this concept is simply the reverse of the concept of adding "like" fractions that have the same denominator. The process is also a form of simplifying a fraction.

Ex. $\dfrac{10X + 5}{5}$ $=$ $\dfrac{10X}{5} + \dfrac{5}{5}$ $=$ $2X + 1$

Checking: $5(2X + 1)$: Multiplying the quotient and divisor together, and this is also a factored form of the of the multinomial dividend. Distributing the factor of 5 to each term in the second factor which is an expression that is a binomial since it is two terms being combined:

     $5(2X) + 5(1)$
      $10X + 5$: checks, since this is equal to the dividend

Ex. $\dfrac{30X^3 + 15X^2 + 10X}{5X}$

   $\dfrac{30X^3}{5X} + \dfrac{15X^2}{5X} + \dfrac{10X}{5X}$

   $6X^{(3-1)} + 3X^{(2-1)} + 2X^{(1-1)}$: $X^{(1-1)} = X^0 = 1$, and this instance of variable X
    $6X^2 + 3X^1 + 2$ is essentially eliminated as being a factor.

Checking: $5X(6X^2 + 3X^1 + 2)$ indicating the distributing of the 5X factor:
     $5X(6X^2) + 5X(3X^1) + 5X(2)$ distributing:
      $30X^3 + 15X^2 + 10X$: checks with the original numerator or dividend

Dividing A Several Term Expression By A Several Term Expression

This is about dividing a several or multiple term expression, often called a multinomial, by another several or multiple term expression. First, so that you can accept the concepts and process, here is a simple verification using only numeric constants, partial sums, equivalent fractions, and summing fractions, and without using any (algebraic, or symbolic placeholder) variables:

$$\frac{200}{100} = 2 = \frac{100 + 100}{100} = \frac{100}{100} + \frac{100}{100} = \frac{100}{(50 + 50)} + \frac{100}{(50 + 50)} = \frac{100 + 100}{50 + 50}$$

: expressed as a multinomial expression divided by a multinomial expression

Now an example using variables:

since $(x+2)(2) = 2x + 4$, then mathematically, dividing both sides by the (x+2) factor we should have (2):

$$x + 2 \overline{\smash{\big)}\, 2x + 4}$$

: the quotient should obviously be equal to 2, since dividing a product by one factor yields the other factor of that product.

To get the first term of the quotient, divide the first term of the divisor (or denominator if you will) into the first term of the dividend (or numerator if you will). Then continue as in normal division.

$$\frac{2x}{x} = 2$$

$$\begin{array}{r} 2 \\ x + 2 \overline{\smash{\big)}\, 2x + 4} \\ - \underline{(2x + 4)} \\ 0x + 0 \end{array} \quad = \quad \begin{array}{r} 2 \\ x + 2 \overline{\smash{\big)}\, 2x + 4} \\ + \underline{-2x - 4} \\ 0x + 0 \end{array}$$

: after using distribution (of -1) to rid the grouping symbols.
: remainder of 0, division is complete

Checking by multiplying the quotient and divisor: $2(x+2) = 2x + 4$: checks, since the product equals the dividend

Also: $$\begin{array}{r} x + 2 \\ 2 \overline{\smash{\big)}\, 2x + 4} \\ - \underline{2x} \\ 0 + 4 \\ - \underline{+4} \\ 0 \end{array}$$, and $2(x+2) = 2x + 4$

A method to make the process of the division of values that contain variables easier is to first place the terms containing the variable in an order of descending or decreasing (indicated) powers or exponents. Do this for both the dividend and divisor. If a power of a variable is absent or unused, replace it with a "0 placeholder". That is, express as a term that (unused) power of the variable, and with a 0 constant.   This effective 0 value added in to the sum will obviously not change the value of the dividend (or numerator) or divisor (or denominator).   For example, if the numerator or denominator is:

$3 + x + 5x^3$ rearrange this for the division process as:

$5x^3 + 0x^2 + x^1 + 3$:the last term could also be indicated as having a factor that is a power of x: $3X^0$, since $3x^0 = 3(1) = 3$

Actually, you can think that you have been dividing polynomials by polynomials all along without even realizing it. This can easily be seen by representing the strict (number only) numeric values of the numerator and denominator in an expression form where each term is the positional product of each digit of that number:

Ex. $\dfrac{385}{11} = 35$ Creating a positional sum by expressing each digit as the positional product of the digit value and its corresponding decimal weight in the decimal number:

$$\dfrac{385}{11} = \dfrac{3(100) + 8(10) + 5(1)}{1(10) + 1} = \dfrac{3(10\text{^}2) + 8(10\text{^}1) + 5}{1(10\text{^}1) + 1}$$

Here above, it is quite obvious that instead of a "polynomial in x (a variable)" we have a "polynomial in 10". If you were to replace each power of 10 above with a corresponding power of x, the results would be the same.

```
                3(10^1)  +  5_____
1(10^1) + 1 | 3 (10^2) + 8(10^1) + 5
            - [ 3 (10^2) +  3(10^1) ]
                0(10^2) + 5(10^1) + 5
                      - [+5(10^1) + 5]
                        0(10^1)+ 0
```
:start by dividing 3(10^2) by 1(10^1), these are the first terms of the dividend and divisor
: Note, the negative sign preceding the [ ] grouping symbols and it can be thought of as the factor (-1) to distribute and then do the combining.

Checking, using multiplication of the divisor and quotient so as to see or compare if it equals the dividend:

```
(1(10^)1 + 1)  (3(10^1) + 5)          distributing:
 3(10^2) + 5(10^1) + 3(10^1)  + 5      combining:
      3(10^2) + 8(10^1) + 5            distributing:
           300 + 80 + 5                combining:
                385
```

Another check is to use the order of operations more closely, simplifying the expressions within the grouping symbols first:

```
(1(10^)1 + 1)  (3(10^1) + 5)       using distribution on the inner grouping symbols:
    (10 + 1)( 30 + 5 )             combine (add):
       (11)(35)                     distribute (multiply)
          385
```

As like during the process of dividing two plain numbers, when dividing multinomials, if the difference after subtracting is 0, the division is complete. If the first term (ie. monomial) or variable in the difference has a lower power than that of the first term of the divisor, then the divisor cannot divided (not even once) into that difference, and then the remaining part of the dividend becomes the remainder.

It should also be mentioned that it is mathematically possible, but maybe not practical, to divide each term of the dividend (or numerator if you will) by the entire divisor (or denominator if you will). Here, the entire multinomial divisor, perhaps set within grouping symbols for clarity, is used to represent a single value divisor.

EQUATIONS

Two expressions that represent the same value can be expressed or set as being numerically equivalent (ie., "equal", =) in value, and this expression of equivalence is called an equation. An equation can be mathematically worked with and manipulated much like a conceived imaginary numerical weighing scale or balance. Whatever you do mathematically to one (entire) side of the equation, you must also do to the other (entire) side of the equation so as to keep both expressions or sides equal and the entire equation in balance.   More specifically, whatever change in total value you apply to one side, you must also apply that same change in total value to the other side. All of this will be of great importance since it will allow us to solve for the equivalent value or expression of a variable or unknown value used in that equation.

When both sides do not represent the same value, they are said as "not equal", and the equation is "unbalanced". When the two sides or expressions are equal, there is no mathematical difference between their values, and the equation is therfore balanced. When the two sides or expressions are not equal, there is a mathematical difference between them, and therefore, those expressions or sides are unbalanced with respect to each other. One side is greater in value than the other, and therefore, one side is less in value than the other. Given an equation, the difference of a side value with respect to the other side value = side value - the other side value. A difference can also be thought of as the amount of separation or unbalance between two things such as the two sides or expressions of an equation.

Ex. Given the two expressions of: Eq.1: (5+2)   and   Eq.2: (3+5) , are they equivalent in value?

First simplifying each expression to find their resulting net worth, total, sum, value or weight:

Eq..1:   5+2   Eq.2:   3+5
     7   8

   7 - 8 = -1   :The amount that Eq.1 is different in value than Eq.2 ,
                Since the result is not 0, the equations are not in balance.

   8 - 7 = +1   :The amount that Eq.2 is different in value than Eq.1

   Notice, the absolute (or signless) value of each difference (or numerical separation), or numeric separation
   or "distance" between them is the same value of 1.

Since the net worth, sum or weight of each expression does not have the same value, there is a difference in or of value among the sides, and therefore, we can not equate those two expressions. We can however indicate or express their mathematical relationship with a special (mathematical, values) relational or relationship operators:

     7 < 8   : < , when read from left to right, it is called the "less-than" symbol.   ,   "seven is less-than eight"
     8 > 7   : > , when read from left to right, it is called the "greater-than" symbol. , "eight is greater-than seven"

        In either case, the arrow's smaller point end always points to, indicates or expresses the smaller or
        less value, and the open or large end of the arrow is near the greater or larger value.

     7 |=| 8   : expressing that 7 is not equal to 8, and-or that 8 is not equal to 7

Ex. Given the two expressions of: Eq. 1:   (10-3)   and   Eq. 2:   -(257-249) , are they equivalent in value?

First simplifying each expression to find their resulting net worth, total, sum, value or weight:

Eq.1:   +10-3   Eq.2:   -(-257+249)
      +7   +256 -249
                      +7

Checking for any difference: (+7) - (+7) = 0, hence there is zero or no difference of the values of each expression. Since the net worth, sum or weight of each expression has the same value (here, 7), we can equate those two expressions and create an equation with sides of those equations:

10 -3 = -(-256 +249)

Here is a graphical representation of a balanced equation where: left side = right side: [FIG 22]
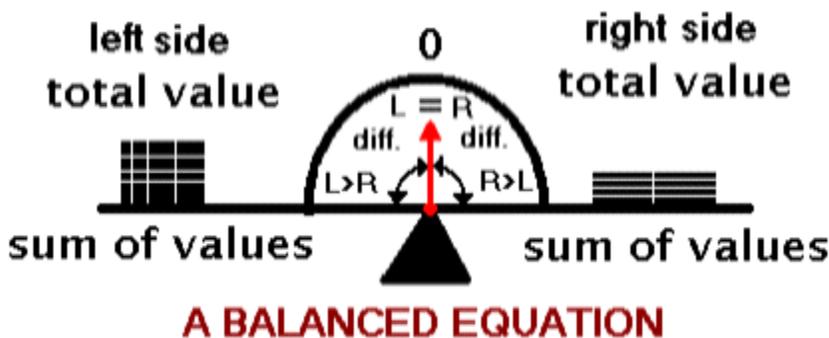
When the scale is balanced, both sides are equivalent in weight, and if any weight is added to or removed (ie., subtracted) from just once side, there will be an unbalance.   When two values on either side of the scale are not equivalent (ie., =|=), there is a difference (diff.) between those two values. If we subtract one value from the other, we can find if there is any difference between those two values, and the actual value of that difference.

Given: value1   and   value2   Checking to see if those two values are equal or different:

value2 - value1   =   difference   : difference is the amount that value2 is different than value1

Note also that:

(value1 - value2) = difference

Here, difference will be the same (signless or absolute) value as:
(value2 - value1) = difference
, but will now have a different sign indicating if value1 is greater or less than value2. Expressed this way, the sign of the difference is in reference to the first term,and that is value2.

value2 and value1 can be in placed in balance or equivalence when the specific amount of their unbalance, here the difference value, is combined with value1, and their difference will then be 0. This is much like a checking a subtraction problem when the difference is added to the subtrahend, so as to equal the minuend.

Now given:

(value1+ difference)   and   value2   or: value1   and (value2 + difference)   if the signed difference is used

Checking to see if those two values are equal or different:

value2 - (value1 + difference) = 0   : since there is no difference, these two values are equivalent:
value2 = (value1 + difference)   : both sides of this equation balance. This can be expressed as:
value2 =   value1 + difference   : as like all variables, these should be considered as signed values which
                                                can be either positive or negative in sign.

Ex. If value1 is 3, and value2 is 7:

We immediately can see that these two values are not equal since one value is larger or greater than, or with respect to, or in reference to the other value. Likewise, one value is therefore less than the other value:

value2 > value1   and-or: value1< value2   and   value2 /=/ value1   : mathematical or numerical relationships

7 - 3 = 4   : +4 is the difference between 3 and 7, which could also be said as the difference between 7 and 3.
                      7 is 4 greater than 3. Likewise it could be said that 3 is 4 less than 7.
                      Note: 3 - 7 = -4 , here the negative sign of the difference means that the initial term and
                      reference value of +3 is 4 less than +7.

7 - (3 + 4) = 0
7 = (3 + 4)
7 = 3 + 4   : a balance scale having a weight of 7 units on one side will be in balance if the other side
7 = 7   has a weight of 3 units and another weight of 4 units.

Below are some examples of equations and mathematical manipulations on them.

Ex.   14 = (7)(2)   : each side can be simplified to a value of 14, therefore the equation is true, and not
                              unbalanced and false.
   14+6 = (7)(2)+6   : Lets add 6 to each side
    20 = 14 + 6   : after combining on the left side, ORD step 3 (ie. mult.) on the right side
    20 =   20   : after combining the right side, ORD step 4 (ie. combining). The resulting equivalent values
                              on both sides of the equals sign indicate that both sides were, and are still in balance after the
                              same mathematical operations were applied to them.

Ex.   (8+2) = (5)(2)   : each side can be simplified to a value of 10
     (3)(8+2) = (5)(2)(3)   : multiplying 3 to each side
       +24+6 = +30   : after distributing
          +30 = +30   : after combining

Ex. 75 = (5)(15)

   +15   +1
    $\dfrac{75}{5}$ = $\dfrac{(5)(15)}{5}$: dividing each side by 5, and canceling (canceling out, dividing
   +1   +1   both the num. and den. by a common factor(s))

    15 = 15

Ex.   100 = (25)(4)   taking and expressing the square root of both sides:

   $\sqrt{100}$ = $\sqrt{(25)(4)}$ factoring the radical:

    10   = $\sqrt{25}$ $\sqrt{4}$ simplifying (here, rationalizing) each radical:
    10   =   (5)   (2)   distribute or multiply:
    10   =   10

Let's say expression1 equals some value that we can set equal to and call as: x. If another different expression, such as equation2 also equals this value of x, then we can equate expression1 and expression2 since we know they are equal to the same value, here x. Showing or expressing this mathematically:

expression1 = x = expression2   therefore:
expression1 = expression2

Ex.   If, or given: (7 + 3) = 10   and $\sqrt{100}$ = 10   therefore, the expressions can be equated
since they both equal the same value of 10:

$$(7 + 3) = \sqrt{100}$$
$$10 = 10$$

ALGEBRAICALLY EXPRESSING THE MORE ADVANCED MATHEMATICAL OPERATIONS

Previously in this book, examples of concepts, expressions, equations and formulas, with many using a pure or strict numerical form were given, such as for powers, roots, logarithms, etc. Some simple algebraic or introductory algebraic-like methods were also utilized where needed. Below, in a more pure algebraic sense, the mathematical and corresponding relationships of a base (b), exponent (e) or (x), and number (N) are expressed in a pure or strict algebraic mathematical form using variables, and in three fundamental mathematical operations and equations. Also take note of the various locations of the variables (N, b, x) within each equation, and that each variable also has the same value in each equation:

$N = b^x$: **Powers**

In a power expression or equation, the number (N) is often called the power value, (b) is often called the base or "base of the power value", and (x) is often called the exponent or the "indicated power".

$b = \sqrt[x]{N}$: **Roots**

In a radical expression or equation, the base (b) is often called the root, (N) is often called the radicand, and the exponent (x) is often called the index (i) or "indicated root".

$x = \log_b N$: **Logarithms**

In a logarithmic expression or equation, the exponent (x) is often called the logarithm, (b) is called the base of the logarithm, and (N) is often called the Number. Number is actually equal to a power of that indicated base: $b^x = N$

Expressing the above equations in a somewhat unconventional format or way so as to help make the three relationships more clear using just the same three names or identifiers for the parts of all the equations:

powers	roots	logarithms	: expression type
power = root^logarithm	root = $\sqrt[logarithm]{power}$	logarithm = $\log_{root} power$	

Expressing the above by using the more formal, common, standard identifiers or names for the values, along with an equivalent variable or identifier:

power   =   base^exponent   =   Number

root   = $\sqrt[index]{radicand}$ =   base = $\sqrt[exponent]{power}$ = $\sqrt[exponent]{number}$

logarithm   = $\log_{base} Number$ =   exponent

We see in all three equations above that each of the three values can have a new or alternate meaning as a different part of each equation of each of the three mathematical operations, but it's actual value is the same in all three equations:

These will have the same numeric values:

base = root
exponent = logarithm = index,
power = Number = radicand

Given all the corresponding values for (N), (b), and (x) from one equation, they will satisfy (ie. keep both sides in balance and equivalence) all of the three equations given above. For example, if b=10, x=2, and N=100:

Powers   ,   Roots   ,   Logarithms

$N = b\char94 x$, $b = \sqrt[x]{N}$, $x = \log_b N$

$100 = 10\char94 2$, $10 = \sqrt[2]{100}$, $2 = \log_{10} 100$

Given any two values of: (b), (x), and (N), there is only one possible and corresponding third value. The instance of, and the specific set of three corresponding values is unique and essentially happens only once, and all the corresponding values are specifically related according to the equations shown above.

Some of the following has been mentioned about exponents, but here it is presented in algebraic terms:

$b1\char94 x = b2$ therefore, $b2\char94(1/x) = b1$ checking that this is true using substitution for b1:

$b1\char94 x = (b2\char94(1/x))\char94 x = b2\char94(x/x) = b2$: as shown above. Also:

$b1\char94 x = b2$ therefore, $b2\char94(1/x) = \sqrt[x]{b2} = b1$: the exponents of b1 and b2 are reciprocals.

$b1\char94 x = b2$ therefore $b1 = \sqrt[x]{b2}$: or $b1\char94 x = N$ and $\sqrt[x]{N} = b1$

Ex. $3\char94 2.5 = 15.588\underline{5}$ and $15.5885\char94(1/2.5) = \sqrt[2.5]{15.5885} = 15.5883\char94 0.4 = 3$

BASIC METHODS OF SOLVING FOR A VARIABLE

To solve for the value of a particular variable that's part of an equation is to isolate the variable on one side of the equation, and then the other side of that equation will be the equivalent value of that variable. First, if possible, simplify each side of the equation using the order of operations and-or the CDC method discussed in this book. Some other helpful steps are to place all terms that have the variable in question on one side of the equation, and place all the other variable terms and constants on the other side of the equation. This moving, shifting or arranging of terms to the other side of an equation is referred to as transposing (T). To transpose, add the negative of the term to both sides of the equation, then after combining like terms, the variable that was transposed will have a numerical coefficient of 0, and therefore, that variable is effectively eliminated from that side of the equation since anything times zero is equal to zero, and adding zero does not affect the value of that side.

equation

expression1 = expression2

term1 + term2 = term3   :   To solve for term1, or a variable that is within term1, isolate term1 to one side of the equation; and here, by keeping term1 where it is and removing term2 from that side of the equation by the transposition (change position) process of adding the negative of that term to both sides of the equation:

term1 + term2 -(term2) = term3 -(term2)   using distribution, here of and by: -1, to clear grouping symbols:

term1 + term2 -term2   = term3 -term2   combining like terms:
    Note, +term2 -term2 = (+1)term2 (+) (-1)term2 =
        (+1 + -1)term2 = 0 term2 = 0

term1 = term3 -term2

Ex.   Solve for (x) given: x+4 = 7

x+4 = 7
x+4 -4 = 7 -4   :T. +4, so we then add the negative of (+4) = - (+4) = -4 to each side of the equation
x = +3   : after combining

Once the value of a variable is found, it can be checked by substituting (setting equal to, replacing) the value back into the variables place (placeholder) or position in the original equation. That is, the variable itself is to set equal to that found value. If both sides of the equation are found to be equal, then the value of the variable is correct.

Using the last example where x=+3 :

 x+4 = 7   : original equation, substitute the solved or calculated value for x:
+3+4 = 7   combine:
   7 = 7   : checks

Ex. Solve for X given: X - Y = 5

$X - Y = 5$ First, let's T. (-Y) = Transpose (-Y) = add the negative of (-Y) = -(-Y) = +Y to both sides
$X - Y + Y = 5 + Y$ combining (C.) like terms:
$X + 0Y = 5 + Y$
$X = 5 + Y$: We find that X is equal to an expression containing a variable (here it's Y) whose actual value
                    has an affect on determining the corresponding and specific value of X.

       If a variable being solved for is being multiplied by some multiplying factor, other than one, that variable can be isolated by dividing both sides of the equation by its multiplying factor or numerical-coefficient.

Ex. 25X = 50, solve for X

$\dfrac{25X}{25} = \dfrac{50}{25}$: to rid X of its multiplying factor of 25, simply divide by 25.
                  This would also be equivalent to multiplying each side by (1/25).

$\begin{array}{cc} +1 & +2 \\ \dfrac{\cancel{25}X}{\cancel{25}} = & \dfrac{\cancel{50}}{\cancel{25}} \\ +1 & +1 \end{array}$: cancel

$+1X = 2$ or simply: $X = 2$

       If the variable to be isolated is being divided by some value, the variable can be isolated by multiplying both sides of the equation by this value, and then using canceling to effectively rid it from that side.

Ex. $\dfrac{X}{2} = \dfrac{50+Y}{1}$: solve for X

$\dfrac{(2)(X)}{(1)(2)} = \dfrac{(50 + Y)(2)}{(\ \ 1\ \ )(1)}$: to rid (x) of its "dividing factor", here 2, multiply both
                           sides by this "dividing factor" and then cancel it out:

$\dfrac{(2)(X)}{(1)(\cancel{2})} = \dfrac{(50 + Y)(2)}{(\ \ 1\ \ )(1)}$: cancel on the left side, and using distributing on the right side:

$X = (+2)(+50) + (+2)(+Y)$:indicated distribution

$X = +100 + 2Y$:after distributing

Ex. Solve for X given: $\dfrac{X}{5} - \dfrac{X}{7} = \dfrac{6}{35}$

There are several ways that X can be solved for, and for the first, we will combine the like terms of: (1/5)X and (-1/7)X :

$\dfrac{X}{5} - \dfrac{X}{7} = \dfrac{6}{35}$ combine like terms on the left side (here, essentially combining fractions with the LCD = 35):

$\dfrac{7X}{35} - \dfrac{5X}{35} = \dfrac{6}{35}$: showing the newly created and equivalent fractions on the left side

$\dfrac{(+7X - 5X)}{35} = \dfrac{6}{35}$: indicating the combining of the numerators of the like or similar fractions that have the same denominators

$\dfrac{+2X}{35} = \dfrac{6}{35}$ now rid X of its' dividing coefficient factor of (35), and multiplying coefficient factor of (2): This can also be done in a single step by multiplying both sides by the reciprocal of 2/35 which is 35/2.

$$\dfrac{\overset{+1}{\cancel{(35)}} + 2X}{(1)\ 35_{+1}} = \dfrac{6\ \overset{+1}{\cancel{(35)}}}{35_{+1}(1)}$$: cancel fractions (rid denominators)

$2X = 6$ Isolate X, rid X of its multiplying factor of 2, by dividing by that value of 2:

$$\dfrac{\overset{+1}{\cancel{2}}X}{\underset{+1}{\cancel{2}}} = \dfrac{\overset{+3}{\cancel{6}}}{\underset{+1}{\cancel{2}}}$$: cancel

$X = +3$

The method below uses the CDC method. Here, the fractions are to be eliminated or canceled first rather than combining them. To eliminate the fractions on one side of the equation, multiply both sides of the equation by a common denominator (LCD preferred) of that side of the equation. To eliminate all fractions of the entire equation at once, then multiply both sides by the LCD of all the fractions.

$\dfrac{X}{5} - \dfrac{X}{7} = \dfrac{6}{35}$ multiplying both sides by 35:

$$\dfrac{35(X}{1(5} - \dfrac{X)}{7)} = \dfrac{6\ \overset{+1}{\cancel{(35)}}}{35_{+1}(1)}$$: cancel on the right side, now distribute on the left side:

$\dfrac{+35X}{5} - \dfrac{35X}{7} = \dfrac{6}{1}$ cancel on the left side:

$$\dfrac{\overset{+7}{+\cancel{35}}X}{\underset{+1}{\cancel{5}}} - \dfrac{\overset{-5}{\cancel{35}}X}{\underset{+1}{\cancel{7}}} = \dfrac{6}{1}$$: cancel

$+7X - 5X = 6$ combine, add the like terms numerical coefficients or factors:

$+2X = 6$ isolate X, divide both sides by 2:

$$\dfrac{\overset{+1}{+\cancel{2}}X}{\underset{+1}{+\cancel{2}}} = \dfrac{\overset{+3}{+\cancel{6}}}{\underset{+1}{+\cancel{2}}}$$: cancel

$X = +3$, Checking using substitution:

$\underline{X} - \underline{X} = \underline{3} - \underline{3} = 0.6 - 0.428\underline{6} = 0.1714\underline{3} = \underline{6} \quad$: checks
$5 \quad 7 \qquad 5 \quad 7 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 35$

Ex. The formula relating Fahrenheit (F) and Celsius (C) temperature is given below. Solve for variable F.
If water normally boils at 212° Fahrenheit, then at what corresponding Celsius temperature does it boil?

$C° = \dfrac{5}{9} (F° - 32°)$ after multiplying both sides by 9 and canceling:

$9C° = 5 (F - 32°)$ dividing each side by 5 and canceling:

$\dfrac{9C°}{5} = F° - 32°$ after transposing (-32) by adding the negative of (-32) = -(-32) = (+32) to each side, and afterwards, switching sides:

$F° = \dfrac{9C°}{5} + 32°$ or $F° = 1.8C° + 32°$

Notice that for every degree change in Celsius temperature that the corresponding Fahrenheit temperature changes not by 1 degree, but by 1.8 (almost 2) degrees. As to the boiling point of water:

$C° = \dfrac{5}{9} (212° - 32°)$

$C° = 5 \dfrac{(180°)}{9} = 5 (20°) = 100°$: Celsius temperature scale, boiling point of water.
(At 0° C = 32° F , water freezes)

Ex. Show that $\left(1 + \dfrac{1}{x}\right)$ = one plus the reciprocal of x, is equivalent to: $\left(\dfrac{x + 1}{x}\right)$ or = $\dfrac{(1 + x)}{x}$

$\dfrac{x + 1}{x} = \dfrac{x}{x} + \dfrac{1}{x} = 1 + \dfrac{1}{x}$: after putting each term in the numerator over the common denominator. You can also show they are equal by combining fractions in the first expression given for this example.

Ex. If x = 2: $1 + \dfrac{1}{2} = 1 + 0.5 = 1.5$ and:

$\dfrac{2 + 1}{2} = \dfrac{3}{2} = 1.5$

Ex. If x = 100: $1 + \dfrac{1}{100} = 1 + 0.01 = 1.01$ and:

$\dfrac{100 + 1}{100} = \dfrac{101}{100} = 1.01$

Ex. If: a = b + c   this can be mathematically expressed as:

$$a = b + c \qquad \text{dividing each side by a:}$$

$$\frac{a}{a} = \frac{b + c}{a} \qquad \text{which can be expressed as:}$$

$$\frac{a}{a} = \frac{b}{a} + \frac{c}{a}$$

$$1 = \frac{b}{a} + \frac{c}{a} \qquad \text{we see that (b) and (c) essentially represent a fraction or percent of (a):}$$

$$1 = 100\% = \left(\frac{b}{a}\right)\left(\frac{100}{1}\right)\% + \left(\frac{c}{a}\right)\left(\frac{100}{1}\right)\%$$

Ex. 5 = 3 + 2

$$1 = \frac{3}{5} + \frac{2}{5} = 0.6 + 0.4 = 60\% \text{ of } 5 + 40\% \text{ of } 5 = 100\% \text{ of } 5$$
$$\text{: 3 and 2 are expressed as relative values with respect to the sum = 100\%=1}$$

When the variable being solved for is part of a radicand, rationalize the radical by raising it, and also the other side of the equation (to keep their balance and equivalence), to an indicated power equal to the index (indicated root of the radicand), and then convert the radical expression to its equivalent exponential form for easy simplification.

Ex. $2\sqrt{+X+5} = 7$ first, squaring both sides of the equation:

$(2\sqrt{+X+5})^2 = (7)^2$ placing the radical form into exponential form:

$((+X+5)^{(1/2)})^2 = +49$ power to a power, distribute the power:

$(+X+5)^{(2/2)} = (+X+5)^1 = +49$

$(+X+5) = +X+5 = +49$ T. +5:

$+X +5 -5 = +49 -5$ combine terms:

$X = +44$

Be sure to check this result in the original equation since raising values to powers, though algebraically correct, can possibly introduce some extraneous (extra, and usually undesired) results. As a simple verification, consider this:

$$10 - Z = 0 \qquad \text{after solving for Z we find:}$$
$$Z = +10$$

If both sides are squared, the solution of Z should still be the same since the equation is still in balance:

$$Z^2 = 10^2 \qquad \text{or:}$$
$$Z^2 = 100 \qquad \text{Solving for Z by taking the square root of both sides, we now find:}$$
$$Z = \pm 10 \qquad \text{: the solution of (-10) will not satisfy the original equation, and it is}$$
$$\text{therefore an extraneous root.}$$

Below is an example where understanding a problem can be more difficult than mathematically solving it, and the problem was not even intentionally written to be difficult.

50 workers belong to a machine shop. Some of them can make nuts, and some can make bolts. Some workers are known to be able to make both. If 35 workers said they can make nuts, and 40 workers said they can make bolts, how many workers make both nuts and bolts?

The total sum of nut makers (N), and bolt makers (B), where some of these people can make both:

Total = N + B
  75   = 35 + 40   : total "virtual" number of makers or "jobs" possible, where only 50 are actual people and makers

However, for each given day only 50 real or actual workers out of this total of 75 (nut or bolt makers, which include dual (both) makers) work at the machine shop making either nuts or bolts, but not both. That is, regardless of the actual number of workers who are making nuts or bolts that day, this sum is always 50 workers. Subtracting this value of 50 will leave the number of people who can make both nuts and bolts.

75 = N + B + workers who can make both nuts and bolts
75 - (N + B)   =   workers who can make both nuts and bolts
75 -   50   =   virtual workers - actual workers = both
both   =   25 workers

Even though the problem was solved using a simple equation, it may need some clarification. First, consider the possibility that none of the workers can make both nuts and bolts, then the total "possibilities", "jobs" or "positions" (nut makers or bolt makers) is always 50 (instead for example of the 75 shown above), and cannot be any less since there is always the same 50 workers at the machine shop. Now, if any workers do both, the total "possibilities" will then exceed 50, and this excess above 50 can only be those who do both. The subtraction shown will find this excess or people who can make both nuts and bolts. Note also that if every one of the 50 workers can make both nuts and bolts, that the maximum "positions" or "possibilities" is: 50 people can make nuts, and 50 people can make bolts = 100 people can make both.

As a simple verification of the results of this problem, consider just 2 workers belonging to the machine shop, and each will either make nuts or make bolts during their work at the machine shop, that is, both workers are known to be a nut maker and a bolt maker. How many workers are both nut makers and bolt makers?

Total = N + B
  4   = 2 + 2   : total possible or virtual "positions"

Since only 2 people are at the machine shop:

  Both = Total - 2   : 2 workers at the machine shop
  Both = 4 - 2
  Both = 2   : 2 workers can do both (make nuts or bolts)

Though not discussed much further in this book, problems like the above can be represented and analyzed graphically with the aid of what are known as Euler (Leonhard Euler) or Venn (John Venn) diagrams. These diagrams represent sets (similar things) by using simple geometric figures, and their relationships and relative values are seen by observation and-or by using a special mathematical notation for sets. A simple Venn diagram of the above example might look like this: [FIG 23]
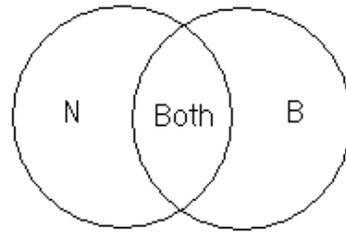
N = Nut Makers
B = Bolt Makers

GRAPHING AN EQUATION TO VIEW AND SOLVE IT

Sometimes an equation cannot be solved easily using normal algebra. By graphing the equation you can get a close estimate of the solution. A graph of the relationship of variables is also very powerful and a useful visual tool for quickly understanding the numerical values and relationship between two variables over a wide range of possible values. For example, to view if one variable is increasing or decreasing and-or at what rate (ie. "how fast") as the other variable changes (increases or decreases) in value. Once an equation is graphed by making many calculations with it and plotting the results or points, it also then effectively becomes an automatic visual calculator or computer that can quickly solve for a corresponding value of either the independent or and-or dependent variable. It is even possible to make a mechanical computer with the curve drawn, and such as using the intersection of two movable or user adjustable lines, one vertical (for the x or independent value) and one horizontal (for the y or dependent value).

Solve for (x) in :   2x + 4 = -2

Though algebra can be used to easily solve this equation for x, it will be only used to check or validate this demonstration of using a graph to solve an equation. To solve this equation, you are to find what value of (x) will make the left side equal to the right side of -2.

By transposing (-2), the equation can be algebraically expressed as:

   2x +4 +2 = -2 +2
   2x + 6 = 0   : x can be mathematically solved here, but also we want to solve this equation graphically
                             Expressed this way, we can see that there is a certain or specific value of x that will
                             make the expression on the left side equal to 0.

Here, we are to find the value of (x) that will make the left side equal to the right side of 0, still, the value of (x) that solves this equation will also solve the original equation since the new "derived" equation was derived (ie., created directly from) from the original equation. It is simply another mathematical expression of that original equation. This can easily be verified after solving for (x) mathematically. When the equation is expressed this way (set equal to 0), the value of (x) that satisfies or solves this equation is called the "root" of the equation, although don't confuse this concept with that of radicals.

Since the equation given is a linear (has a line-like graphing, of the proportional mathematical variable relationship) or "first-order" equation (where the exponent of the variable is only one = first = 1st), there will only be one root or solution. In general, if the power of the variable is 1, it is a linear equation. Linear equations always graphically plotted (ie. scribed, indicated , graphically expressed locations (ie., address) of corresponding variable values) as marks or points of a straight line.

To plot or graph an equation, there must be a mathematical relationship between two variables. In this discussion, the relationship is a mathematically proportional or linear ("one to one" or "balanced") type of relationship. As one "independent" variable changes in value by some factor (ie. a multiplier applied to the variable), then the other "dependent" variable will then change at the same constant rate since its value is essentially multiplied by that same constant factor value. Consider:

          y = 1x   : this is the same as: y = x, and this will actually graph as a horizontal line as long
                          as the multiplying factor or coefficient to x is 1, but if it changes to another value,
                          then it will no longer be horizontal, but will have a vertical part, either upward
                          if the coefficient is positive, or downward if the coefficient is negative. The
                          greater this multiplier, the "steeper" the line, because the rate of change of the
                          y values with respect to the x values is mathematically more such as for : y = 5x

We can also set the entire equation, or its equivalent result, as equal to another variable such as (y) for plotting purposes. Rather than just expressing a mathematical relationship between (x) and the result or numeric "output" of the expression,

and we will now have a mathematical relationship between two expressed variables such as (x) and (y). (x) can be thought of as the input or "independent" variable, and (y) can be thought of as the output or "dependent" (dependent or depending upon the value of x) variable.

$$2x + 6 = 0 = y \qquad \text{or by switching sides:}$$
$$y = 2x + 6$$

By setting the equation equal to 0, the root or solution will actually be where the graphical line (considered as a special instance of a curve, or curve plot on a graph) of the equation crosses the (x) or horizontal axis which is where the (y) value, of that corresponding point or part of the curve, is always 0. When finding corresponding values of (y) and (x), that is, finding the coordinates (corresponding values which define a points location) of each point on the line, it is best to find points both above and below the (x) or horizontal axis so that a smooth line (or curve) can be drawn (by hand, with the aid of a straight-edge drawing guide or a ruler) through or across the axis when using the points as a drawing guide. To draw a line, at least two points on it must be known, and a third or more points gives some verification that the two other points and line is accurate. You could begin by selecting random-like values for (x), and then mathematically finding (solving for, using the equation) each corresponding (y) value:

Make a list of several corresponding values of the (x) and (y) variables to be graphed or expressed visually as the (coordinate, or coordinated) location of points on a (flat) plane or rectangular (square-like) coordinate (location) system. These points are identified by that corresponding pair of values for the variables such as: (x , y). The coordinates (corresponding locations) are fundamentally in reference to an arbitrary point chosen as the starting or reference point, usually identified with coordinates (x=0,y=0), and which is sometimes called the "axis center point" of the coordinate system. The horizontal and vertical dimensions are called the axis (lines) of the system, and these are numerically scaled for values and measurements of each variable.

x	y = 2x+6
0	+6
-1	+4
- 2	+2
-5	-4

: each pair of corresponding values of the variables can be represented graphically as a point whose location, address or coordinates is identified as point(x,y) = p(x,y)

Plotting the points:

[FIG 24]

point (x , y) = p(0,+6)

line
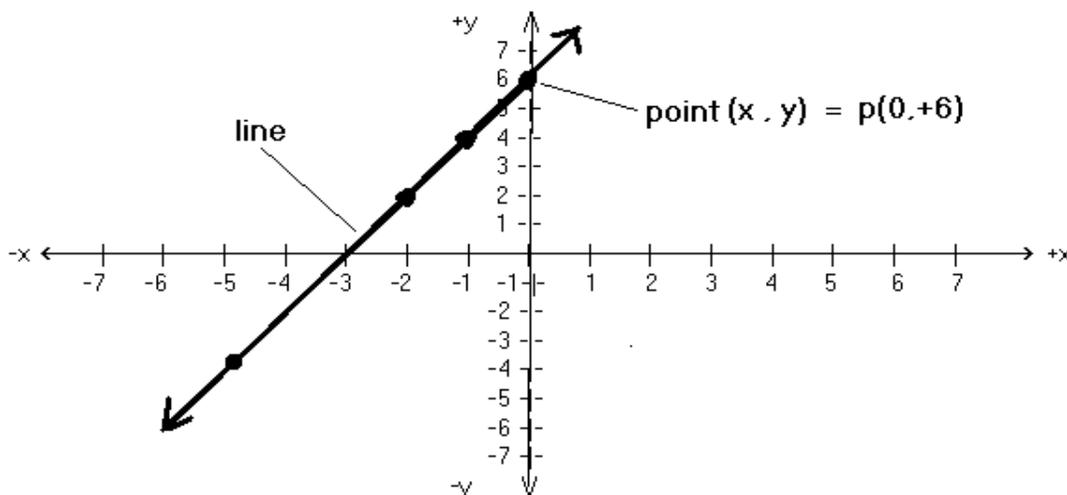
The solution (visually, and by the chart or list of the corresponding values of the variables) appears to be about: x = -3. Let's check by substituting this value into the original equation:

$$2x + 4 = -2$$
$$2(-3) + 4 = -2 \qquad \text{distributing:}$$
$$-6 + 4 = -2 \qquad \text{combining;}$$
$$-2 = -2 \qquad : \text{checks}$$

The point where the line crosses the x-axis is called the x-axis intercept. Here, the corresponding value of y is always 0. If not already done, solve the equation for y, and then set this equation and y equal to 0.   This is important since it will create a completely solvable equation with only one variable in it.

y = 0 = 2x + 6   : setting y equal to zero, to find the corresponding value of x. Solving for x we have:

x = $\frac{-6}{2}$ = -3   : after transposing +6, and dividing both sides by 2
                      The x-intercept, or x-axis intercept, point is: P(x,y) = P(-3,0)

Likewise, given y = 2x + 6, we can solve for variable x and set its value equal to 0. Then we can solve for y since there will be only one variable in this equation since variable x will be essentially eliminated (by using a constant or numeric value for it, such as 0). y is then solvable, or is said as being "completely solvable" since it will have a numeric value:

x = $\frac{y - 6}{2}$ setting x=0 for the purpose of finding its corresponding y value:

0 = $\frac{y - 6}{2}$ when x=0, solving for y so as to find the corresponding y value of that point:

y = 2(0) + 6

y = 0 + 6 = +6   : The y-intercept point is: P(x,y) = P(0,+6) as seen in the graph above.
                      The intercept points also can be used to draw the line, and-or to verify if
                      a previously plotted and graphed (drawn) points or line is correct. Any
                      more points should also be plotted on that same existing line.

SOLVING EQUATIONS CONTAINING MULTIPLE VARIABLES

If an equation contains multiple variables, a specific variable and its value can still be solved for if all the other variables can be placed on the other side of an equation and be expressed "in terms" of the given variable . This is known as algebraic substitution (as opposed to substituting actual numeric values for numeric substitution). The overall method is to produce a new equation which will have only one unknown or variable of which it will then be solvable.

Ex. In a certain animal sanctuary, there is a total of 60 animals consisting of cats and dogs. There are twice as many cats as dogs. How many cats, and how many dogs are there?

We see that we have a problem with one known value: 60 animals, and two unknown values: the number of cats and the number of dogs. From the given statement, creating a simple algebraic equation:

60 animals = number of cats + number of dogs

Let's let variable D represent the number of dogs, and variable C represent the number of cats.

Notice that the number of cats was stated in terms of the number of dogs. For each (1) dog, there are two (2) cats. There are twice as many cats as dogs.

First, it is incorrect to state or express this situation mathematically as: 1 dog = 2 cats or 1D = 2C or 1D/2C ("one dog per two cats") as a ratio. If used that way, dog, D, and cats, C, are being incorrectly used as units of measurement, rather than as variables or numeric values that can change in value. Saying, that there is one dog per two cats is normally acceptable, but if you actually express or divide the values: 1D/2C = 0.5(D/C) , hence the units would be (D/C), and again, D and C are to be variable here, rather than units. It would be better to express the ratio as: 1 dog/2 cats.

The actual number of cats: (C) equals the number of dogs times two, hence the number of cats depends on the number (D) of dogs, well mathematically anyway:

C = 2D   : C depends on D, and C can be called a dependent variable whose value depends (is
                          mathematically related to, depends upon, and is determined by) on the value of an
                          independent variable, here D.

                          C = 2D can also be expressed as: D = C/2

60 = C + D   since C = (2D), and using algebraic substitution for C:
60 = 2D + D   : hence, now a solvable equation (because it has just one unknown value, here variable D)
60 = 3D   isolating D by dividing each side by 3 :
D = 20   : number of dogs

C = 2D   : original equation
C = 2(20)   : using (numeric) substitution for the value of D
C = 40   : number of cats

OR:

60 = C + D   after solving for C:
C = 60 - D
C = 60 - 20
C = 40   : number of cats.

If you used algebraic substitution for D = C/2, you would also arrive at this value:

60 = C + (C/2) = 40 + (40/2) = 40 + 20 = 60

Ex. A box contains 75 coins consisting of pennies and nickels with a total sum of $1.75. There are twice as many pennies as nickels, how many pennies and how many nickels are there?

Let P = value of a penny = (0.01)
Let N = value of a nickel = (0.05)

Let X = number of pennies
Let Y = number of nickels
Total coins = X + Y = 75

Total monetary sum due to pennies + total monetary sum due to nickels   = 1.75
                      X(P)   +   Y(N)   = 1.75   substitution the monetary values:
                    X(0.01)   +   Y(0.05)   = 1.75

Since there are twice as many pennies as nickels, Y must be multiplied by 2 to equal X: X = 2Y

Now, X can be algebraically substituted or replaced by 2Y, therefore producing a solvable equation which has only one unknown (variable):

2Y(P)   + Y(N)   = 1.75
2Y(0.01) + Y(0.05) = 1.75   distribute:
 0.02Y + 0.05Y   = 1.75   combine like terms (add their numerical coefficients):
         0.07Y   = 1.75   isolate Y:

 Y = 1.75 = 25   : there are 25 nickels
        0.07

Since   X + Y = 75
            X + 25 = 75
            X = 75 - 25
            X = 50   : there are 50 pennies

Or: X = 2Y = 2(25) = 50   : as noted above

Checking, using the original equation:

 X(0.01) + Y(0.05)   = 1.75
50(0.01) + 25(0.05) = 1.75
     0.50 + 1.25   = 1.75
          1.75   = 1.75   : checks

Here is another way, perhaps a simpler way of solving this problem, and which uses only the number of coins involved, and it's very similar to the previous example:

We were given:

X + Y = 75   and as stated in the original problem:
X = 2Y   by algebraic substitution:
2Y + Y = 75   combining like terms:
   3Y   = 75   solving for y, isolate it by dividing both sides by 3:

Y = $\frac{75}{3}$ = 25   : number of nickels

X = 2Y = 2(25) = 50   : number of pennies

Check: X + Y = 75
         50 + 25 = 75
              75 = 75   : checks

SIMULTANEOUS EQUATIONS

One method to solve multiple variable equations that are all related to a particular situation or instance is referred to as the "addition and subtraction" method. The basic requirement is to have N equations for N unknowns (variables). These initial equations must be independent of one another, that is, none can be algebraically derived or created from another because that would essentially be expressing the same equation as just another way.

Things that happen together or function together, and sometimes at the same exact time or "instantaneously together", are called simultaneous and their (numerical) relationship is that they behave and change simultaneously or together, and may be dependent on the values of each other. A set or group of related multiple variable equations is often called "simultaneous equations" when the solution for each variable is also simultaneous (ie. the same) for all of the given equations. After solving any equation in a set of simultaneous equations for a particular variable(s), its value will satisfy all the other equations..

Here is a system of simultaneous equations that has only one unknown (variable):

Equation 1:   5 + N = 8   : N is a placeholder (ie. a variable) for some unknown value yet to be found
Equation 2:   (7.9)(N) = 23.7
Equation 3:   3N + 2 = 11
Equation 4: N^2 - 2 = 7

Find the value of N. Since the system of equations shown contains only 1 variable, solving for the value of the variable can be done using just one equation, and with or without the assistance of any of the other equations. This will also solve all the equations of the system of equations shown.

5 + N = 8   T. (5), add the negative of it to each side = -(+5) = -5
5 + N - 5 = 8 - 5   combining:
N = 3

Checking using equation 4:

 N^2 - 2 = 7
 3^2 - 2 = 7
    9 - 2 = 7
         7 = 7   : checks

Before a multi-variable example is shown, some verification or practical proof of the "addition and subtraction" process often used comes from the basic fact that adding, subtracting, multiplying, or dividing both sides of an equation by the same value still keeps the equation in balance or "true". For instance, given these two equations:

Eq. 1: (3+2) = 5   and   Eq. 2: (4+6) = 10   Adding 10 to each side of the first equation:

Let A = cost of each apple
Let P = cost of each pear

Optionally considering each value with units of cents, rather than dollars. Therefore:

Eq. 1: 6A + 8P = 110 cents
Eq. 2: 3A + 2P = 35 cents

Before the addition (or subtraction) of these two equations, you must first multiply (or divide) either or both equations by some value so that when they are either added or subtracted that one variable will essentially be eliminated from the new equation formed, hence producing a solvable equation which has only one variable or unknown value. Whenever you multiply or divide both sides of equations by the same value, this actually creates a (derived) "equivalent equation" in the sense that the solution will be equivalent or the same for both. Related equations where the solution is the same for both are also called simultaneous equations as mentioned previously.

Ex. Eq. 1:   2x + 3 = 7   multiplying both sides by 2 :
                 2(2x + 3) = (2)(7)   we have created an equivalent equation of:

   Eq. 2:   4x + 6 = 14

    For both Eq. 1 and Eq. 2 directly above, the solution is the same value of x=2.

Extra: If you were to graph (plot or place, and draw) these two (line or linear) equations (as discussed previously, and ahead in this book), they would intersect (cross, meet, join) at a single point. Since the second equation was created from the first, the solution here is actually a point on the "x" (horizontal or independent variable) axis, and which is also the "x" axis intercept (where y=0) for both lines. Algebraically re-arranging the equations for plotting:

From eq. 1, we get: 2x - 4 = 0   = y   : let y = 0 (and the eq.), and then plot this relationship.
From eq. 2, we get: 4x - 8 = 0   = y   : let y = 0 (and the eq.), and then plot this relationship.

Note however, that if an expression, not an equation, such as these is added by a value, that the new expression will yield a line that is parallel to the original line expression. The reason that it will be parallel is that this added in constant value will essentially shift the line (or curve) vertically (up and down) in reference or respect to the original line. The vertical steepness (or "slope") will still be the same for both lines. Consider:

Ex.   2x + 3   after adding 5:
       2x + 8

Setting both expressions equal to (y) for plotting:

Eq. 1:   y = 2x + 3
Eq. 2:   y = 2x + 8

Since parallel lines do not intersect at some point, there is no (simultaneous) solution to this system of equations.

Continuing the original example and equations:

Multiplying the second equation by 2 since 6A/3A = 2, or simply 6/3 = 2 after canceling A in the num. and den.

Eq. 2:   2(3A + 2P) = 35(2)   distribute:
New Eq. 2:   6A + 4P   = 70

   6A + 8P = 110   Eq. 1

Therefore: $c_1 - 2b = \dfrac{c_2 - b}{3}$ transposing the right hand side:
(Could also begin to solve for (b) by multiplying
both sides by 3, and then canceling.)

Or: $c_1 - 2b - \left(\dfrac{c_2 - b}{3}\right) = 0$ with some distribution:

$c_1 - 2b + \dfrac{-c_2 + b}{3} = 0$ combining fractions on the left hand side:

$\dfrac{3c_1 - 6b - c_2 + b}{3} = 0$ multiplying each side by 3 and canceling:

$3c_1 - 6b - c_2 + b = 0$ combining like terms:

$3c_1 - c_2 - 5b = 0$ transposing (-5b), and then dividing each side by 5, and switching sides:

$b = \dfrac{3c_1 - c_2}{5}$:since c is a constant and there are no other variables in the expression, the numeric value of (b) is solved for

In a similar manner to the above process, variable (a) could have been found. Since (b) is now known, (a) can be found by substituting this value of (b) into one of the original equations and then solving for (a).

There are a few other methods to solve a system of equations besides the last two methods shown. For example, there is the "graphical method" discussed previously where the solution is the point of intersection (crossing) of the graph (plotting, drawing) of the equations. However, the result is usually only a good approximation since it's a non-algebraic (ie. non-exact) method that does not produce a specific or exact numeric result. Still, don't underestimate graphs since for instance they have the power to quickly show the relationships of variables over a wide range of possible values and can be used for quick estimates.

Problems involving percentages might seem to be awkward at first, but if you can write some simple equations, the problem usually resolves to one of basic algebra.

Ex. If 5% of N is 30, what is 15% of N ?

Writing equations with the information given:

1) $N(0.05) = 30$: note that the percentage rate is expressed here in (strict numeric) decimal form for the equations
2) $N(0.15) = X$

By algebraically solving for N in the first equation, we can algebraically substitute its equivalent value into the second equation that has two unknown values, and therefore make that equation solvable with only one unknown value:

From the first equation: $N = \dfrac{30}{0.05} = 600$ substituting this value for N into the second equation:

$N(0.15) = X$
$600(0.15) = 90$: hence X = 90   and   15% of 600 is 90

An alternate approach to finding X is to use the concepts of (equal or similar) proportions, and which will be discussed shortly ahead, but that method can simply be thought of as an equivalent fractions concept where if one value is magnified (basically by a multiplying factor, greater or less than one [ie. technically, demagnified]), all the other

N can also be found using a proportion type of equation.

Since 100% = 1.0 = 1

$$\frac{90}{0.15} = \frac{N}{1} \qquad \text{or} \qquad \frac{30}{0.05} = \frac{N}{1}$$

$$N = 600$$

Ex. In a bag, there are some marbles. Of the total number of marbles, 25% (of the total) are red, 10% are white, and exactly 13 are blue. What is the total number of marbles?

Notice that the number of red and blue marbles was not given, however their values were indirectly stated in terms of the total number of marbles, and more specifically, a portion or percentage of the total number of marbles.

Let us represent the total number of marbles with the letter or variable M, then we have:

Total Marbles = red marbles + white marbles + blue marbles   or:
M =   red   +   white   +   blue
M = (25% of M) + (10% of M) +   13   which can be expressed as:
M =   25%M + 10%M + 13   expressing the percentages as their decimal number equivalent:
M =   0.25M + 0.10M + 13   combining like terms on the right side:
1M = 0.35M + 13   transposing +0.35M, add the negative of this term to each side:
1M - 0.35M = 0.35M + 13 -0.35M   combining like terms:
0.65M = 13   after dividing both sides by 0.65 we have:
M = 20   :a total of 20 marbles in the bag

Note that the total percentage of red and white marbles in the bag is: 25% + 10% = 35%. The percentage of blue marbles in the bag is therefore the remaining (of 1.0 = 100%) percentage of the marbles left:

100% - 35% = 65% of all the marbles are blue.   : or   1.0 - 0.35 = 0.65

checking: (Total Marbles)(0.65) = (20)(0.65) = 13

$$\frac{\text{blue marbles}}{\text{Total marbles}} = \frac{13}{20} = 0.65 \qquad \text{and the problem could have been solved using:}$$

$$\frac{13}{M} = 0.65 \qquad \text{solving for M:}$$

$$M = \frac{13}{0.65} = 20 \qquad : \text{total marbles}$$

The number of red marbles is therefore: 25% of 20 = (0.25)(20) = 5
The number of white marbles is therefore: 10% of 20 = (0.10)(20) = 2
red marbles + white marbles + blue marbles = 5 + 2 + 13 = 20 marbles

ALGEBRAICALLY VERIFYING THE LOG RULES

Below are algebraic verifications of the fundamental log rules.

Given: $N1 = b\text{^}x$ and $N2 = b\text{^}y$: power equations, therefore:

$N1\ N2 = b\text{^}x\ b\text{^}y = b\text{^}(x+y)$ by the log definition:

$\log_b (N1\ N2) = x + y$ the equivalent expressions for x and y are:

From $N1 = b\text{^}x$, we have by the log definition: $x = \log_b N1$.

From $N2 = b\text{^}y$, we have by the log definition: $y = \log_b N2$.

Using substitution in:

$\log_b (N1\ N2) = x + y$

$\log_b (N1\ N2) = \log_b N1 + \log_b N2$: LOG PRODUCT RULE

Ex. Given $\log (X + Y)$, factor it to where the log product rule can be used:

Factoring out X from each term of the given binomial, we now have two factors:

$\log (X (1 + Y/X)) = \log X + \log (1 + Y/X)$:

Letting $N = N1$, and given: $\log N + \log N + \log N$

Since repeated addition of any value can be represented using multiplication of that value, we have:

$1 \log N + 1 \log N + 1 \log N = (1+1+1)\log N = 3 \log N$

By treating (log N) like a variable and adding the numerical coefficients of each of the like variables, we can let a variable (such as x shown below) represent this sum of numerical coefficients (c) of (log N). Unlike for repeated addition as shown above, the numerical coefficient(s) need not be 1. Expressing this as an algebraic equation, we have:

$x \log N = (c_1 + c_2 + c_3 + ...) \log N = c_1 \log N + c_2 \log N + c_3 \log N + ...$

Note, you can also consider this as factoring (log N) from each of the terms, leaving a sum of the numerical coefficients.

Ex. $x \log N = \log N + \log N + 0.3 \log N = (1 + 1 + 0.3) \log N = 2.3 \log N$: x = 2.3

Now, the concept of the log product rule will be used on the following equation:

$x \log N = \log N + \log N + \log N + ...$

The number in a log expression, can also be an expression.

Ex. ln (a + 1) = 3.218875   solve for a: Using the log definition, and N = number = (a+1):

$$\ln N = \ln_e N = \log_e N$$, e is a special number in mathematics, like PI is special.
e is about 2.71, and PI for a circle is about 3.14
When e is used for logarithms, it is called the
"natural logarithm (nl = ln) base"
$$\log_e (a+1) = 3.218875 = \ln (a+1)\ 3.218875$$

e^3.218875 = (a + 1) = a + 1   Solving for a:
a = e^3.218875 - 1   Using a calculator which can easily calculate powers of (e):
a = 25 - 1
a = 24   : A calculator can easily check this if it can calculate natural logarithms.
$$\ln (a + 1) = \ln (24 + 1) = \ln_e 25 = 3.218876$$

SOLVING FOR THE BASE OF A POWER

Ex. Solve for x, the base of an indicated or expressed power, given:

x^(2/3) = 10   raising each side to the 3/2 power, we can isolate x as: x^1 = x :

(x^2/3)^(3/2) = 10^(3/2)

x^(6/6) = 10^(3/2)

x^1 = x = $2\sqrt{10\wedge 3}$ = $(2\sqrt{10})^3$ or $2\sqrt{10\wedge 3}$ = $2\sqrt{1000}$ = 31.62

x can also be solved using logarithms:

x^(2/3) = 10   taking the log (use same base) of both sides:

log x^(2/3) = log 10   using the log exponent rule:

(2/3) log x = log 10   solving for log x by dividing by its multiplying factor:

log x = $\dfrac{\log 10}{(2/3)}$ = $\dfrac{\log 10}{0.67}$ from the definition of logs (here x = N, the number):

x = 10^((log 10)/0.67) = 10^(1/0.66667) = 10^1.5

x = 31.62   and   31.62^(0.677) = 10

Note, you could also begin to solve for x by first simplifying its fractional exponent as:

x^(2/3) = 10
x^0.66667 = 10   and after solving for x: x = 10^1.5   as shown above.
If you were to also take the 0.66667 root of both sides of: x^0.66667 = 10^1, the result
would also be the same value for x.

Method 2:

   log 2^(5 - x)   =   log 8

      (5-x) log 2   =   log 8

        5 -x   = $\dfrac{\log 8}{\log 2}$

      x = 5 - $\dfrac{\log 8}{\log 2}$ = 5 - 3 = 2

Ex. Solve for x:

        2 = 4^x

   log 2 = log 4^x
   log 2 = x log 4

        x = $\dfrac{\log 2}{\log 4}$ = $\dfrac{\log 2^1}{\log 2^2}$ = $\dfrac{1 \log 2}{2 \log 2}$ = $\dfrac{1}{2}$: = 0.5

Note $4^{(1/2)}$ = $4^{0.5}$ = square-root(4) = $\sqrt[2]{4}$ = 2

Note also that this is an example where if you can express the numbers (N1 and N2) as indicated powers having the same base (2 in this example), then the result is simply the quotient of those indicated powers.

Ex. Find x given: 5^x + 6^x = 61

A more advanced analysis will be discussed, and you may skip over this discussion for now if you want. Since the (x) variables cannot be combined, the method below solves for one instance of (x) in terms of the other instance. Of course you cannot (immediately) solve for a variable when that same variable is also part of the solution for itself. To overcome this limitation, a guess, calculated or not, for the value of (x) is first used with the hope that by repeatedly using the formula, that a very close value of (x) will gradually and eventually be found. When this happens, it is said that the result is converging (becoming closer to, approaching (a certain final value or "limit"), "focusing" or "zeroing in on" and becoming equal) to the true value of (x), and the difference between each new value of x approaches 0, hence continue this process till this difference is as close to 0 as needed. This method of solving for a variable is called successive approximation with recursion (reusing the same formula over and over with each new closer approximation of x to be used in it). More examples of successive approximation will be shown throughout this book. Successive approximation will not always work as shown below:

First, as a very simple example to help consider the above described method:

Find x in the equation: 5 + x = 8. Rather than mathematically solve and express the equation in terms of, and equal to x, you can take an initial guess, of say, 1 for x. 5+x = 5+1 , and since this left side equals 6 and does not equal the right side of 8, the difference with respect to, or compared to the right side is: (8-6)= 2. Therefore, the difference in the sides is not 0, and you can keep increasing the value of x and comparing the left side to the right side of 8. This comparing is done mathematically by taking their difference.   If the difference between both sides is 0, they are therefore equal and the actual value of x has been found, and that is 3.

SOLVING FOR AN INDEX VARIABLE

The easiest way to solve for an index (the indicated root of the radicand in a radical expression or root operation) is to first rationalize the radical by raising it to an indicated power equivalent the index, then convert the radical to its exponential form and simplify, or vice-versa. This effectively transforms the problem into one of solving for an exponent.

Ex.   What "root" of 1024 is equal to 4 ?

Expressing this mathematically:

$x\sqrt{1024}$ = 4   raise each side to the (x) power:

$(x\sqrt{1024})$^x = (4)^x   convert the radical to exponential form:

(1024^(1/x))^x = 4^x   using the power to a power rule:

1024^(x/x) = 4^x

1024 = 4^x   : This could have easily been found by inspection by considering:
                                   "What is the exponent of a base of 4 so that the power would equal 1024".
                                    OR:

$x\sqrt{1024}$ = 4   expressing the radical with its equivalent exponential form:

  1024^(1/x) = 4   raising each side to its x power:

  (1024^(1/x))^x = 4^x   using the power to a power rule, multiplying exponents:

  1024^(x/x) = 4^x   simplifying:

  1024 = 4^x   : the same equation as shown above
                                         Taking logs of both sides:

log 1024 = log 4^x   using the log exponential rule:

log 1024 = x log 4   dividing each side by log4:

$\dfrac{\log 1024}{\log 4}$ = $\dfrac{x \log 4}{\log 4}$: isolating x

x = $\dfrac{\log 1024}{\log 4}$ = $\dfrac{3.0103}{0.60206}$

x = +5   , hence $5\sqrt{1024}$ = 4 ,   and 4^5 = 1024

If you were to write a simple formula for the above process, it would look like this:

  i = $\dfrac{\log radicand}{\log root}$: **FORMULA FOR THE INDEX , THE INDICATED ROOT OF A RADICAND**

EQUATING RADICALS (Solving For An Index)

Ex. $2\sqrt{100}$ = $x\sqrt{1000}$ converting both sides to their equivalent radical form:

$100^{(1/2)}$ = $1000^{(1/x)}$ taking logarithms of both sides, be sure to use the same base:

$\log 100^{(1/2)}$ = $\log 1000^{(1/x)}$ after using the log exponent rule:

$\dfrac{\log 100}{2}$ = $\dfrac{\log 1000}{x}$ after solving for x:

x = $\dfrac{2 \log 1000}{\log 100}$ = $\dfrac{6}{2}$ = 3

hence: $2\sqrt{100}$ = $3\sqrt{1000}$

checking: 10   =   10

Writing a general formula for the above procedure:

$i2$ = $\dfrac{i1 \log N2}{\log N1}$: EQUATING RADICALS (INDEX FORMULA)
                                 N1 and N2 are the radicands of the radicals.

Some other observations and derivations can be made here. Considering the above formula, we then have:

$\dfrac{i2}{i1}$ = $\dfrac{\log N2}{\log N1}$: when two radicals are equal, the ratio of the indexes is equal to the ratio of the logarithms
                                 (with the same base) of the numbers. N1 and N2 are the radicands of the radicals.

Some of the following may, or may not, appear obvious according to the logarithm and radical definitions. We know that the log of a value is an exponent (here of any base that is used consistently in a problem), hence:

$\dfrac{i2}{i1}$ = $\dfrac{exponent2}{exponent1}$ = $\dfrac{\log N2}{\log N1}$: base^exponent2 = N2   and   base^exponent1 = N1

power of N to the same root of N such as in the above two formulas, you will see that the exponents and their values are reciprocals of each other, and this leads to the reciprocal-like results of the two formulas.

Here is a solved example for the index (x), where N=100, and x=6.644:

$6.644\sqrt{100}$ = log 100   = 2   : again, simplifying the right hand side, as shown to a value of 2, is the easiest way to solve this equation, as: $x\sqrt{100} = 2$

As an extra note, when the index of a radical is less than one, it actually represents an indicated power greater than one for that radicand.

Ex. Here is the half-root (1/2 = 0.5) of N, and it's actually equivalent to a larger value than that base of N:

$0.5\sqrt{N}$ = $N^{(1/0.5)}$ = N^2 Ex. $0.5\sqrt{10}$ = 10^2 = 100

Of course, if N is less than one , a (proper) fractional value, then the expression would actually represent a value less than N since repeated multiplication of a fractional value is essentially a fraction of that fraction, and which results in a lesser value. For example, $0.5\sqrt{0.3}$ = $0.3^{(1/0.5)}$ = 0.3^2 = 0.3^2 = (0.3)(0.3) = 0.09

SIMPLE EQUATIONS CONTAINING RECIPROCALS

Given: $\frac{1}{G}$ = c G^x   Here, G and cG^x are reciprocals of each other.   (c) is a constant.   The expression of: (1/G) is the reciprocal of G. The expression of: c G^x is a multiple of G^x since (c) is multiplying factor to G^x. (c) and G can have any value except 0. Solve for c, G and x:

$\frac{1}{G^1}$ = c G^x   : note here that G^x = (1/c)(1/G^1) = (1/cG^1), multiplying both sides by G:

1 = cG^(x+1)   dividing both sides by G^(x+1), and switching sides:

c = $\frac{1}{G^{(x+1)}}$ For example, if x = 1:   c = $\frac{1}{G^2}$ = $\frac{1}{G^1}$ $\frac{1}{G^1}$

Ex. Given: $\frac{1}{5}$ = c 5 , x=1, c = $\frac{1}{G^{(x+1)}}$ = $\frac{1}{5^2}$ = $\frac{1}{25}$ = 0.04   : G=G^1=5

Note also that 1/cG = G^x , here,for this example: 1/((0.04)(5)) = 1/0.02 = 5

After mathematically solving for G^(x+1), such as by taking the reciprocal of both sides::

G^(x+1) = $\frac{1}{c}$ taking the (x+1) root of both sides:

$(x+1)\sqrt{G}$ = $(x+1)\sqrt{\frac{1}{c}}$ = $\frac{(x+1)\sqrt{1}}{(x+1)\sqrt{c}}$ = $\frac{1}{(x+1)\sqrt{c}}$

Given:

c G^x = $\frac{1}{G}$ solving for x:

SOME EXAMPLES OF ALGEBRA WITH GEOMETRIC FORMULAS

Geometery is a word that means the measure (meter, metery, metric) and analysis of earthly (geo), physical (real) or virtual (as if real) things or ideas.

Since this book has a trigonometry section, the first example will be directed towards finding the formula for the area of a triangle.

Ex. What is the formula for the area of a triangle?

Previously in this book, some basic concepts about area have been discussed in the topic of Powers.   Below are some more fundamental details about the concept of area.

Distance is a measurement of how many units, or "long" (ie. length), something is, or the length between two points or locations, hence it is a measure of the amount of separation, which is therefore a difference, between things. Distance or length is a linear or line-like concept. The length or distance between two points is found by measurement and-or the difference between those two points having a known linear value. In short, linear (having a line-like quality or relationship) means one-dimensional which basically means having only a single direction for its structure and measurement. Lines are one-dimensional geometric concepts. Lines are infinitely thin, that is, in theory or concept, they actually have no thickness, and are infinitely long, and each extends infinitely outwards in one specific direction. Being one dimensional, a line and its can graphically represent one dimension and its concept. Likewise, one dimensional concepts are therefore often associated with lines and are said as being linear. A line can also be considered as being composed of an infinite number of infinitely small points, all being side by side or next to each other.

One dimensional means that something can only be measured, so as to quantify it, in only one way, such as the length of a line segment that is a small part of a line. A line segment is a defined or (length) bounded region, section or segment of a line.

Two dimensional basically means that something can be measured, so as to quantify it, in two ways, such as a length (dimension) and width (dimension) of a defined or bounded area that is part of the infinite area of a plane. Planes are essentially infinitely "flat surfaces" that have a length and width dimension, but no thickness or third dimension. Two dimensional measurements will naturally build upon and use the one dimensional concept of length. Some examples of two dimensional structures are squares, circles and triangles which are a small defined or bounded (having defined boundaries) part of an infinitely large plane. A plane can be thought of as an infinite number of side by side parallel lines so as it creates a width, thickness, second dimension. Parallel lines will never intersect or cross each other. Each of the two dimension of a plane are defined and understood as being perpendicular (90° of rotation apart, right-angle, exactly "sideways" or "cross") to each other.

Three dimensional means that something can be measured in three ways such as the length, width, and height (or "thickness") of a bounded or defined volume, space or solid structure that is part of the larger and infinite volume of space. Three dimensional measurements will be done using the concepts of two dimension measurement of which are based on the linear or one dimensional concept. Therefore, it is possible to conceive a solid or volume as being a bounded or defined segment, section, or region of an infinite number of side by side parallel planes. The third dimension of a volume is defined and understood as being perpendicular to the other two dimension of a plane.

A drawn horizontal (left to right) line can be thought of as graphically representing the first dimension, and a vertical (up and down) line can be thought of graphically representing the second dimension. A line extending straight through the paper can be thought of as graphically representing the third dimension.

The perimeter (the bounding and defining sides, and-or its corresponding length values) of a triangle or the circumference of a circle, both two-dimensional objects, can still considered as a single (total) length value even though it is not a single (straight) line or segment but rather the sum of them, and that the objects are two-dimensional. This total distance or sum of various lengths can then be thought of and represented as one single line segment.

common fixed "reference" line in this (location) system. Still, a problem now would be identifying a line among an infinite number of parallel lines, and that is not practical. This reference line need not be perpendicular to these parallel lines, but it is usually set perpendicular (a "right-turn" or "right-angle" = a quarter of a full 360° circular turn = ninety degrees = 360°/4 = 90°) since it is essentially the least parallel-like of all possible parallel lines. It is also done in order to create a rectangular coordinate (measuring) system. Rectangular (which means "right-angular") is a term that is used to describe something that is: "square-like", where each dimension or side is rotated 90 degrees in reference to each other.

Some initial concepts for creating a coordinate system on a plane.   [FIG 26]
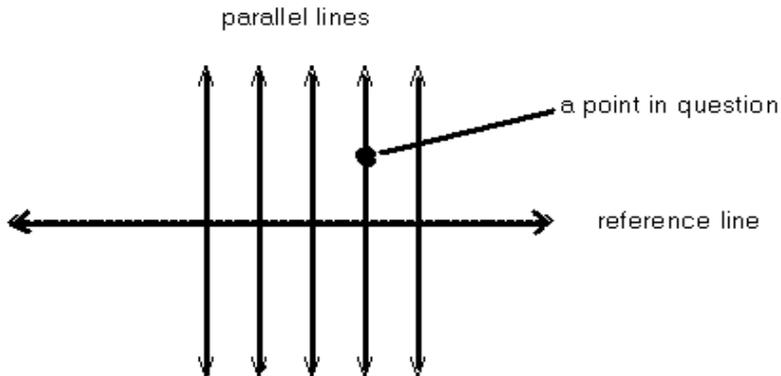
As mentioned, finding the location of a point on a plane using this system, shown in the above figure, is still difficult since there is an infinite number of parallel lines to identify so as to locate a specific parallel line, and then to locate the specific point in question along that specific line. To overcome this problem of identifying and-or address a line, you could assign a particular parallel line as a reference (of measurement) line and then identify each line by its practical (and not the infinite and hence undefinable line number) length or distance from this reference line using a standard (agreed upon) unit of measurement. The outcome of all this is that a points location on a two-dimensional plane or area is identified by it's two perpendicular distances (which could be represented as dotted lines) from two reference lines.

Each point on a plane or surface is at a unique (only one instance of) intersection of two specific lines in a rectangular measuring and addressing system.   This perpendicular system is the simplest and most practical method to locate (give an "address" to, or identify and refer to) a point on a plane or surface. The location or point where the two reference lines (of measurement in this system) intersect or cross is called the reference point (that the location of all other points on that plane will be in reference or respect to) or **origin** ("original", source, reference, or starting) point of the system, and it is therefore has an address, "offset" or distance value of 0 with respect to the two reference lines of the system.   The location of this point, and any other points in that system, is typically written as a pair of coordinates (its two corresponding location values). The location of the origin of the system is written as: (0,0) or p(0,0) which can be read as: "point zero comma zero".   The two perpendicular reference lines drawn through the origin are commonly called **axis lines** of the coordinate system. Axis is a word based on "axel" and "center" location such as of a wheel. These two lines represent the reference lines of measurement for each of the two each dimensions of a plane or rectangular the coordinate system. Each of the two coordinates of a point are effectively the (perpendicular) distance to (or from) the other coordinates axis line of reference. If given an address of (x,y), the x-coordinate is the points distance from the y-axis, and the y coordiante is the points corresponding distance from the x-axis.

The coordinates of a point (p) are often expressed using the basic format of:

p(length or position in dimension1 **,** length or position in dimension2)

meters, centimeters, etc), it is common sense to call this unit of measurement a "square unit" or "unit square".   A square-unit is the basic unit of measurement for measuring surfaces.   In mathematical notation:

1 square unit = 1 unit square = 1 sq. unit = (1 unit) x (1 unit) = (1)(1)(unit)(unit) = 1^2 unit^2 = 1 unit^2 =   unit^2

Consider the algebraic notation when width = length, as for a square: (length)(width) = (length)(length) = length^2
Also, if that length was just one unit long, that length unit squared is also the unit of area measurement:
length^2 = unit^2 = squared unit

The actual measured size of a portion of a plane or surface is called the surface's area, area of surface, or just area. The word "area" is sometimes loosely used as meaning surface itself, instead of a measurement of it. Since a surface is measured in reference to a square-unit, the actual number of units of a measured surface or "area" will be equal to a specific quantity or number of those square-units of measurement.

If you had 4 separate surfaces, each with an area of 1 unit^2 = 1 square-unit, their total area would be:

Total Area = (1 unit^2) +   (1 unit^2)   + (1 unit^2) +   (1 unit^2)   adding like units:
Total Area = (1 + 1 + 1 + 1) unit^2   =   4 unit^2   : "4 units square" = 4 units^square = "4 square units"

If the reference length units used to define a reference square-unit were feet units, the units of surface measurement must therefore be expressed with those same units:

Total Area   =   4 unit^2
Total Area   =   (4)(1 ft x 1 tt) = (4)(ft^2)   : the word feet is actually the multiple or plural of foot (ft)

        = 4 feet^2 =   4 ft.^2   : note, this expression: units^2 , is not a mathematical operation, and it is
        = 4 square-feet   just the mathematical notation used for these types of units of
                              measurement since it was defined this way from two linear units:
                              (1 unit)(1 unit) = 1 unit^2 = unit^2 . That is, the unit of area reference is an
                              area that is bounded or defined as a square, 2 dimensional shape, and the
                              unit of reference is therefore not just a line or one dimensional unit only.

These different shaped surfaces shown below can be measured as a sum of unit areas, and the total area is equivalent to any other possible surfaces having the same total area of 4 ft.^2. This can easily be seen when these separate surfaces are joined together, particularly when the entire shape becomes a rectangle and the area can be easily calculated.
[FIG 29]

1 unit

1 unit   1 unit   1 unit   1 unit

: total area = 4 square-units =
             = 4 units-square = 4 units^2

1 unit

1 unit

1 unit   1 unit   1 unit

: total area = 4 units^2
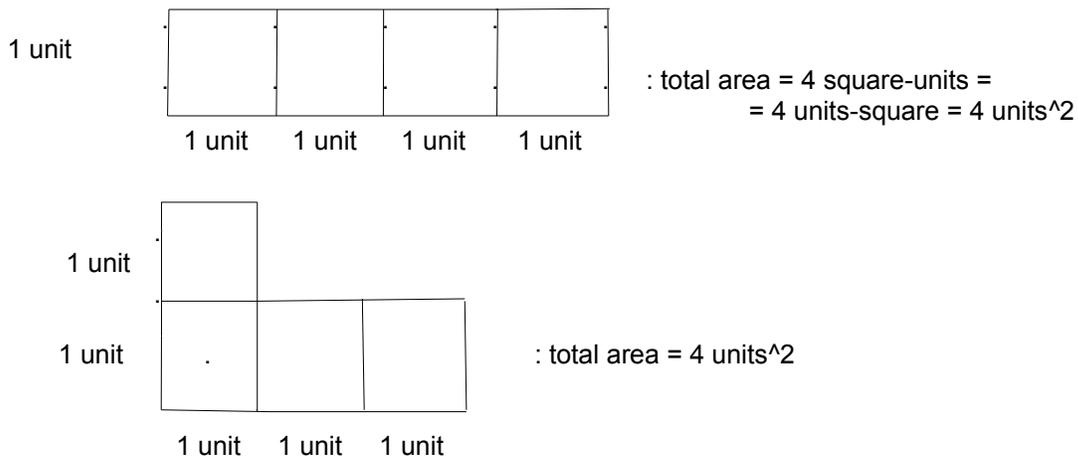
If each square with sides 1 unit long is the unit of measurement for area, that is, each is a square-unit:

Ar = LW square-units = LW units^2

The same formula for the area of a rectangle is also naturally used if the area being measured also has or includes a fractional part of a square unit.   For example if a rectangle is measured to be 5.2 centimeters long, and 2 centimeters wide, the total area of that rectangle is: Ar = LW units^2 = (5.2 cm)(2 cm) = (5.2)(2) square-centimeters = 10.4 cm^2

      As a final description of (two-dimensional) area, the corresponding change or increase in the amount of area is not a linear (balanced, direct, one-to-one, matching, same, constant portion or ratio) concept when its (one-dimensional, or linear) sides change in size or value. The formula for area in relation, respect or in reference to its sides is not a linear expression or resulting value. When a line has or will be increased in length by some factor, you can find its new or resulting length value by adding repeatedly or multiplying a value to the old length and value.   If you add a length to itself, you double its length and value. This is the same as multiplying that value by 2. However, if you were to double the sides of a square, it's corresponding area does not just double, but actually increase by the square (the second power) of this (increasing) factor. The area will then increase by the (multiplying) factor of: (the increasing or multiplying factor of its side length)^2. For this example, where the side length of a square increased by a factor of 2, the corresponding area will increase by a factor of = 2^2 = 4   The new area is 4 times more or bigger. It could be said that the area has been magnified or multiplied by a factor of 4. Here is an algebraic or symbolic mathematical verification:

      A = (s)(s) square-units   : s = side length of a square, where length=width,
                                         A square is actually a special or unique instance and form of a rectangle

      A = s^2 square-units   : **FORMULA FOR THE AREA OF ANY SQUARE = (side)(side)**

                                         Since the variable does not have an exponent equal to just 1, this
                                         mathematical relationship of (s) and A, will not graph as a line.
                                         Their mathematical relationship is said to be non-linear.

Doubling each side of a square we find::

      A = (2s)(2s) units^2 = A1 =
      A = (2)(2)s^2 units^2 = 2^2 s^2 units^2 = 4s^2 units^2 : when each side of an area is increased by a
                                                   factor, that area increases by: (factor^2).

      A2 = ((factor)(s))^2 = (factor^2)(s^2) = (factor^2)(area) = (factor^2) A1

Dividing this new or second area A_2 by the older or original area A_1 that was initially shown and expressed above as just variable A, we can find the number of times that A_2 is bigger than A_1, or the number of times that the original area A_1 has been magnified (a multiplicative or factor increase = increased by a multiplying factor value, rather than being just a simple linear increase by adding some value).

     $\dfrac{A_2}{A1}$ = $\dfrac{4 s^2}{s^2}$ = 4   : the new area is four times bigger or larger than the old area, such as is the result
                                     when that old area has been increased by a factor or four after doubling (2) its sides.

If the side length of an area tripled (increased by 3 times), the area will increase by: (increasing factor)^2 = 3^2 = 9 times more.
      A = (3s)(3s) units^2
      A = (3)(3)s^2 units^2 = 3^2 s^2 units^2 = 9 s^2 units^2 , hence A2 = 9 A1

The mathematical relationship between area and length is said to be a non-linear relationship, and specifically, it is an exponential relationship. Area grows or increases in value much more rapidly than the side lengths of that area. As the

variables being considered. Previously, when the side increased
by 1, the area only increased by 3.

4	16 sq. units	: the side increased by 1, and the area increased by 7
5	25 sq. units	: the side increased by 1, and the area increased by 9

Notice that when the side length increase or changes by 1, a constant value, that the area increases or changes (ie. a difference) not by a constant value (which would make the relationship between area and length a linear mathematical relationship), but by a variable and growing (not constant) value.

Likewise, if the area of a square is only doubled, the corresponding sides are, or were not actually doubled either, as would be the case for some kind of linear or direct relationship, but they are only increased by a (multiplying) factor of:

$\sqrt{2}$ which is about 1.414, and here is the verification:

A = s^2   : formula for area of any square.   Solving for the side variable (s) given the area:

s = $\sqrt{A}$: **FORMULA FOR THE SIDE OF A SQUARE SHAPE OR AREA**

Given: A = s^2   multiplying each side of the equation by 2, so as to double the area:

2A = 2 s^2   factoring 2 (here, for its two sq. root factors):
2A = $\sqrt{2}$ $\sqrt{2}$ (s) (s)   by the associative law :
2A = ($\sqrt{2}$ s) ($\sqrt{2}$ s)   : each side or dimension of a square must have the same length
2A = ($\sqrt{2}$ s)^2   : if area of a square doubles, the sides only increase by the
                                   square root of 2 (about 1.414), and which is lesser value than 2.

In general, if some area and value is said to change by a factor of N, the formula for the sides of a representative square that has the same equivalent area is:

From: s = $\sqrt{A}$ If area increases or is magnified by some factor N:

                                (Note, we cannot use: s = N $\sqrt{A}$, since this is actually only a multiple
                                of $\sqrt{A}$ or= s, and this is not equal to the area value = A)

s = $\sqrt{NA}$ which can be expressed as:

s = $\sqrt{N}$ $\sqrt{A}$: If area increased by a factor of N, the corresponding sides of that (equivalent square
                                   shaped) area will only increased by the factor of: $\sqrt{N}$, and not N.
                                   Shown above, when the area doubled (2), the side length of the square
                                   increased by the (multiplying) factor $\sqrt{2}$ = about 1.414

If it is said that the side length of an area (being thought of, represented or considered as a square) changed by a factor of N, a general expression for area would be:

From: A = s^2   if the side length changed by a factor of N:

A = (Ns)^2   which could be expressed as:
A = N^2 s^2   : we see that when the sides of a square change by a factor of N, that
                           the area of that square will actually change by a factor of: A / s^2 = N^2

If each linear dimension of a volume is increased a factor of (n) times, the volume will increase by a factor of n^3 times:

New Volume = (n) old_length x (n) old_width x (n) old_height   : n = linear magnification or factor
New Volume = n^3 (old_length x old_width x old_height)
New Volume = n^3 old_Volume

For example if the 3 dimensions of a volume, space or solid object are double (2), the old volume is increased by:

n^3 = 2^3 = 8 times.

Another related concept is that the mass and-or weight of a 3 dimensional object will also increase by (linear_magnification)^3 = the cube of the linear magnification of the volume ("or size") of an object.

new_mass = (linear magnification)^3 old_mass   and   new_weight = (linear magnification)^3 old_weight

If any one factor of a volume changes by (n), either multiplied or divided by that factor of (n), the volume will likewise decrease by that same factor of (n). Here is a verification for when the height is half:

V1 = (length)(width)(height) = LWH   if the height is now half, or divided by a factor of n=2:

$$V2 = (length)(width)\frac{(height)}{2} = L\,W\,\frac{H}{2} = \frac{LWH}{2} = \frac{V1}{2}$$:the volume is divided by that same factor of n=2

Due to the previous discussions, we now have more information to help us develop the formula for the area of a triangle.

First, the area of a rectangle is: Ar = (length)(width) , the units of measurement for length and width must also be the same so that the area's units of measurement resolve to square units = units^2 of measurement. Now, if you diagonally (corner to opposite corner, that is, being not on the same side or line itself) divide any rectangle in two, and the result is two identical (same = congruent = identical or "exact same size") triangles. Hence, each one of those two triangle's is equivalent to half the area of a rectangle that it was or can be derived from. Expressing this as a mathematical expression we have the formula for the area of any right triangle:

[FIG 31]
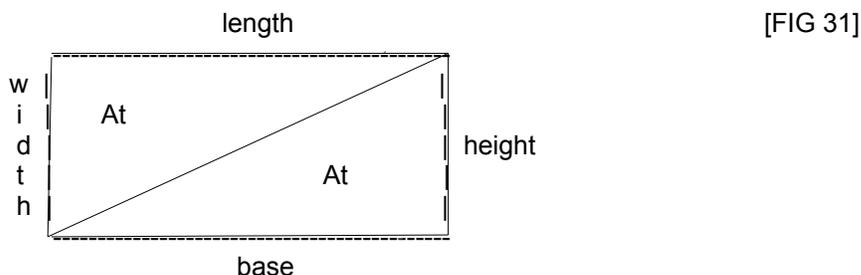
$$At = \frac{(base)(height)}{2}$$: **AREA OF A RIGHT TRIANGLE**

Here, base (side of the triangle) is equivalent to the length of the rectangle, and height (side of the triangle) is equivalent to the width of the rectangle. This formula is essentially that of half (divide by 2) the area of a rectangle which contains "square corners" or "right-turn" angles (90 degrees in value). All right-triangles will have a 90 degree internal angle = 90° angle.

(s + a), and whose height dimension is (h):

The area of the indicated rectangle below in which the parallelogram resides is:

Ar = length x width
Ar = (s + a) x (h)   : here h = height of parallelogram = width of rectangle. Distributing :
Ar = sh + ah

[FIG 35]

h = height

s   a
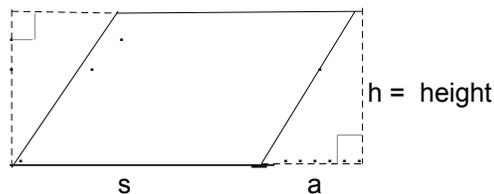
The area of the parallelogram (Ap) can be found if the area (At) of the 2 identical small triangles is subtracted from the total area (Ar) of the rectangle:

 Ap = Ar - 2 At   using substitution:

Ap = [(s + a) x h] - 2$\cancel{\frac{(ah)}{2}}$: canceling like terms, then using distribution:

Ap = sh + ah - ah   combining like terms:
Ap = sh   letting s= b= base of the parallelogram :

Ap = bh   : AREA OF A PARELLELOGRAM

As the parallelogram is tilted away from where the sides intersect, one pair of diagonal angles will decrease, and the other pair will increase by the same angle. The more the sides of a parallelogram are "tilted", the more its height will decrease, and therefore its area will decrease. The parallelogram becomes "flattened".

If you draw a diagonal within the parallelogram you will create two identical (exactly the same or congruent) triangles, and then it is easy to see the general formula for the area of any triangle, and not just for right-triangles, and that it is one-half of the area of the corresponding parallelogram: [FIG 36]

At

At

b

h

: **AREA OF A TRIANGLE**

At = $\frac{Ap}{2}$ = $\frac{bh}{2}$
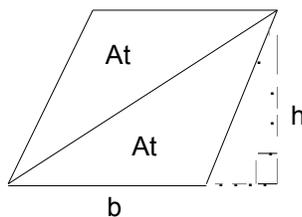
Note, the base (b), and height (h) lines and-or indicated extensions, are defined as perpendicular to each other.

Consider this alternate analysis and derivation of the area of any triangle: [FIG 37]

p, (("pi") or PI), is not a variable but is a symbol for a numeric constant that has a value of about 3.141592654. The appendix section of this book contains a derivation of pi, area, and the circumference of a circle. The value of p is equal to the circumference (C) of any circle divided by its corresponding diameter (diagonal or across measurement) (D) which is the greatest distance across the circle. **p = C / D**. Mathematically, (pi) is the ratio of a circles circumference to its diameter. It is how many times longer that the circumference is than the diameter. Pi is a strict numeric value that does not have any associated units, and is simply a factor value. Given any circle, its circumference is always larger by a factor of about 3.14 times more than its diameter. The diameter, when considered or represented as a line segment, always passes though the center of the circle from and to the circumference. The radius (R, or r) of a circle is the distance from the center of a circle to its outer perimeter, hence it is equal in length of half the circle's diameter. The word "radius" is rooted in the word "ray", such as a ray of light that extends outward from a source.

For a specific given circle that remains a constant size, the radius, diameter, and circumference are not variable in value, but are constant (specific, unchanging) values for that circle. All circles are similar, but are not identical (the same) if they are of different size and-or have different values or parameters (radius, circumference, diameter, area) in their construction.

$$R = \frac{D}{2} \quad \text{hence:} \quad D = 2R$$

therefore, **C = pD**   : **CIRCUMFERENCE OF A CIRCLE** (in terms of its diameter)

$\quad\quad\quad\quad$ C = 3.14159 D   : a simplified, approximate formula for the circumference of a circle

$\quad\quad\quad\quad$ C = p2R   : Rearranging the values to express a more standard form so that the decimal
$\quad\quad\quad\quad$ **C = 2pR**   number comes first as a typical numerical coefficient of a term.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$: This is the circumference of a circle in terms of its radius.

$\quad\quad\quad\quad$ C = 2(3.14159)R = 6.2832 R   : a simplified, approximate formula for the circumference of a circle

From C = 2pR , after solving for R:

$$\mathbf{R = \frac{C}{2p} = \frac{D}{2}} \quad : \textbf{RADIUS OF A CIRCLE}$$

R = $\dfrac{C}{2(3.14159)}$ =   0.159155 C   : a simplified, approximate formula for the radius of a circle. This indicates
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ that the radius is about 16% of the length of the circumference.

Substituting the values given in the example:

R = $\dfrac{10.75\text{ft.}}{2(3.14159)}$ = $\dfrac{10.75\text{ft.}}{6.2832}$ =~ 1.71091 ft.

Also, since C = pD , **D = $\dfrac{C}{\text{pi}}$**   : **DIAMETER OF A CIRCLE** , pi = about 3.14159

D = $\dfrac{C}{3.14159}$ =   0.31831 C   : a simplified, approximate formula for the diameter of a circle. This indicates
$\quad\quad\quad\quad\quad\quad\quad\quad$ that the diameter is about 32% =~ (1/3 = 0.3333...) of the length of the
$\quad\quad\quad\quad\quad\quad\quad\quad$ circumference, or that the circumference of a circle is slightly more than 3 times its
$\quad\quad\quad\quad\quad\quad\quad\quad$ diameter length.

The following example may be too advanced, involved or lengthy for some people, and they can just skip it for now, and consider viewing it at another time.

$$\frac{1{,}600{,}921.188 \text{ miles}}{1 \text{ day}} \;=\; \frac{1{,}600{,}921.188 \text{ miles}}{24 \text{ hour}} \;=\; \frac{66{,}705.04949 \text{ miles}}{1 \text{ hour}} \;=\; 66{,}705.04949 \text{ miles/1hour}$$

Since there are 60 minutes in 1 hour, we can find the orbit speed or distance traveled per minute if we divide the hourly orbit distance by 60:

$$\frac{66{,}705.04949 \text{ miles}}{1 \text{ hour}} \;=\; \frac{66{,}705.04949}{60 \text{ minutes}} \;=\; \frac{1{,}111.750825 \text{ miles}}{1 \text{ minute}}$$

Since there are 60 seconds in 1 minute, we can find the orbit speed or distance traveled per second if we divide the minute orbit distance by 60, and to keep the fraction in balance or equivalence, it must also be done to the numerator:

$$\frac{1{,}111.750825 \text{ miles}}{60 \text{ seconds}} \;=\; 18.52918042 \text{ miles/second}$$

We don't notice this speed since we are relative (in relation to) to Earth's speed. That is, we are also going a distance of 18.53 miles per second around the Sun so there isn't any new accumulated distance between us and the Earth so as to even notice this speed. The difference between our speed and the Earth's speed is practically 0.

The formula or standard equation for the area (A) of a circle in relation to its radius is derived below:

Given a circle, and creating an infinite number of concentric (all having the same center point) circles, "rings" or "shells" within it, the width of each ring will approach a thin width close to 0 (but never actually reaching 0 since there would not even be a possible ring to consider). The length (ie. circumference) of each ring will go from C=2(pi)R at the outer edge of the circle to 0 at the center of the circle. The area of each ring will essentially be that of an infinitely thin rectangle.   Here is the area of the outer ring:

-- = width	: C = Circumference of the circle
C = 2πR = outer length	: R = radius of the circle

Stacking all the rectangle areas and aligning them to the right, their combined width or height is obviously equal to the radius of that circle, and the structure has the shape of a triangle: [FIG 39]

(unaligned)   (aligned to the right)

R = height of this triangle

C = 2πR   =   C = 2πR
(base side and length of this triangle)
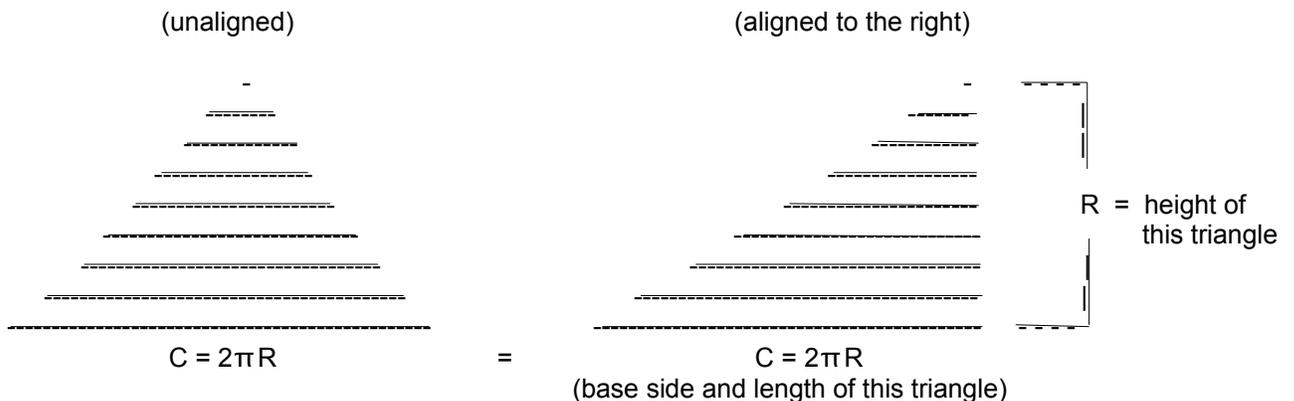
The area of the circle is then equivalent to the sum of the areas of all the thin rectangles, or simply the area of the right-triangle created as seen above when all the rectangles are aligned to their right side. The triangle created is a right triangle and its area is therefore equivalent to the area of the circle from which it came from:

$$\frac{A2}{A1} = \frac{78.54\ in^2}{19.635\ in^2} = 4$$

Showing this in general terms algebraically if the radius (r) of a circle increases by a factor of N:

$$\frac{A2}{A1} = \frac{(pi)Nr^2}{(pi)\ r^2} = \frac{(pi)N^2r^2}{(pi)\ \ \ r^2} = N^2$$: Area will multiply in value equal to the square or
second power of the multiplying factor to the radius.

Using this information, it is now reasonably clear that if it is said that the diameter or radius of a circle is halved, then the area will be 4 times less:

$A2 = (pi)(r1/2)^2 = (pi)(r1^2/4) = (pi)(r1^2) / 4$:where r1 is the radius of the original circle.
The area will be divided by 4, hence 4 times less.

Dividing by 4 is the same as multiplying by (1/4) = 0.25

r2 is noted at being half of r1: r2 = r1/2 = 0.5 r1,   Hence   r2/r1 = 1/2 = 0.5

If r1 was changed by this factor value of 0.5, then area will change by this factor value squared:

$$\frac{A2}{A1} = N^2 = 0.5^2 = 0.25 = (1/4)$$: A2 is a "fourth" of A1, as mentioned above

As a final note, it must be mentioned that the mathematical relationship between area (A) and perimeter (P) of any two objects is not always a corresponding or direct relationship. The relationship can be an inverse relationship, that is, It is possible for the area of an object-1 to be smaller than the area of object-2 even though the perimeter of object-1 is larger than the perimeter of object-2. Consider the simple example below where the units of measurement are considered the same for both rectangles: [FIG 40]

Ex.   object-1   object-2
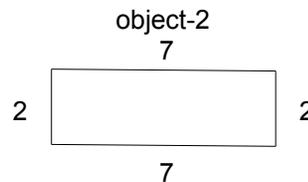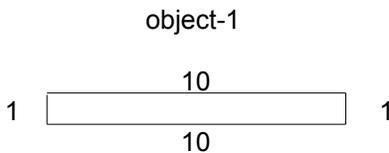
From: Perimeter = sum of each side of an object:

P1 = 10 + 10 + 1 + 1 = 22   P2 = 7 + 7 + 2 + 2 = 18

From: Area of a rectangle = (length)(width):

A1 = (10)(1) = 10   A2 = (7)(2) = 14

The mathematical conclusion of this example can be expressed as:

P1 > P2   : > means "greater than". Arrow points to the smaller value.
"Perimeter 1 is greater than perimeter 2."

A1 < A2   : < means "less than", still, the arrow points to the smaller
value.   The larger or "open" end of the arrow indicates
the larger value.   These symbols and their use are
discussed next. "Area 1 is less than area 2."

Ex.   a /=/ c   : "a is not equal to c"

Ex. Express the quantity of (x+1) being greater than or equal to 5.

   x + 1   >=   5

To solve for values of (x) that satisfy this relationship, solve the inequality just like solving an equation:

     x + 1   >=   5   T. +1, so add or combine -1 to each side :
   x +1 -1   >=   5 - 1   C.   and we find that:
         x >= 4

If you multiply or divide each side of an inequality by a negative signed value, the direction (or "sense") of the inequality sign must be changed. The example below will illustrate and verify this:

Ex. We already know by common sense, or according to a number line, that both of the following statements are true:

          5 > 4   and   -5 < -4   : both statements are right, correct, true

      -8 <-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------> +8   : [FIG 41]
   (lesser) . . . - 5   -4   -3   -2   -1   0   +1   +2   +3   +4   +5 . . . (greater)
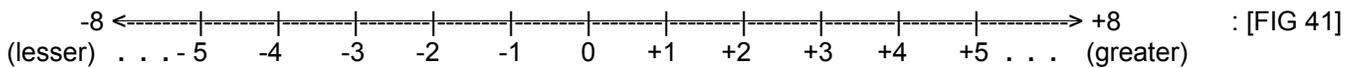
If this number line was a road, this is what some direction signs located at any point on it would look like:

   (greater -->)   : values that are greater, are in the rightward direction, (regardless of your current position)
   (<-- less   )   : values less are in the leftward direction, (regardless of your current position)

Multiplying each side of both of the inequalities expressed above by (-1), but not changing the "sense" or relationship:

          5 > 4   and   -5 < -4
      (-1)(5) > (4)(-1)   (-1)(-5) < (-4)(-1)
         -5   >   -4   5   <   4   : both relationships are wrong, incorrect, false

According to the number line, both of these statements are now false. To true these statements, simply change the direction of the relationship indicated between the members (expressions) or sides of each statement. The results will then be equivalent to the true statements shown at the beginning of the example.

remaining.   A computer has a special memory (data or information storage implemented via electronic circuitry) location that indicates ("flags") the mathematical sign (pos. + , = 1 , or neg. -, = 0) of the result of an arithmetic operation after performing it. Another memory location indicates if the result's value is 0 which means no difference. This will allow for conditionals such as: "equivalence", and the "greater-than-or-equal" dual-conditional where both the "sign" and "zero" flags are then both checked after the subtraction, and both need to be set to 1 for that conditional statement to evaluate (or result, "return") as true = 1 to the computer program being used to control or direct the computer system to do things. If a certain bit at a memory address or location, such as a "flag register" and "flag bit" is set to 1, it is sometimes said as that bit or location is "set", otherwise if is 0, it is said as "not set".

As an example consider this pseudo-code (generic code expressed with literal statements, generally adaptable for any computer language) of a conditional statement, and the possible syntax for it:

 if (condition is true) then do something   or: ?(condition) then (do something)   , Ex: if (P2 < P1) then do something

The "do something" in the above statement is computer program code (coding, instructions, statements, "steps", commands, often written in plain text) to perhaps change the value of a variable, or code to branch (go to) to another portion of the program to do something else, or to evaluate the logic (truth or falseness) of another conditional statement. If the condition that the relationship stated is true, it will be "flagged" (remembered or indicated) as 1=true, then the "do something" code will be processed by the computer. If the relationship stated is false, it will be "flagged" as 0=false, then "do something" will not be performed and that portion of computer code will simply be ignored by "ignored", "jumped over", or "skipped over", and the program will then continue processing ("running") at the next computer code.

With modern-day "high-level" (plain text) computer programming languages, the eventual and actual computer (strict numeric) code (steps) to find the difference and branch to the "do something", or not, is then converted and written as an equivalent machine language program.   Machine language and programs are a "low-level", fundamental computer language which is strictly a numeric (digital, binary) code language so as to be compatible with the electronics of the computer, and specifically the CPU which is the "Central Processing Unit" to execute (ie., "run") programs, usually built as an integrated circuit ("chip") inside the computer system. Higher level languages such as BASIC convert the much more mentally easier to understand, more readable (as ASCII text and understandable words) programs into machine code.

A plain text, readable "high-level" computer language that the computer programmer types in will later be processed with the aid of another general computer program called a compiler program that gathers and compiles (ie., combines to a single piles) together the "readable" program parts (variables, values address, etc.) and transform it into its equivalent, low-level, machine program that an electronic machine, such as a computer, can effectively understand or work with, and which is very much like (logically to physical electronic voltages) controlling, setting and resetting, many on-off switches of that machine.   Note that it is still possible for a programmer to initially write (faster) actual machine code or language programs, and a helpful program to do this is with a program called an "assembler", of which itself often has its own (minimal) syntax called "assembly language", and these programs are then called assembly (language) programs. In pure (digital, numeric input, control) machine language, "strict (ie., only)" numbers that are input and stored as digital, binary numbers in its memory would be used for commands and-or data to process (ie., use, work with).   For example, 00000100b = 4d might be the documented instruction (ie., command, or "opcode" = operation code = machine code) to add the values (ie., the operands of the addition operation) in two memory locations and-or (operand) "registers" and place the output or result in another memory location, and the equivalent for the assembly language program would use a mnemonic (pronounced as "nuh-maa-nik", and is associated with the word memory, and is another word, abbreviation, or symbol(s) so as to represent and be easier to use) for that instruction or command such as ADD, and this helps the programmer understand what is being done in the program and makes it easier to write and-or edit later.   In a high level language such as C, we can more easily read and write (for example, using more common English language) expressions and-or program statements (much like a complete set of instructions or a sentence) for what we want done, and the compiler program will quickly create all the corresponding machine code from that for us.   Here, the machine is the computer's central processing unit (CPU, digital circuit, on chip(s)) and the rest of the computer system that will run (ie. process, execute, utilize) that computer program. For example, the C statement: c = a + b; will add the operands (a) and (b) that are each stored at a memory location that is identified and accessed by the program as (a) and (b) respectively, and the result is assigned to variable (c) that it and-or its value is assigned to another memory location to hold (numeric,

(either 0 or 1) and-or logic value (true=set=1, or false=notset=0) at the input will determine the binary and-or logical output of either 0 or 1. Simple binary logic or binary **unary** logic can be thought of as a **single** switch that can be ON=1 or OFF=0. Negative or inverse logic is when a binary value is converted to its opposite value where 1 becomes a 0, and a 0 becomes a 1. A single transistor can be used to create either a switch (ie., a gate) to control current flow, much like a relay, or to be an "inverter" which is also called a "not gate" or "(binary) complement" of which will invert the logic or state (ie., logic status) of a binary value. The above circuits are also called "logic gates" for it can be said that it is the input logic (truth, binary on, or binary off, open or closed, set or not-set, or logical 1 or 0) and gate (ie., like a "logic or binary filter", here implemented as circuits) that determines what input(s) are allowed to enter, and to then produce a specific output (ie., 0 or 1). Note for example that it is possible to construct an AND gate with more than two or more inputs, and that all would have to be set to 1 so as to produce a 1 output. A NAND gate is a Negative-AND gate of which inverts the logic of the output of the initial AND gate. If the logic result of the AND section is 1, it will be inverted by an inverter and sent to the output as a 0.

Before the invention of the modern-day electronic, digital or binary computers, there were mechanical, analog computers. These often contained gears and rods. One of the oldest found mechanical computers is known as the **Antikythera mechanism** which is a (hand cranked powered, turned) mechanical computer having thin, fine bronze gears and which could calculate and-or predict many astronomical events in the past or future. It was made by the ancient Greeks and has been scientifically estimated to have been made around 75BC to 100BC, due to the carbon dating of wood pieces, and other items having a known antiquity date. The device was found in a submerged shipwreck in 1901 near the Greek island Antikythera in the Mediterranean Sea, and it is considered as an incredible discovery. The delicate device was encrusted by debris and had to be x-rayed so as to find much further details about it. An obvious speculation is what other mechanical computers, metallurgy, machine and-or gear knowledge and technology was available at its time and that which came before this, and was the production of such devices very limited, and were there many less complex, and less costly mechanical devices available to the general public. Later, similar fine metal gear work would be found in the clocks made in the 1500's. Surely, hundreds of years of astronomical observations, calculations and knowledge had to be already known so as to produce a **automatic computing** device such as the Antikythera mechanism. By 1000AD, the Arab civilization of the middle-east made further progress in the sciences, including mechanical devices, most likely assisted by the decimal, positional numeric system first invented in India and then refined and used by the Arabs so as to express formulas and make calculations. Numeric counters and-or displays before the electronic digital age used gears and-or wheels. The Antikythera mechanism is a "hard coded" computer, that is, it is a (mechanically) pre-programmed (by using specific gear sizes, etc) device to do only specific (astronomical) calculations that it was designed to do. Many computer processes are repetitive and-or cyclic, and a wheel or gear is a mechanical device that can be used to do and-or control such processes. As for electronic computers, at the core of their repetitive and or cyclic processes is an electronic oscillator (ie., creates pules, a uniform clock-like timing) which replaces the older technology, mechanical rotating wheels or gears. An electronic computer is not analog (analogue, analogy, similarity or representation, to account, accurate), but rather digital (ie., discrete steps, (fine) approximation, high precision) and capable of being very fast.

Ex. If you have two apples and five oranges, the ratio of apples to oranges (meaning the quantity of apples compared with, or with respect or in reference to, the quantity of oranges) is said to be "two to five", which is mathematically expressed as: 2/5 = 0.4/1 ("zero point four, to one", or simply 0.4) after reducing the fraction to lower terms. It is incorrect to convert this decimal value to its equivalent percentage and say that 40% of all the fruit is apples. NOTE: This value of 0.4 really means that the quantity of apples only is 0.4 times larger than the quantity of oranges only. The value of "all the fruit" is actually the number of both apples and oranges, and which is (2+5) = 7 fruits.

The ratio of apples to oranges is: $\dfrac{apples}{oranges} = \dfrac{2}{5} = 0.4$: apples and oranges, being different fruits, cannot be physically or mathematically compared or combined to produce more of either. The expressions here just represent the quantities of the apples or oranges. The words of: "apples" and "oranges" are being used here like mathematical variables that are only numeric in value.

Mathematically the number of: apples = 0.4 oranges

If the number of apples equaled the number of oranges, their ratio would be 1. For the result value of 0.4, it can be said that, for every 1 orange, there are 0.4 apples. That's almost half an apple, or 0.5 apples for ("per") each orange

As mentioned, tt is incorrect to convert this decimal value to its equivalent percentage and say that 40% of all the fruit (both apples and oranges) is apples. This value of 0.4 really means that the quantity of (only) apples is (only) 0.4 times larger than the quantity of (only) oranges. Since 0.4 is less than one, it is a fraction, hence the number of apples is only a fraction and will be less than of the number of oranges. The can be clearly seen since 2 is less than 5, and therefore, 2 is only a fraction or part of 5.

The ratio of oranges to apples is: $\dfrac{oranges}{apples} = \dfrac{5}{2}$: "five to two", or by canceling:

$\dfrac{oranges}{apples} = \dfrac{2.5}{1}$: "two-point-five to one", or simply 2.5 times more oranges
This form is an expression for a "fundamental or unit" ratio.
Mathematically: oranges = (2.5) apples

Clearly with the numbers shown that for every 1 apple, it can be said that there are 2.5 oranges. Likewise, it can be said that there are 2.5 oranges per 1 apple. Note that 0.4 and 2.5 are reciprocals of each other and that these two corresponding ratio's may sometimes be noted as being "reverse" or "inverse" ratios. If you converted both of these ratios to their corresponding percentages, their combined percentage is: 40% + 250% = 290%. Clearly, this is incorrect since the total quantity of fruit cannot exceed its whole or entire self = 1 = 100%. Since apples is part of the total quantity of fruit, to find the percentage of (all) the fruit that is apples only, the quantity of apples must be mathematically compared, by a division by the total quantity of fruit, hence by taking the ratio of apples with respect to the total quantity of fruit (and not just oranges):

$\dfrac{apples}{fruit} = \dfrac{apples}{apples + oranges} = \dfrac{2}{2 + 5} = \dfrac{2}{7} = 0.2857...$: The ratio of apples to, or of, all the fruit is 0.2857...
"~ 28.6% of all the fruit is apples", and the ratio of apples to oranges is 0.40 = 40%

The remaining fruits' (oranges in this example) percentage can be found by:

Whole = Sum of parts   and for this example:
Whole = (part apples) + (part oranges)   algebraically:
part oranges = Whole - part apples
part oranges = 1.0 - 0.285714285   : can optionally use percentages: 100% - 28.57%
part oranges = 0.714285714   : "71.43% of the fruit is oranges"

value of 5, and the corresponding part, say part A2 of construction2 has a value of 35, how many times bigger is that part and-or construction since all of its parts were also magnified by this same value so as to have a similar and larger construction?

$$\frac{\text{part of construction2}}{\text{corresponding part of construction1}} = \frac{A2}{A1} = \frac{35}{5} = 7 \quad : \text{"seven times" bigger. All the parts, and and the construction, are n=7 times bigger.}$$

This ratio value indicates that part A2 is seven times bigger than the similar (same) or corresponding part A1. Since the constructions are similar, each part has also been magnified or increased by the same (factor) value and the ratios of all corresponding parts will be this same value of 7. All the parts of construction2 can be said as being proportional or "of or with the same portion" to all the corresponding parts of construction1. Likewise, construction2 is therefore also said as being proportional to construction1. When constructions are proportional like this, even the ratio of any two parts of construction2 is equal to the ratio of the two corresponding parts of construction1, and these parts and-or their values are also said to be proportional or having the same portion (fraction of) or ratio value. In general, the ratio of any two parts within a single (unmagnified) construction does not necessarily have or need this same ratio value, however, this ratio of parts within one construction will be equal to the corresponding ratio of the two similar parts in a magnified or similar construction.

The reciprocal of (n) could be thought of as the "demagnification" factor. Given (an) and (n), to solve for (a), we divide (an) by (n), which is the same as multiplying by the reciprocal of (n), which is (1/n):

$$a(n) = (an) \qquad \text{dividing both sides by (n):}$$

$$\frac{a(n)}{n} = \frac{(an)}{n} = a \qquad \text{or by indicating the reciprocal of (n) more clearly:}$$

$$\frac{(1)}{(n)} a(n) = \frac{(1)}{(n)} (an) = a$$

Ex. For the last example, the magnification factor was 7. If we multiply A2 by the reciprocal (here, 1/7) of this magnification (specifically here, by a shrinking, or reducing) factor, we will find A1.

Since $\frac{A2}{A1} = 7$, $\quad A2 = 7\,A1$, $\quad A1 = A2/7 = A2(1/7) = (A2)(0.14286) = 35/7 = 5$

We can take the ratio of the values of these two parts (a and b) of one construction before and after a magnification. One value will become the numerator, and the other value will become the denominator. This will essentially create two fractions that are equal since both the numerator and denominator have been multiplied by the same value. Expressing this mathematically:

$$\frac{a}{b} = \frac{an}{bn} = r \qquad \text{or perhaps a this subscript notation that indicates which construction that instance of the part, side or variable (value of that side) is in reference to:}$$

$$\frac{a1}{b1} = \frac{n(a1)}{n(b1)} = \frac{a2}{b2} = r$$

(r) is the common or constant ratio value of all these and any other equivalent fractions, and similar (but magnified) constructions. Note, as shown above, (r) is not the magnification value that is to be (or was) applied or to all the parts of any one construction (so as to make a larger construction) since (n) is that value. (r) is sometimes called the "constant of proportionality" which indicates a constant (same value) numerical relationship among (with respect to) any two, or more, specific parts of any one construction, or of the same specific parts of a magnified (similar) construction.

First, since the percentage of tin is more than that of copper in the mixture, the weight of the tin in that mixture will be likewise be more than the weight of the copper in that mixture.
.
From the information given in the question, we can write some equations:

      85% of total weight = 5 lb   : can think of 85% from: part weight/total weight = 5lbs/total weight = 0.85,
      15% of total weight = x lbs   therefore mathematically: (0.85)(total weight lbs) = 5lbs

      (total weight) (0.85) = 5lb   : tin , 0.85 is the decimal equivalent of 85% , to be used for the calculations
      (total weight) (0.15) = xlb   : copper

From these, we algebraically have:

$$\text{total weight} \ = \ \frac{5lb}{0.85} \quad \text{and:}$$

$$\text{total weight} \ = \ \frac{xlb}{0.15}$$

Equating the two (equivalent) expressions for total weight, we have a common expression of an equation for a proportion type of problem. With practice, this could be written just from the information given in such problems:

$$\frac{5lb}{0.85} \ = \ \frac{xlb}{0.15} \qquad : \text{This can be read as: "5lbs is to 85%, as xlbs is (equivalent or equal) to 15%".}$$

Solving for xlb in the equation of (equivalent) proportions, or "proportion equation", we have:

$$xlb \ = \ \frac{(0.15)(5lb)}{0.85} \ = \ 0.882lb \qquad : \text{pounds of copper in the mixture}$$

The reciprocal of each side of the equation for the above proportion will also yield the correct result for xlb, if you happened to write the fractions and equation that way:

$$\frac{0.85}{5lb} \ = \ \frac{0.15}{xlb} \qquad : \text{This can be read as: "85% is to 5lbs, as 15% is to xlbs".}$$
$$xlb = 0.882lb$$

Though the total weight could have been found from the original equations written, we can also find the total weight from the left hand side of the proportion equation by performing the indicated division:

      total weight =   5lb/0.85 = 5.882lb   : since (amount or weight of copper) = (total amount or weight)(0.85)

However, finding the total weight this way (with the specific right side expression known to be equal to the total weight) from the proportion equation may not of been intuitive (natural to you) that this was the only equation needed to solve the problem.   As a check to this result, we can also create an expression for the total weight from:

      total weight = (weight of tin) + (weight of copper)
      total weight =   5lb   +   xlb   = (5lb + xlb) = (x + 5)lb

Substituting this value into the original expressions:

      (x + 5)lb (0.85) = 5lb   : tin

However, it is incorrect to say that 66.7% of the entire mixture is sugar since the 1 cup of another ingredient was not even included in the calculation. Since there are more ingredients, the percentage of sugar in the mixture will then be less than 66.7%. To find the actual percent that is only sugar in the entire mixture is to have the ratio of the amount of sugar with respect to the total amount used for all the ingredients in the mixture. The units of each value must all be the same and consistent. For this example, cups are the units of measurement:

$$\frac{\text{amt. sugar}}{\text{amt. of all ingredients}} = \frac{2 \text{ cups}}{2 \text{ cups} + 3 \text{ cups} + 1 \text{ cup}} = \frac{2 \text{ cups}}{6 \text{ cups}} = 0.333 = 33.3\% \text{ sugar}$$

$$\frac{\text{amt. flour}}{\text{amt. of all ingredients}} = \frac{3 \text{ cups}}{2 \text{ cups} + 3 \text{ cups} + 1 \text{ cup}} = \frac{3 \text{ cups}}{6 \text{ cups}} = 0.50 = 50\% \text{ flour}$$

$$\frac{\text{other ingredient}}{\text{amt. of all ingredients}} = \frac{1 \text{ cups}}{2 \text{ cups} + 3 \text{ cups} + 1 \text{ cups}} = \frac{1 \text{ cups}}{6 \text{ cups}} = 0.16\underline{7} = 16.7\% \text{ other ingredients}$$

Summed together, we see that the sugar and flour comprise 33.3% + 50% = 83.3% of the mixture.
The sum of all the ingredients or parts portions is: 0.333 + 0.50 + 0.167   = 1.0 = 100%   : the whole or entire mixture

If twice the amount of each ingredient is used to double (2) the amount or size of the mixture, there would be 2x2=4 cups of sugar, 3x2=6 cups of flour, and 1x2=2 cups of another ingredient. The percent of sugar, and flour in this mixture is then:

$$\frac{\text{amt. sugar}}{\text{amt. of all ingredients}} = \frac{4 \text{ cups}}{4 \text{ cups} + 6 \text{ cups} + 2 \text{ cups}} = \frac{4 \text{ cups}}{12 \text{ cups}} = 0.333 = 33.3\% \text{ sugar}$$

$$\frac{\text{amt. flour}}{\text{amt. of all ingredients}} = \frac{6 \text{ cups}}{4 \text{ cups} + 6 \text{ cups} + 2 \text{ cups}} = \frac{6 \text{ cups}}{12 \text{ cups}} = 0.50 = 50\% \text{ flour}$$

We see that the portion, or relative amount, of sugar in each mixture is always 33.3%, and the portion of flour in each mixture is always 50%. Each of the two mixtures is said to consist of the same fractional value or portion of a given ingredient. This is the basic essence of the concept of proportion - when the portions of something are always consistent regardless of the actual sizes or specific amounts (of units) being considered. Also, the ratio of the portions of sugar and flour in any one and all similar mixtures or "constructions" of the recipe (plan, steps, method, ingredients)) will always be the same value of:

$$\frac{\text{sugar}}{\text{flour}} = \frac{0.333}{0.50} = 0.667 \quad : \text{for any similar (magnified=increased or demagnified=decreased)}$$
batch or construction using that recipe

Because of the portions (ie., percentages) of sugar and flour are to be always the same value, the actual amounts of sugar and flour used for any similar mixture of this recipe are also said as being proportional to one another. Since the ratio of the portions of sugar to flour is always 0.667, a constant, all similar ratios of sugar to flour in various sized mixtures or instances of this recipe will be equivalent, and these equivalent ratios are called (equal, same) proportions.

Using the actual amounts of each ingredient in a mixture, the amount of sugar will always be 33.3% of the mixture, and the amount of flour will always be 50% of the mixture, and the ratio of the amount of sugar to the amount of flour will always be 0.667

Checking: $\frac{2 \text{ cups}}{3 \text{ cups}} = \frac{2(2)}{3(2)} = \frac{4}{6} = 0.66\underline{7}$:66.7% sugar

Considering just the mixture of the sugar and flour, what percent of this mixture is sugar?

$$\frac{\text{sugar}}{} = \frac{2}{} = \frac{2}{} \quad 0.40 = 40\% \text{ sugar}$$

construction 1   construction 2   construction 3

$$\frac{c}{a} = \frac{d}{b}$$: ratios of corresponding parts of two similar constructions

$$\frac{3}{2} = \frac{3}{2} = 1.5$$

.

$$\frac{e}{c} = \frac{f}{d}$$

$$\frac{5}{3} = \frac{5}{3} = 1.6\underline{7}$$

Likewise, for example:

$$\frac{e}{a} = \frac{f}{b}$$

$$\frac{5}{2} = \frac{5}{2} = 2.25$$

Clearly, the ratios are different (due to the magnification value (n) applied to the values of all the parts of one construction, so as to create a similar construction), but still, the ratio (r) among specific parts of any one (similar) construction will always be the same as that of any another similar construction. Here, for example, one specific ratio of sides happens to be:

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = 1.0$$: note for example that a and b are sides of the same construction, c and d are the corresponding (similar, or in reference to a and b) sides of another, such as the first construction.

It is important to note that we can mathematically derive ratios among corresponding sides of two similar constructions (here, in the example they happen to be triangles):

From: $$\frac{a}{b} = \frac{c}{d}$$ These are equivalent fractions and corresponding ratios of two sides of each construction. Here, each is an equal ratio value, this (before magnification is even considered) is the fundamental meaning of things or parts of a construction being proportional, or kept in proportion, among themselves. If a construction is magnified, all of its parts are automatically kept in proper proportion having the same fractional value of the whole. From this, we can mathematically derive other ratios between two constructions. From the above equation, if we multiply each side by (b), and divide each side by (c) we get:

       $$\frac{a}{c} = \frac{b}{d}$$: Ratios of corresponding sides of two similar constructions. Due to the various possible magnifications (n) possible for similar constructions, equations of their sides, as shown here, may not have the same values, as shown previously. The only thing that can be said here is that given any two specific similar constructions, the ratio of corresponding

or : $$\frac{c}{a} = \frac{d}{b}$$ sides will be the same. This value is also the "magnification factor" (n, m, or f) for only these two specific similar constructions.

       One method often taught for solving for a variable in an equivalent fraction or proportion type problem is to "cross multiply" the numerators to the denominators, effectively ridding the expressed fractions on both sides of the equation, but

physical portion or part of each new similar construction (here, apples or baskets) to have the same mathematical portion or fractional value with respect to the other parts of that construction. Mathematically, a proportion is essentially an equivalent fraction concept, and you can solve for the variable as if you were creating an equivalent fraction, otherwise, just solve it just like an equation.

First, in a true ratio form. the numerators and denominators are the quantity of similar things or quantities with the same or like units. The result (the ratio value) is a unitless or pure mathematical value:

new number of apples = a ratio = _ new number of baskets_   : this ratio here is actually the magnification (n)
   old number of apples   old number of baskets   value of similar (proportional) constructions

Mathematically, we can get another ratio from the above proportion:

 old number of apples = _new number of apples__ = this a different ratio (r) and value than (n) is
  old number of baskets   new number of baskets

Continuing the example:

Equation Method:

   450 apples   =   _x apples_   isolate x, here, we will multiply each side by (24 baskets):
   15 baskets   24 baskets

      +1.6   +1
   (24 baskets)(450 apples)   =   (x apples)(24 baskets)   : c. cancel
   (1   )(15 baskets)   (24 baskets)(1   )
               +1   +1

   (+1.6)(450 apples) = x apples   d. distribute:

   720 apples = x   or by switching sides:

    x = 720 apples   : 15 baskets is to 450 apples, as is, 24 baskets is to 720 apples , or
                                450 apples is to 15 baskets, as is, 720 apples is to 24 baskets

Equivalent Fraction Method:

   450 apples   =   _x apples_
   15 baskets   24 baskets

The denominator on the right side of the equation is larger than the denominator on the left side by a factor of:

  +1.6
   24 baskets   =   +1.6
   15 baskets
   +1

To make a proper equivalent fraction, the numerator must also be changed (here multiplied) by the same factor.

   (450 apples)(+1.6)   =   720 apples

450 apples_   =   (450 apples)(+1.6)   =   720 apples
 15 baskets   (15 baskets)(+1.6)   24 baskets

number of B parts

A = 0.333333 B   and:

B = ___A___ = 3A   : when given just two parts in a proportional system, the numerical coefficients
        0.333333   of those values are reciprocals of each other

Here is a note about proportions:

Ex. You have the equivalent fractions or proportion of:

  _X_ = _1_   , you can say that:
  10   2

"X is one-half of 10"   or =   "one-half of 10 is X"   : remember, "is" means "equals"

As a verification to these statements, consider this:

  _X_ = (1/2) = 0.5
  10

    Solving for X, we get: X = (0.5)(10) = (1/2)(10) = _10_   : "X is one-half, or 0. 5, of 10". Any value
                                                                    2   divided by 2 is half of that value.

Ex. You have the equivalent fractions or proportion:

  _5_ = _1_   : You can say that: "five is one-half of x"   or = "one-half of x is 5"
   x   2

Letting 1/2 = 0.5 be the quotient of 5/x,:

  _5_ = 0.5
   x

Multiplying the divisor and the quotient to yield that dividend of 5, we see that the word "of" in "one-half of x is (equal to) 5"
is mathematically expressed as a multiplication symbol, and the word "is" is expressed as the equals symbol:

    (0.5)(x) = 5   after dividing both sides by 0.5:
      x = 10

Ex. A cup of a liquid is defined as 8 fluid-ounces (oz.), which is actually a measure of volume. Weight measurements can
also have units of ounces and is a measure of force. This similarity is due to that 1 fluid ounce volume of water is defined
as having a weight 1 ounce. However, if the substance is not water, then 1 fluid ounce volume of that substance will no
longer weigh 1 ounce of weight. How many (fluid) ounces are in 1/6 of a cup?

First, a more general or fractional approach:

Writing what we know about the relationship between cups and fluid-ounces of measurement:

    1cup = 8oz.   : 1 cup equals 8 ounces   , this is an equation where each equivalent side
                                                          is expressed as a quantity of different units.
                                                          1 is not equal to 8, but 1cup is equal to 8oz

W in. $=$ $\dfrac{(1.125 \text{ in.}) (45.6 \text{ in.})}{0.25 \text{ in.}}$ $=$ $\dfrac{51.3 \text{ in.}^2}{0.25 \text{ in.}^1}$ $=$ 205.2 in.

W $=$ 205.2 in. $=$ $\dfrac{205.2 \text{ in. } (1 \text{ ft. })}{(12 \text{ in})}$ $=$ 17.1 (1 ft.) $=$ 17.1 ft.

: (1ft/12.in) is a "unity ("1") fraction"
Each inch is, (=), a fraction of a foot.
From 1ft = 12 in, equivalency or "unity":
Dividing both sides by 12 to solve for 1in:
1in = 1ft/12 = 0.083333ft, and to
have multiple inches, such as 205.2 here,
we multiply both sides of this by 205.2 to
have the equivalent number of feet.
From the expression above, we also get
these equivalent fractions:
(1ft/12in) = (0.08333ft/1in)
Here, the "magnification factor" is:
0.083333. Mathematically. from these
equivalent fractions. we can get:
(1ft/0.08333ft) = (12in/1in) = 12 which is
the reciprocal of the "magnification factor":
(1/0.083333) = 12

Checking: $\dfrac{1.125 \text{ in.}}{0.25 \text{ in.}}$ $=$ $\dfrac{205.2 \text{ in.}}{45.6 \text{ in.}}$ $=$ 4.5   : the ratio of the width to the height in each similar construction

The similar construction magnification ratio (n) or factor is:

$\dfrac{H}{h}$ $=$ $\dfrac{45.6}{0.25}$ $=$ $\dfrac{W}{w}$ $=$ $\dfrac{205.2}{1.125}$ $=$ 182.4   or   (1/182.4) = about 0.005482456 for demagnified constructions

Up to now, the examples of proportions have been of the direct proportion type. Another type, perhaps not as common, is the inverse proportion type. With inverse proportions, a quantity is mathematically inversely related to another quantity. One way to think of an inverse relationship is to think of a quantity where getting "more when there is less" , or when getting "less when there is more". Algebraically, we know that a quotient (Q) is directly related to a numerator (N) value , and inversely related to the denominator (D) value:

$$r = Q = \dfrac{N}{D} \qquad : r = \text{ratio}$$

If the numerator increases, the ratio increases. If the numerator decreases, the ratio decreases. This is a direct mathematical relationship. If the denominator increases, the ratio decreases. If the denominator decreases, the ratio increases. This is an inverse type of mathematical relationship.

If an equation expresses the relationship between only two variables, and they are known to be inversely proportional or related to each other, then the numerator variable is not needed to express their relationship and it is replaced by (1), or changed to (1) using division if necessary:

$B = \dfrac{1}{A}$: essentially then, each variable is the reciprocal or "inverse" of the other. These variables are said to be (mathematically) inversely related to each other.

$A = \dfrac{1}{}$ mathematically: 1 = AB   : the product of two corresponding reciprocals

4 people

1 job   =   1 job   : since 1 job = (people)(time), and using substitution of the problems values:

1 ((people)(hour)) = ((4 people)(x hours))   solving for x hours :

x hours = _1((people)(hour))_ = 0.25 hours , or = 15 minutes
                    4 people

Note that for both instances:

(people) (time) = 1   : = 100% , one job completed

(people A) (time A) = 1 = (people B) (time B) = 1 job completed   : here, A=of job A, B=of job B
(4 people)(0.25 hours) = (2 people)(0.5 hours) = (1 people)(1 hour) = 1 = 1 job finished or completed

We can now make ratios which will then express the problem as an inverse proportion or inverse equivalent fractions:

From: (people A) (time A) = 1 = (people B) (time B)   we mathematically have:

people A = _time B_   : A BASIC FORMAT EXAMPLE FOR AN
  people B   time A   INVERSE PROPORTION

Notice that the ratios are inverted, instead of having corresponding units in their numerators and denominators. In more general terms, if A is inversely related to B, as C is inversely related to D, the proportion is of this form:

From: A = _1_ , and C = _1_   mathematically:
                B   D

AB = 1 = CD
   AB = CD   mathematically we can then get this proportion (but it's an inverse one):

_A_ = _D_   Using the values from the example:
  C   B

people A = _time B_
  people B   time A

_1_ = _0.25_
  4   1

0.25 = 0.25   : since the ratios are equal, the values are correct

relationship among those variables.

Ex. The cost of a certain rectangular piece of material was $24 and this cost was directly related to its' size (LW = length x width), that is, the cost was directly related to the area of the material. The length of the material was 4ft. and the width was 2ft. Calculate the cost per square unit of area of this material. The basic equation or formula for this is:

$$\text{total-cost} = c\,(LW) \qquad\qquad \text{: where c is the price per square foot.}$$
$$\$24 = c\,(4ft)(2ft)$$
$$\$24 = c\ 8ft^2 \qquad\qquad\qquad \text{solving for c:}$$

$$c = \frac{\$24}{8\ ft.^2} = \frac{\$3}{1\ ft.^2} \qquad\qquad \text{: \$3 per square foot is the cost per square foot, and the basic formula becomes:}$$

total-cost = (cost per square unit)(total number of square units)

$$\text{total-cost} = (\$3/ft.^2)(LW) \qquad or = \quad (\$3/ft.^2)\ \text{Area}$$

If another section of that same kind of material was 5ft. long , and the width was 7ft., what would be the cost?

$$\text{total-cost} = \left(\frac{\$3}{1\ ft.^2}\right)(5ft.)(7ft.)$$

$$\text{total-cost} = (\$3)\left(\frac{35ft.^2}{1\ ft.^2}\right) = (\$3)(35)$$

$$\text{total-cost} = \$105$$

As a check, solving this problem using proportions:

$$\frac{\$24}{8ft.^2} = \frac{x}{35ft.^2} \qquad\qquad \text{solving for x:}$$

$$x = \frac{(\$24)(35ft^2)}{8ft^2} = (\$24)(4.375)$$

$$x = \$105$$

In terms of the proportions, notice that the constant of variation, ($3 per square foot or= $3/1ft^2) above, is the constant of proportion or proportionality for this specific system, and it is the cost per square foot of a certain material.

If a variable is directly related in value to another, when one increases, the other will increase. When one decreases, the other will decrease.

If a variable is directly proportional to another, the variables are directly related, and when one increases by a factor or multiple value, the other will increase by that same factor or multiple value.

Ex. a = b , If (b) increases by a factor of (n), (a) will increase by (n), and so as to keep the equation in balance:
    na = nb

Ex a = b , If (b) decreases by a factor of (n), (a) will increase by (n), and so as to keep the equation in balance:

$$\frac{a}{n} = \frac{b}{n}$$

$\dfrac{10}{5} = \dfrac{50}{25}$: here the ratio of the numerator to the denominator parts of each fraction is 2, and that the parts of the fraction on the right are 5 times bigger since (num2/num1) = (50/10) = 5, and (den2/den1) = (25/5) = 5, hence a magnified set, instance or version of the portions.

$\dfrac{10\,(5)}{5\,(5)} = \dfrac{50}{25} = 2$, also consider the reverse, where the multiplier is the reciprocal of 5: $\dfrac{50\,(1/5)}{25\,(1/5)} = \dfrac{10}{5} = 2$

(1/5) = 0.2 , hence a demagnified set, instance or version of the portions.

We can get (which can algebraically be found from the initial proportion shown) :

$\dfrac{5000 \text{ ft.}}{5280 \text{ ft.}}$ = $\dfrac{x \text{ mi.}}{1 \text{ mi.}}$ = x   : x is the number of units which are known to be miles.
We see that x is basically the ratio of the left hand side.

On the other hand, multiplication is the basic essence of converting from a quantity having larger (sized) units to a quantity having smaller (sized) units since there are many more of the smaller units per each larger units of measurement. For example:

Given: 1 mi. = 5280 ft.

To convert 5 miles to its equivalent value of feet, you can set up a proportion type of problem, or simply multiply each side of the equation by 5 since 5 miles is a factor of 5 times larger than 1 mile; (5 = 5mi./1mi.) :

(5)1 mi. = 5(5280) ft.
  5 mi. = 26400 ft.   putting this in a general formula, for a "miles to feet" conversion formula:

  x mi. = x (5280) ft.   : which is the same as multiplying both sides by (x), and the equation will still be in balance

Here's how to convert between miles and inches if you do not know the direct "conversion factors" between miles and inches, but do know the conversion factor between feet and inches:

From: 1 mi. = 5280 ft.   : eq. 1
And: 1 ft. = 12 in.   : eq. 2 ,   multiplying each side of this equation by 5280:

          5280 ft. = 63360 in.   substituting this fact into eq. 1:

           1 mi. = 63,360 in.   multiplying both sides by x:

         x (1 mi) = x 63,360 in

           x mi = x 63,360 in   : a general formula for converting miles to inches

Or, since as shown previously:

           x mi. = x (5280)ft.   : converting miles to feet

And likewise:

         x (1) ft. = x (12) in.   after mult. both sides by x, and substituting this into the above:

         x mi. = x (5280) ft. = (5280)(x ft.) = (5280)(x 12 in.) = x (5280)(12) in. =

         x mi. = x 63360 in.   : a formula to convert miles to inches

Perhaps a similar and simpler derivation is from: 1 mi = 5280ft, then x miles would be equivalent to:

         x mi. = x (5280)ft. = x 5280 (1 ft) = x 5280 (12 in) = x 63360 in.

Here, a proportion is used to solve for the number (N) of inches in a given amount (x) of miles.

         $\dfrac{1 \text{ mi.}}{}$ = $\dfrac{x \text{ mi.}}{}$ after solving for N inches:

$$\frac{1 \text{ gallon}}{231 \text{ in.}^3} = \frac{x \text{ gallons}}{1728 \text{ in.}^3}$$

Therefore, 1 ft.^3 of water = 1728 in.^3 = 7.4805 gallons of water , and:

One ton (2000lbs) of water is equal to : 2000lb / 8.3456 lb / gallon = 239.6<u>5</u> gallons =~ 240 gallons
One cubic foot of water, or 7.4805 gallons of water weighs: (7.4805gal)(8.3456 lbs/gal) = 62.43lbs
One ton (2000lbs) of water is equal to : 2000lb / 62.43 lbs per cubic ft. = 32.03<u>6</u> =~ 32 cubic feet of water
     Taking the cube root of a cube with 32.036 cubic units, here: ft.^3, is 3.17<u>6</u> ft per side: (l, w, or h).
     This is just slightly more than a yard (3ft) per side, and also just slightly less than 1 meter (3.28 ft.) per side.

Ex.   A basement is to be resurfaced with concrete. The dimensions of the basement are as follows: Length=30 ft., Width=12 ft., and the thickness or Height of the concrete is to be 1 inch deep. How many cubic yards of concrete will be needed?

First, we will convert all the measurements to their equivalent measurements with units of yards.

Since   3 ft. = 1 yd.   we can write a proportion problem as:

$$\frac{1 \text{ yd.}}{3 \text{ ft.}} = \frac{x \text{ yd.}}{n \text{ ft.}}$$ solving for x yd:

$$\frac{(n \text{ ft.})(1 \text{ yd})}{3 \text{ ft}} = x \text{ yd.}$$: Hence to find x yd. of a given length of feet, simply divide the number (n) of feet by 3.   .

Here are the converted values: L = 30 ft. = 10 yd.
                          W = 12 ft. = 4 yd.

For the Height, we will first convert its value with units of inches to an equivalent value with units of feet, and then proceed as usual:

$$\frac{1 \text{ ft.}}{12 \text{ in.}} = \frac{x \text{ ft.}}{1 \text{ in.}}$$

x ft. = $\frac{(1 \text{ in. })(1 \text{ ft.})}{12 \text{ in.}}$ = (1/12) ft. = 0.083333333 ft.   : 1in =~ 0.0833ft

Dividing by 3, or multiplying by (1yd/3ft), to convert feet to yards, we have:

H = 0.083333333 ft. = 0.027777777 yd.

Since   (L yd.)(W yd.)(H yd.) = x yd.^3   , we have:

(10 yd.)(4 yd.)(0.027777777 yd) = 1.111111111 yd.^3   : the minimum needed
                                            Slightly more than 1.11 cubic yards.

Below is a small table that can be used for making typical conversions between the metric and American systems of units of measurement for length or distance:

Metric   English-American

 1cm = 0.393700787 in.   : cm=centimeter, in.=inch. 1cm = (1/100)m = 0.01m , Note that 1 in = 2.54 cm, exact

Dividing the equivalent meter value for centimeters by the equivalent meter value for millimeters, we can find out how many times larger that a centimeter is than a millimeter:

$$\frac{1\ cm}{1\ mm} = \frac{0.01m}{0.001m} = 10$$

Therefore, a centimeter is 10 times larger than a millimeter. Considering the division this reciprocal way:

$$\frac{1\ mm}{1\ cm} = \frac{0.001m}{0.01m} = 0.1$$

We find that a millimeter is only a tenth of a centimeter.   This can be mathematically verified from the above equation by multiplying each side by 1 cm:

$$1\ mm = 0.1\ (1\ cm)$$
$$1\ mm = 0.1\ cm = \frac{1}{10}\ cm$$

It can also be said that there are 10 millimeters in a (1) centimeter. This can be mathematically verified from the above equation by multiplying each side by 10:

$$10\ mm = 1\ cm \qquad or = \qquad 1\ cm = 10\ mm$$

This could have also been found from the original division shown above after multiplying each side by 1mm.

Before leaving this topic of changing units, consider these example equations below.

1ft. = 12in.   Having some multiple of feet to convert to inches, its a matter of repeated addition or simply a multiplication. Multiplying each side by (x):

$$x\ (1ft) = x\ (12in)$$

$$xft = x12\ in \qquad : \text{to convert a number of feet to its corresponding number of inches}$$

Given 1ft = 12in , and dividing both sides by 12:

$$1in = \frac{1ft.}{12} = 0.0833333\ ft \qquad \text{when having some multiple of inches to convert to feet:}$$

$$xin = \frac{x\ (1ft\ )}{(\ 12)} = \frac{xft}{12} \quad or = \ 0.0833333x\ ft$$

You can also use equivalent fractions or "proportions" to solve conversion problems.

AN OBSERVATION ON RECIPROCALS

When two values are reciprocals of each other, say (a) and (b), we know that their product equals 1.

   (a)(b) = 1   and   a=1/b   and   b = 1/a

But what do we get when we divide these reciprocals by each other?

   $\dfrac{a}{b}$ = c

   $\dfrac{b}{a}$ = d

It may or may not be obvious or intuitive, but the quotients of dividing two reciprocals, here (c) and (d), are also reciprocals. For example:

   Given: a=0.5 and b=2:

   (0.5)(2) = 1   and   0.5 = 1/2   and 2 = 1/0.5   : Verifying that 0.5 and 2 are indeed reciprocals
                                                                 since their product equals 1.

   $\dfrac{a}{b}$ = $\dfrac{0.5}{2}$ = 0.25 = c

   $\dfrac{b}{a}$ = $\dfrac{2}{0.5}$ = 4 = d

Observe that 0.25 and 4 are also reciprocals:

   (c)(d) = (0.25)(4) = 1,   and:

   1/c = 1/0.25 = 4 = d,   and:

   1/d = 1/4 = 0.25 = c

So in a way, we are right back where we started with two reciprocals, and the process can be continued by dividing these new reciprocal values, and here they are 4 and 0.25.

Some other observations:

From: ab = 1   ,   a = 1/b ,   b = 1/a   : (a) and (b) are the reciprocals of each other

And: $\dfrac{a}{b}$ = c   : c = the quotient of the reciprocals (a) and (b),   mathematically:
                        It will be shown below that the square root of this quotient (here c) of two reciprocals is equal
                        to the numerator, here a solving for c:
       a = bc = $\dfrac{1}{b}$
                        solving for c:

       c = $\dfrac{1}{b^2}$ after taking the square root of both sides of this equation:

       $\sqrt{c}$ = $\dfrac{1}{b}$ = a   , also, squaring both sides we have: c^2 = a

From: ab = 1

$$\frac{a}{b} = \frac{\frac{a}{1}}{\frac{1}{a}} = a^2 = c$$: a reciprocal divided by its reciprocal is equal to: numerator^2
taking the square root of both sides we have:

$$\sqrt{c} = a$$

An interesting pair of reciprocals is the square root of 10, and the reciprocal of this square root of 10. Both values have the same digits, but they are different by a factor of 10.

$$\frac{a}{b} = \frac{\sqrt{10}}{1/\sqrt{10}} = \frac{3.16227766}{1/3.16227766} = \frac{3.16227766}{0.316227766} = c = 10$$ checking:

$$\frac{a}{b} = \frac{\sqrt{10}}{1/\sqrt{10}} = \frac{10^{0.5}}{1/10^{0.5}} = (10^{0.5})(10^{0.5}) = 10^{(0.5+0.5)} = 10^{1.0} = 10 = c$$

Clearly, when two values differ by a factor of 10, as just shown, their digits will be identical.

$$a = \sqrt{c} = \sqrt{10} = 3.16227766$$

You can also use other values besides 10 for the radicand (N), and the two values ($\sqrt{N}$, and $1/\sqrt{N} = \sqrt{1/N}$) will differ by a factor of that specific radicand., but their digits will not be (visually) identical as when the radicand is 10. Also expressed is that the reciprocal of a root of a number is equal to the root of the reciprocal of that number:

$$\sqrt{1/N} = \sqrt{1} / \sqrt{N} = 1/\sqrt{N}$$: The root of the reciprocal of a number is equal to the reciprocal of the root of that number.

Note that::

$$\sqrt{1/10} = \sqrt{0.1} = \sqrt{1/10} = \sqrt{1}/\sqrt{10} = 1/\sqrt{10} = 1/3.16227766 = 0.316227766$$ and

$$\frac{\sqrt{10}}{\sqrt{0.1}} = \frac{3.16227766}{0.316227766} = 10$$ checking by multiplying the divisor and quotient together:

$$10\sqrt{0.1} = \sqrt{100}\sqrt{0.1} = \sqrt{(100)(0.1)} = \sqrt{10}$$

As an extra note or concept involving reciprocals, of which much has already been discussed:

If you have say 10 items, each (1) item can be said or indicated as: 1 in 10, 1 of 10, 1 out of 10 , and these can be numerically expressed as:

$$\frac{1}{10}$$ in standard fractional form as an expression or= 0.1 in standard decimal fractional form as a single numeric value

Hence given any number of items, say N, the numeric value of each item is simply the reciprocal of that number N = (1/N)
Ex. If given 5 items total, each item can be numerically expressed as: (item) / (total items) = 1 / 5 = "1 of 5" = 0.2

QUADRATIC EQUATIONS

   In this book, we have already discussed and solved equations of the first degree or order. That is, the exponent of the variable being solved for has only been the first (1) power or degree. Below is a discussion on solving equations of the second order or degree, more formally known as quadratic equations. Here, the exponent of the variable being solved for is 2. The word quadratic comes from the word prefix of quad which means 4. Since these equations have a "squared variable" often associated with square shapes with 4 equal sides; (s) and area is (s^2), it makes some sense to call them quadratic. Another reason that they are called quadratic equations is in reference to their graph. When the graph of the quadratic equation is plotted on a perpendicular or rectangular-coordinate system of measurement and location on a plane, the curve will eventually occupy two or more quadrants (a 1/4, or a quarter section) of the rectangular-coordinate system. When plotted, the geometric shape of a quadratic equation or curve is called a **parabola**. "Parabolic" is a word that basically means "two-sided", and this curve is two-sided as being symmetrical about the axis of the parabola curve. A common use for the parabola shape is with the shape of the large main (light collector) mirror of a reflector telescope where all the dim or faint incoming light is reflected to be gathered together and concentrated into a much smaller area or point of brighter or more intense light which can then be magnified and viewed by the eyepiece lens. This point is formally called the focus, or focus point of the curve or system, and which basically means the central locus or location of the parabola curve.

The general format of the basic linear (line) expression and equation is:

  ax^1 + b = ax + b   : (a) is a multiplying factor, or numerical coefficient to the variable (x).
                  (a) cannot equal 0, since the variable would essentially be eliminated
                  since anything times 0 is 0. (a) can be 1, and the expression would
                  simply be: (1)x^1+ b   =   x + b
                  (b) is some arbitrary constant added in and it could be 0, and the
                  expression would simply be: ax + 0 = ax
                  If (a)=1, and (b)=0, the expression would simply be: x

If the value of this expression was set equal to y, such as for graphing or plotting the relationship of, or between, y and x, and where y can be thought of as the output of the expression, and x as the input of the expression. The mathematical relationship between y and x is a linear (ie. proportional) type of relationship.

If the basic linear expression is multiplied by a constant (c), the result is another linear expression "in (variable) x":

  c(ax^1 + b) = acx^1 + bc

That is, the expression still consists of a term which has (x) to the first power, and which is multiplied by a numerical coefficient, and has a (constant) value term added to the first or "variable term".

However, if it is multiplied by the variable (x) that is also within and defines that specific linear expression, the result is a quadratic expression:

  x(ax^1 + b) = ax^2 + bx

As an extra note to consider: This is equivalent to repeatedly adding the expression or its resulting value a total of (x) times. For example:

  x(ax^1+ b) = (ax^1 + b) + (ax^1 +b) + **. . .** + (ax^1 +b)   : here, the number of terms depends on the value of x
                           If x=3:

  3(a3^1+ b) = (a3^1 + b) + (a3^1 +b) + (a3^1 + b) = a3^2 + 3b = 9a + 3b

We can add in an arbitrary constant (c) term, and set the expression equal to another variable such as (y) to hold the

as the (x) variable increases, hence the "steepness of the curve" or slope at that point on the curve is said to be 0. Setting (y) of the derivative equation (which is basically an equation for the slope of the curve at a given point or value of x), or "slope [rate of change of the variables] equation" equal to 0, you can then easily solve for its corresponding value of (x), and then use this value of (x) in the actual quadratic equation to find its corresponding value of (y). For example, if the (x) variable was actually time (t), and the (y) variable was actually temperature (T), you can mathematically find or predict the exact point in time where the temperature was either at minimum or maximum value, and the corresponding value of that temperature.

The two axis of a rectangular measuring and location system will divide that plane into 4 areas called quadrants. (Quadrants (Q) are numbered counter-clockwise from the upper right quadrant which is considered as the first quadrant (Q1). The other quadrants are identified as Q2, Q3 and Q4.

Many other things can be observed from the curve of an equation. For example, in the above drawing, the co-ordinates of the minimum point, or any point, within the fourth quadrant will have the form of:

> Numerical signs of the co-ordinates of a point in the 4th quadrant are:
> p(x is positive in value, and y is negative in value)

Another observation is that the two roots (or "zeros") or x values where the curve crosses the x-axis and where y=0 will both be positive in sign since they are on the positive side of the x axis.

The word root (or "root of an equation") is often used to describe the solution of a quadratic equation expressed in the general format where y is set equal to 0. The description is used since the solution of any squared value, such as the variable (here x^2), is to take the square-root of it so as to have the "plain", "unsquared" and root value.

Another thing that can be observed is that the curve leftward of the minimum point slopes downward (as x increases). Here, the overall mathematical relationship between (y) and (x) is an inverse relationship. As (x) increases, the corresponding (y) co-ordinate of each point on the curve decreases. The curve to the right of the minimum point slopes upward. Here, the overall mathematical relationship between (y) and (x) is now a direct type of relationship. As (x) increases, the corresponding (y) co-ordinates of each point on the curve is also increasing. Due to this behavior of the slope of the curve not being constant, the mathematical relationship between variables (y) and (x) is not constant or linear in value, and is said to be a non-linear relationship.

The graph also indicates the value of (y) when the corresponding value of (x) equals zero. This point, p(0, y), on the curve is called the y-axis intercept.

When b=0 in a quadratic type of equation, the x^1 term is eliminated, and the resulting equation is reduced to a partial quadratic equation which can be easily solved:

$ax^2 + c = 0$ transposing c :

$ax^2 = -c$ dividing both sides by a:

$x^2 = \dfrac{-c}{a}$ taking the square root of both sides:

$x = +- \sqrt{\dfrac{-c}{a}}$: note that since (c) and (a) are constants, the x-axis intercepts, p(x,0) where y=0, the value of x will be a constant

And for a real root (ie. not having an unknown or imaginary value), the value of the radicand must be positive in sign since the square root of negative numbers are not defined as being a real number and are considered as imaginary

is not close to 0), the adjustment process will need to be used.

This equation can also be expressed as:

new guess = guess + adjustment   or:
closer guess = guess + adjustment

new guess will then be substituted into the guess position,
and this "refining" or "zeroing in (getting closer, a low difference)"
process can be continued.

After each step you are to take the value of the left side of the equation and substitute it into the right side of the equation and so as to automatically make adjustments and create a new closer value. When the new updated or modified guess or the numeric result of the equation is reasonably close to the true number that is being guessed or solved for, the process can stop. As you get close to the number, the adjustments should be "smaller and smaller" in absolute value. The concepts of "higher or lower" and "smaller and smaller" adjustments are automatically built into the mathematical method.

Continuing the example:

By replacing (x) in the denominator with its corresponding value (equation), we will have an equation where the computed value of (x) is nearer to the actual or true value of (x). Here, this method will create a "continued fraction":

$$x = -2 + \cfrac{15}{-2 + \cfrac{15}{x}}$$

: and so on. You will still have to guess at x at some point. It could be 0 if you want, or some value that you know is close to the true value of x. As you can imagine, if the initial guess or value chosen for x as a possible solution is close to the actual solution, the less computation needed to find the true result.

If you want to know some more about continued fractions, see Methods For Solving Square Roots in the appendix.

Let's call or assign the (x) in the denominator x_n and the new, and more accurate solved for, or computed value of (x), as x_{n+1}. In general, with the example given, if you already know or previously calculated a value for (x) such as from the continued fraction method just shown, then x_{n+1} can be found (without solving a lengthy continued fraction) by substituting the value of (x) for X_n and using the original equation again:

$$x_{(n+1)} = -2 + \frac{15}{x_n}$$
: (where x_n = x)   , or as noted above, $x_{n+1} = \frac{15}{(2 + x_n)}$

: here, n and n+1 are not variables but are subscripts of the variable x that indicate a specific instance and-or value of that variable x. Hence, you cannot combine these variables: x_n and $x_{(n+1)}$ cannot be combined.

Note, to use this second formula noted, you must have calculated all previous values of (x) from this formula only. Likewise, to use the first formula noted, you must have calculated all previous values of (x) from that formula only.

Since an initial value of (x) might not be given or known, a guess close to -5.0 would be great for this example, but in general, if you don't like to guess, being in the "true" or exact realm of algebra, simply set (x) equal to any value up to infinity, of course using a value near infinity will usually take you longer to get sufficient accuracy in the result. A computer can do this type of mathematical repetition for you in usually less than a second of time. Note that a computer will usually signal an error and may stop processing the program if it (ie. the electronics) tries to divide by 0, hence, you may have to use an initial value of something other than 0 if you use a computer and-or a specific program being used by the computer. When doing the computation by hand, setting x_n equal to infinity in the example shown will set the value of that

When: $(x2 - x1) = 0$, the difference between the new and old values is 0, or is as small as needed, and the result has been found.

Factoring is another method to solve for the variable in a quadratic equation, however, its only sometimes feasible (especially when the coefficient of the x^2 term is not equal to 1) and therefore, it will not be discussed in great depth. It goes something like this: You can factor an (n) order equation (n=2 for a quadratic equation, a "second order equation") into a product of (n) linear equations (where the highest power of x=1).

First, here is the extended product of two binomials "in x":

$(x + a)(x + b)$ = $x^2 + bx + ax + ab$ = $x^2 + (a + b)x + ab$: clearly, this side is a quadratic expression

If you are given the quadratic expression or trinomial "in (powers of) x" (or whatever variable you happen to be working with) such as on the right hand side above, we now know that it can be factored to a product of two (linear, to the first power) binomials (sum of two terms) "in x" as shown on the left hand side of the equation. The first term of each binomial factor will be the square root of x^2, which is (x). Notice that when the factors (here, a and b) of the third term are summed, that this sum equals the coefficient of (x) in the second term. If the product (ab) is positive in sign, the second terms in the two (linear, first order) binomial factors are either both positive or both negative in sign (remember, a negative times a negative is a positive). When (ab) is positive and the sum of (a) and (b) is negative, that is, if the coefficient of (x) in the second term is negative, then both of the second terms in the binomial factors are negative in sign:

$x^2 - (a + b)x + ab$ = $(x - a)(x - b)$

Ex. $x^2 -9x + 14$ = $(x -7)(x -2)$: ab = (-7)(-2) = + 14   and   (a+b) = (-7) + (-2) = -7-2 = -9
Note, you will need to figure out which factors of +14 will actually sum to -9; algebraically, which factors of (ab) will sum to (a+b). Here is a pseudo algebraic form of this factorization method:

$x^2 + $ (sum of factors of third term)$x + $ (product of factors of the third term) = $(x+a)(x-b)$

, and where (sum) is the sum of the coefficients of (x^1), and (product) is the product of the coefficients of (x^1).

When (ab) is negative, and the sum of (a) and (b) is positive, the highest factor of (ab) is positive and the lower factor is negative. In general, the second term in only one of the two binomial factors will be negative in sign:

$x^2 + (a+b)x - ab$: here, expressing that when the product of (a) and (b) is negative, hence the term is negative. That is, think of (a) and (b) here as numeric values rather than variables which can generally have any (positive or negative) sign. This is the best we can do to mathematically to represent or express values that are signed constants, for this analysis.

$(x + a)(x - b)$: when (a) is positive and greater than (b) which is negative

Ex. $x^2 +5x -14$ = $(x +7)(x -2)$: (+7)(-2) = -14   and   (+7) + (-2) = 7-2 = +5

$(x - a)(x + b)$: when (b) is positive and greater than (a) which is negative

When the sum of (a) and (b) is negative, the highest factor of (ab) is negative and the lower factor is positive. In general,

$x + 3 = \dfrac{0}{x+2} = 0$

$x = -3$: root , on the graph, this corresponds to point(-3 , 0)

Doing the same for the other binomial factor, we find:

$x = -2$: root , on the graph, this corresponds to point(-2 , 0).

In short, since the product shown above is 0, the method essentially assumes one or both factors are equal to 0.

Checking: $x^2 + 5x + 6 = (-3)^2 + 5(-3) + 6 = 9 - 15 + 6 = 0$ and:
$x^2 + 5x + 6 = (-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$

A quadratic equation or curve will only cross or intercept the y-axis once. Setting x=0, and solving for the corresponding value of y in either equation shown above, we find the y-axis intercept to be point(0 , 6).

There is a direct and preferred method to solve for the specific value(s) of the variable in a quadratic equation, and it is the same method so as to produce the two roots. This method uses the quadratic formula:

$x = \dfrac{-b \ +\!- \ \sqrt{b^2 - 4ac}}{2a}$: **QUADRATIC FORMULA** , given: $ax^2 + bx^1 + c = 0$,
$a \ \neq \ 0$, because the x^2 term must be present to be "quadratic" equation.
(b) and (c) are usually constants for a given "quadratic equation in (x)".

A derivation of this is as follows:

$ax^2 + bx + c = 0$: basic quadratic expression, transposing c:

$ax^2 + bx = -c$ dividing both sides by a:

$\dfrac{ax^2 + bx}{a} = \dfrac{-c}{a}$ the left side can be expressed as a sum of fractions:

$\dfrac{ax^2}{a} + \dfrac{bx}{a} = \dfrac{-c}{a}$ after canceling out the common factor of (a) in the first term:

$x^2 + \dfrac{bx}{a} = -\dfrac{c}{a}$

(*) The perfect square of a binomial is expressible in the general algebraic form of:

$(x+b)^2 = (x+b)(x+b) = x^2 + xb + xb + b^2 = x^2 + 2bx + b^2$, or=: $(1)x^2 + (2b)x^1 + (b^2)x^0$

The last expression, a trinomial, is equivalent to a perfect square of a binomial value, and specifically for this example, it is (x+b). Likewise, the square root of a trinomial that happens to be a perfect square of a binomial, is equal to that binomial:

Since: $x^2 + 2bx + b^2 = (x+b)^2$ taking the square root of both sides:

$\sqrt{x^2 + 2bx + b^2} = (x+b)$: the square root of a trinomial that is a perfect square

Now given only $x^2 + 2bx = 0$, what is a formula for the third term that can be added in (to both sides of course) so

Also: (x+3) = 1   , (-2) +3 = -2 + 3 = 1   : checks

For completeness, when the second term of a squared binomial is negative, the expansion becomes:

(x -b)^2 = (x -b)(x -b) = x^2 -1xb -1xb +b^2 = x^2 - 2bx + b^2

, so given any trinomial, if the second terms value (algebraically and formally expressed as: +2bx or -2bx) is twice the square roots of the first and third terms, it is a perfect square of a binomial. The sign of the second term of this binomial will correspond to the sign of the second term of the trinomial.

In the above discussions, it is assumed that the coefficient of x^2 (usually indicated as (a) in ax^2) is 1. If it is not, then "divide through" each side by it, which is essentially dividing all terms on both sides of the equation by this coefficient (usually noted as (a)) if possible. At the above location in this book indicated as (*) in this derivation of the quadratic formula, the left side has an x^2 term and an x^1 term (which we know are the necessary terms of a formal quadratic trinomial or equation), hence we can solve for a value that when added in (to both sides of course) to that side, will make that side an expression equivalent to a perfect square of a binomial. The net result after some simplification is that we will have only one (solvable) unknown which is the variable we are solving for and its' equivalent value on the other side of the equation.

Even after dividing (b) by (a), in the trinomial, (b) which is the coefficient of the x^1 term is still twice (2) of what it needs to be for the binomial to be squared that represents the given trinomial.   Dividing the co-efficient of this x^1 term by 2 so that we can find the value of (b) for the second term of the binomial squared that will be equal to the given trinomial. Algebraically expressing this process:

$$\cfrac{\dfrac{b}{a}}{\dfrac{2}{1}} \quad = \quad \dfrac{b}{2a}$$ squaring this, so as to help during some following rationalization of roots, we get:

$$\left(\dfrac{b}{2a}\right)\left(\dfrac{b}{2a}\right) = \dfrac{b^2}{4a^2}$$ adding this to both sides of the equation, so as to keep the balance on both sides:

$$\dfrac{x^2}{1} + \dfrac{bx^1}{a} + \dfrac{b^2}{4a^2} = \dfrac{b^2}{4a^2} - \dfrac{c}{a}$$

Now, the left side is equivalent to a perfect square of a certain binomial that represents or is equivalent to a given trinomial ("in x"). By taking the square roots of the first and third terms, (which are the (perfect) "squared" terms) we will have the terms of that binomial.

$$\sqrt{x^2} = x \qquad \text{and:}$$

$$\sqrt{\dfrac{b^2}{4a^2}} = \dfrac{\sqrt{b^2}}{\sqrt{4a^2}} = \dfrac{\sqrt{b^2}}{\sqrt{4}\,\sqrt{a^2}} = \dfrac{b}{2a}$$: or= 0.5 (b/a).   ,   hence:

$$\left(\dfrac{x}{1} + \dfrac{b}{2a}\right)^2 = + \dfrac{b^2}{4a^2} - \dfrac{c}{a}$$: the left side is a binomial squared that is equivalent to the trinomial ("in x") of ax^2 + bx + c less the constant term] .   Algebraically, the right side is also equal to this trinomial.

Adding the fractions on the right side, (LCD = 4a^2):

(x+2)(x) = 15   placing this into the std. quadratic eq. form, here by using distribution:
x^2 + 2x = 15
x^2 + 2x - 15 = 0   in relation to the standard quadratic equation form: a=1, b=2, c = -15

Substituting these values into the quadratic formula:

$$x = \frac{-b \ +- \ \sqrt{b^2 \ - \ 4ac}}{2a}$$

$$x = \frac{-(+2) \ +- \ \sqrt{(+2)^2 \ - \ 4(1)(-15)}}{2(1)}$$

$$x = \frac{-2 \ +- \ \sqrt{64}}{2} = \frac{-2 \ +- \ 8}{2} \qquad hence:$$

$$x = \frac{-2 \ + \ 8}{2} \quad or \quad x = \frac{-2 \ - \ 8}{2}$$

$$x = \frac{+6}{2} \quad or \quad x = \frac{-10}{2}$$

x = +3   or   x = -5

As is the case for the solution of many problems with quadratic equations, the useful solution is usually the positive root only. For the last example, there is simply no physical length that can ever be negative in value, and for here, (-5) satisfies the quadratic equation mathematically, but it does not satisfy the given real or physical situation.

Checking:

x^2 + 2x = 15
(3)^2 + 2(3) = 15
9 + 6 = 15
15 = 15 checks

width = x = 3
length = (width + 2) = (x + 2) = 3 + 2 = 5

Area = (length)(width) = (5)(3) = 15   : checks

Ex.   This example will discuss and derive the constant called the **Golden Ratio** = (GR).   This value is about 0.618033988749894... and is used in many types of constructions (both man-made and natural) due to the supposed beauty it brings about.   The Golden Ratio constant is an irrational value (like "pi" = 3.14159265... is) since it cannot be expressed as a ratio (ie. as a fraction) of two values since it does not in-fact have an exact value that could be easily representable mathematically since its digits never end, hence an infinite amount of digits. The derivation of the value of the GR is based on the magical-like concept when the ratio of N1 to N2, is equal to the ratio of N2 to the sum of N1 and N2. Graphically representing these values as line segment lengths:

‾‾‾‾‾   ‾‾‾‾‾‾‾
   N1   N2

‾‾‾‾‾‾|‾‾‾‾‾‾‾‾
  (N1 +   N2)   : Let N3 = (N1 + N2)
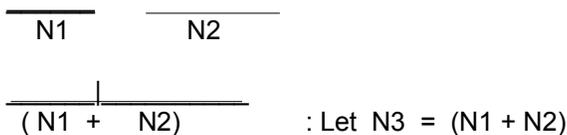
GR^2 + GR -1 = 0   : here, a=1 , b=1, c=(-1)   , after placing these into the quadratic formula and simplifying:

$$GR = \frac{-1 +- \sqrt{5}}{2}$$

As an extra note: The sum of all the (integer powers of GR) = (GR)^1 + (GR)^2 + (GR)^3 + **...** = 1/GR = 1.618...

N2 / N1 = 1.618 and N3 / N2 = 1.618   or the reciprocal ratios: N1 / N2 = 0.618 and N2 / N3 = 0.618

If we let: a=N1 , b=N2, and c=the total length of (N1 + N2) , we obtain the proportion or equivalent fractions of:

$\frac{a}{b} = \frac{b}{c}$ from this we mathematically obtain:

b^2 = ac   taking the square root of both sides:

$b = \sqrt{ac}$: hence (b), (ie. N2), is the "**geometric mean**" (= square root
of the products) of (c) = (N1 + N2) = N3, and (a) = (N1).
Extra note ex.: the geometric mean of 10 and 20 is $\sqrt{(10)(20)} = \sqrt{200} = 14.142...$
$= \sqrt{10}\sqrt{20}$ or= $\sqrt{100}\sqrt{2} = 10\sqrt{2}$
= 14.142

The geometric mean of two vales tends to be closer in value to the least value, and this is
due to the nature of squared values that increase rapidly in value.

In a simple way, you can think of the geometric mean of (10) and (20) as this example:
If you had squares with areas of: 10 square units and 20 square units , their sides are
about 3.162 and 4.47 units long respectively. The average of these two sides is about
3.817 which corresponds to the side of a square having an area of about 14.57

Another way to think of the geometric mean is this: Given an area with sides (a) and (c),
and multiplying (a) and (c) together to find that area, we have: (a)(c), and to find the
side length of an equivalent square shaped area which will have the same value for all
its sides, we take the square root of this area value: b = $\sqrt{(a)(c)}$ b is therefor the side
of a geometric equivalent area of another area having sides of (a) and (c).

A geometric mean could be considered like a geometric average, however, this average
is of another form and calculated a different way because of the way the squared values
grow or increase from one value to the next, and it is not an "arithmetic" or "linear"
average that is located exactly between the two given values,   For example, if we have
the integer or "arithmetic" values of 2, 3 and 4, the average of 2 and 4 is 3. Now given
the square of those values: 4, 9, and 16, the average is not 9. The linear average of 4
and 16 is actually: (4 + 16)/2 = 20/2 = 5. The geometric average is:
$\sqrt{(4)(16)} = \sqrt{64} = 8.$ This geometric series would be: 4, 8, 16, and we see that the
common ratio= r=2. If we take the square root of 8, the corresponding series of these
roots would be: 2, 2.828427... (not 3), and 4.

$$N2 = \sqrt{(N1)(N1 + N2)} = \sqrt{(N1)(N3)}$$ and also:

$$N2 = \sqrt{N1^2 + N1N2}$$

Expressing this in successive approximation form:

$$x2 \;=\; 3\sqrt{12 - x1\text{\textasciicircum}2} \qquad \text{:where x2 is a better approximation than x.}$$

Continue till the difference between x2 and x1 is as small as desired. For this example, the true value of (x) is 2. Another recursion formula that can be derived from the original example is:

$$x2 \;=\; 2\sqrt{12 - x1\text{\textasciicircum}3} \qquad \text{: Note that the value of the second term cannot exceed 12 (ie. > 12) because}$$

it would cause the creation of a radicand that is negative in value and we can't solve for the square root of a negative value.

For equations other than linear (where the variable is only to the first power), it is possible, as shown previously, that there is more than one solution to an equation. Though not discussed in this book, there are some formally defined ways to determine the number of roots of an equation. By graphing an equation that is placed in "root form" (ie. algebraically set equal to 0) you can determine the number of roots of an equation, and you can also find the approximate values of those roots. You can then use each of these approximate values in a successive approximation equation. Using the last example, the "root form" of the equation for plotting (and for finding the roots or solutions) is:

x^3 + x^2 - 12 = 0   : the result of the expression is set to 0, hence y=0, and we are then solving for a
                                   specific value(s) of x such that its corresponding value of y is equal to 0.

By also setting (y) equal to this expression, we create an equation and numerical relationship between two (or more) variables. This relationship can now be graphed as:

y = x^3 + x^2 - 12

The examples above were mainly quadratic or second degree equations since they are encountered more often in mathematics than equations of any higher degree. Another use for being familiar with different types of equations and their graphs is in design. The graphs or curves of many equations, or some combinations of equations, have a beauty in their own right. A part of the equation's curve can be used as the model for some form of construction, and can always be described, or duplicated to any size if given the (same) source equation and "boundary points" (ie. the two "end" or "extreme points" bounding or defining what specific part) of the curve is in question. By using the concepts of calculus, the (bounded) curve's length, and other useful facts (such as the area beneath a curve, or area between two curves) of that portion of the curve can be calculated.

Given two equations, each with their own graphing or curve shape, they can be expressed as being mathematically summed together and the result is a point per point, or instance per instance, amplitude (ie., the y values) summation of each, and it will produce a third or resulting curve or waveform.   As a very simple example to this consider two waves colliding, combining or summing such that the peak (say +1 in value) of one wave is combined with the trough (say -1 in value) of another wave, and the result is: (+1) + (-1) = 0, no signal and-or energy. This is the case when the waves have the same frequency and amplitude, and are 180° (ie., opposite) out of timing or phase at the intercept or combining location. This concept is sometimes called destructive interference, as opposed to when the signals would combine to produce a larger signal (ie., amplitude, strength) and which is called constructive interference. For example: (+1) + (+1) = +2.   In electronics, the electric signals (waveform, frequency, amplitude or voltage strength) might be combined in or across a resistor or transistor. The third or resulting wave is sometimes said as being composed of two or more other waves. As a simple and practical verification to all this, consider how two colors of light can be combined to produce a third color.

SOME ADVANCED EXAMPLES OF USING ALGEBRA

These examples use the mathematical operations previously covered in this book, and they also show some new concepts related to that particular type of example, and are good mathematical tools to have available. They will be helpful to your understanding of further topics in this book and elsewhere. A main point of these examples is to give more insight into algebra and how mathematical formulas can be made and utilized.

WHEN TWO OR MORE THINGS ARE WORKING TOGETHER

When two or people work together on a project or job, it is called cooperation, and that job can then get done or completed in a shorter amount of time and less effort and energy required from just one person. An easy to realize descriptive example is this: If it takes 1 person working for a year (365 days) to complete a project or job, then 365 people can complete the project in 1 day. In both cases, it takes 365 "man-days" (ie., a unit of measurement) or= 365 "people-days" to complete the project.

    365 people-days = (number of people)(number of days)   : basic formula to complete this specific job

    365 people-days = 1 person x 365 days   or:
    365 people-days = 365 persons x 1 days   If there are only 10 people on the team cooperating:

    365 people-days = (10 persons) x (X days)   After dividing both sides by 10 people: X days=36.5days
    365 people-days =   10 persons x 36.5 days   : 10 people can complete the project in 36.5 days

In terms of the typical 8 working hours per day for a person, you can multiply the number of days needed to complete the project by 8 so as to find the number of "man-hours" or "people-hours" needed to complete the project:

    1 people-days = 1 people (day x 8hours/day) = 1 people (8 hours) = 8 people-hours
                                                                      Also: people-hours / 8 = people-days
    365 people-days = 365 people (day x 8 hours/day) = 2920 people-hours

Ex. A tank of water takes 10 minutes to be filled by pipe 1 (P1). The flow rate of water through and out of the pipe is 5 gallons (a volume unit) a minute (5gal / 1min). If pipe 2 (P2) is added to the system and its flow rate is 2 gallons per minute, how long will it take both to fill the tank?

Clearly since another pipe, P2, was added to the system, the tank will fill "faster", "quicker" or in less time than 10 minutes as it would with just P1.

Total flow per minute =   Flow of P1 per minute   +   Flow of P2 per minute   : gal/min
Total flow per minute =   5 gal/min   +   2 gal/min
Total flow per minute =   7 gal/min   : flow or flow rate

The total amount of water the tank holds can be found from what we know about with just information or data given for P1:

From :   distance = (rate) (time)   : rate = rate of change, here it's the change in distance per unit time, and
                                                      it's usually called the speed or velocity of motion or moving. For a
                                                      constant distance value, note that rate and time are inversely related.

      volume = Total water = (rate) (time) = (amt. of flow per unit time)(time of flow)   : here, rate is the change in
                                                                                                  volume per unit time
                                                                                                  rate = R = flow rate

$$\text{Total water} = \frac{5\text{gal}}{1\text{min}} \times \frac{\cancel{10}\,\cancel{\text{min}}}{1} = 50\text{ gal} = \text{Total Volume} = V_t = R\,T$$

Total gallons = total amt. due to P1 + total amt. due to P2
Total gallons = (5gal/min)(7.143min) + (2gal/min)(7.143min)

Total gallons = 35.715 gal + 14.286 gal
Total gallons = 50 gal : checks

Another note to make clear here is that time (T) and rate (R) are inversely related. At completion = 100% = 1.0 we have this:

$1 = (R)(T)$: R=total rate, T=total time, mathematically then:

$T = \dfrac{1}{R}$ and $R = \dfrac{1}{T}$: this is very similar to the relationship between cycles or frequency and time in the study of mechanical motion-or electronics: $T = 1/F$ and $F = 1/T$

At completion of a system with several things or parts of it working together, and with each having their own rate of doing or performing it, the time of completion of doing something is the same for all the parts of the system working together. The total rate is simply the sum of all the rates of the parts of that system as shown in the example above:

$Rt = R1 + R2 + R3 +...$ and we have:

$1 = (T)(R1 + R2 + R3 +...) = (T R1) + (T R2) + T(R3) +... = (T)(Rt)$ and:

$$T = \dfrac{1}{R1 + R2 + R3 ...} = \dfrac{1}{Rt}$$

If each pipe took a certain amount of time to fill the tank, with having several pipes working together for the system, the total time would be inversely related, or less than just one pipe in the system.

From $Tt = \dfrac{1}{R1 + R2 + R3 + ...}$ this can be expressed as:

$$Tt = \dfrac{1}{\dfrac{1}{T1} + \dfrac{1}{T2} + \dfrac{1}{T3} +...}$$

If only two pipes, machines or perhaps people are working in the system then we have:

$$Tt = \dfrac{1}{\dfrac{1}{T1} + \dfrac{1}{T2}} = \dfrac{1}{\dfrac{T1 + T2}{T1\,T2}}$$ therefore:

$$Tt = \dfrac{T1\,T2}{T1 + T2}$$: product divided by the sum

Checking using the last example:

$$Tt = \dfrac{(10m)(25m)}{10m + 25m} = \dfrac{250m}{35m} = 7.143m$$: checks with the previous results

DA = SA TA + distance A already traveled or "lead" distance at the start, where T=0   : distance for object A
DB = SB TB + distance B already traveled or "lead" distance at the start, where T=0   : distance for object B

The added term to the distance value is included as a possible adjustment, and it is a constant added to the equation.

At the start of the system in question, where T=0, we know that:

DA = 25m/h Th + 1m   : when T=0h, DA = 1m as mentioned in the statement of this example
DB = 50m/h Th + 0m

At a point in time, which is what we are solving for, the difference between the distances of the two objects is 0m. That is: DA - DB = 0 (or: DB - DA = 0). Likewise, it could be said or easily verified algebraically, that at a certain point in time, the distances of the objects are equal: DA = DB , and the distance or difference between them is 0m.

DA - DB = 0   substituting values:
(25m/h Th + 1m) - (50m/h Th) = 0m   clearing some grouping symbols:
25m/h Th + 1m -50m/h Th = 0m   combining like terms and transposing the +1m:
-25m/h Th = -1m   isolating Th:

Th = $\frac{-1m}{-25m/h}$ = 0.04h

Let's convert this fraction of an hour value to a more practical value of minutes:

$\frac{60min}{1h}$ = $\frac{xmin}{0.04h}$

x = 2.4min

Let's check using substitution:

DA = 25m/h (0.04h) + 1m = 2m
DB = 50m/h (0.04h)   = 2m   : checks, and:

DA - DB = 2m - 2m = 0m   :checks, at the point of intersection or meeting, their difference
                                 in distance, and-or distance between them, is 0m.

This 2m value also tells us the total distance that B will travel in order to catch up to A. A will only travel only 1m (check: 25m/h x 0.04h = 1m) further since it already had a lead or initial distance of 1 mile from the start of this system where Th is considered as 0h for both objects. A's "meet-up" to B distance, or A's distance to go, in this system (where T starts at 0) - A's lead distance = 2m - 1m = 1m.

Note that in the stated problem that object A's distance lead could mean several things. For example, it could mean that the objects started from the same initial physical point or place as B, or it could also mean that A's starting position is in fact 1m further away or distant along the route from that of B's starting position.
If it was stated that A also had a time lead before the start of the current system or situation of both the objects, its' equation would look something like this:

DA = Sm/h (Th + time lead) + distance lead   : extra, this could be thought as: DA = lead distance, when T=0,
                                               and then substitute this in:
                                               DA = (Sm/h) (Th + time lead) + lead distance

For example, if m=3, the change of, or in (y) with respect to (x) is (the factor of) 3.

From the equation: y = 3x ,   we have: 3 = m = $\frac{y}{x}$

The above quotient, ratio, or "magnification factor" indicates that the corresponding value of y will always be three times greater than the value of x. This value also indicates that for each (1) unit change in or of the value of (x), then the corresponding value of (y) will change in value by 3 times that change. If (x) changes by 1, (y) will change by: (3)(change) = (3)(1) = 3. More specifically for any change in (x), the corresponding change in (y) will be 3 times that change. This is the mathematical relationship between (y) and (x), and it is a linear or constant mathematical relationship as indicated by the constant factor value (m), and here for this example or equation, it is 3.

change in y = (3) change in x   since m=3, mathematically then:

m = $\frac{\text{change in y}}{\text{change in x}}$: regardless of what b, the arbitrary constant, is. Let's give some more verification:

Consider one instance of the line eq.:   y1 = mx1 + b   : eq. 1
Consider another instance of the same line eq.:   y2 = mx2 + b   : eq. 2

This second equation can be expressed (using substitution) in relation (or in "terms of") to the first equation as:

First, below, y2 will be expressed in terms of y1, and x2 will be expressed in terms of x1.
Change is essentially, and calculated as, the difference between y2 and y1. It could also
be said as how y1 was changed, or the change applied to y1, so as to have the new value of y2:

From: (y2 - y1) = (change in y1)   : or= (change in y values) = (change in y) , solving for y2:
         y2 = y1 + (change in y1)   : likewise, (x2 - x1) = (change in x1) or= (change in x)

                    y2 = m x2 + b   , and using substitution for the values:
      y1 + (change in y1) = m (x1 + change in x1) + b
      y1 + m (change in x1) = m x1 + m (change in x1) + b

   mx1 + b + m (change in x1) = m x1 + m (change in x1) + b

Transposing +(m)(change in x1), we find this must be true (especially, since we used: m(change in x1) for (change in y1)) because we arrive at the original equation (eq 1) that we know is true:

      y1 = m x1 + b   : b is also known as the "y-axis intercept or crossing" where x=0,
                                      that is, the coordinates of this point are (x,y) = (0,b)
                                      When b=0, we can still see that y1/x1 = m

Also from the above equation, and substituting the values of: (mx1+b) for y1, and (change in y1) for m(change in x1) :

         y1 + m (change in x1) = m x1 + m (change in x1) + b
         mx1 + b + change in y1 = m x1 + m (change in x1) + b   after Transposing. mx1, and b:

         (change in y1) = m (change in x1)

Hence:

r = $\dfrac{\text{y1 + change in y1}}{\text{x1 + change in x1}}$ = $\dfrac{\text{y2}}{\text{x2}}$

Here is graph for a basic example of seeing, calculating and understanding slope values on a graph: [FIG 46]

This graph would be typical of a object that occasionally changed its speed while moving. The slope of the indicated line segments (indicated as A through F) is as follows:

The main changes in the slope (and its mathematical value) of the line are indicated where the curve or line segment on the graph changes to be more or less vertical which indicates an increased or decreased slope value. The slope value visually and mathematically expresses the relationship between the changes in (y) and (x). The slope value is calculated by using the (ratio of) corresponding changes in the (y) and (x) values.   When there is no corresponding change in the relationship of the (y) values when the (x) or independent variable changes, the line segment or curve will be horizontal. The slope of the line is 0 or "flat" meaning the object is not moving and not increasing its distance.

When there is no noticeable change in the current slope or "vertical tilt" of the curve or line segment, then the slope is a constant value, and the mathematical relationship is then said to be "linear" (a linear mathematical relationship) which means a constant mathematical relationship.   Between any two points on any line segment, such as seen in the above graph, the slope is constant. At any point where the slope changes, the (rate of changes) or "speed" has changed. If the speed is greater, the slope on the graph is greater, and the distance traveled will be greater per unit change in x or time.

A: From x=0 to x=1, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (1-0) / ((1-0) = 1/1 = 1
This value can be thought of as: As (x) changes by 1, (y) changes by 1.

B: The next part of the curve, or relationship between (y) and (x), is not as steep or vertical.
From x=1 to x=3, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (2-1) / ((3-1) = 1/2 = 0.5
This value can be thought of as: As (x) changes by 1, (y) changes by 0.5

C: The next part of the curve, or relationship between (y) and (x), is steeper and more vertical.
From x=3 to x=4, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (4-2) / ((4-3) = 2/1 = 2
This value can be thought of as: As (x) changes by 1, (y) changes by 2.

D: The next part of the curve, or relationship between (y) and (x), is not that steep, and is somewhat horizontal.
From x=4 to x=8, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (5-4) / ((8-4) = 1/4 = 0.25
This value can be thought of as: As (x) changes by 1, (y) changes by 0.25 , or "not that much"

E: The next part of the curve, or relationship between (y) and (x), is that the curve or line segment is horizontal.
From x=8 to x=9, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (5-5) / ((9-8) = 0/1 = 0
This value can be thought of as: As (x) changes, (y) is not changing and is a constant value of 0.

F: The next part of the curve, or relationship between (y) and (x), is that the curve or line segment tilts in a vertically downward direction as (x) increases. From x=9 to x=10, the slope is: (change in y) / (change in x) = (y2 - y1) / (x2 - x1) = (3-5) / ((10-9) = -2/1 = -2. This value can be thought of as: As (x) changes by 1, (y) changes by -2, hence (y) decreases by (-2) for each change in (x).

Ex. Graph the line: y = 2x + 1

Substitute three values for (x) and solve for the corresponding values of (y). Each corresponding value of (x) and (y) will be the coordinates ((coordinating, coordinated, or corresponding) "address" or location) of a point on a line within the rectangular co-ordinate system that is usually used for identifying points and places (locations) on a plane. A rectangular co-ordinate system, as described previously, is simply two number lines that cross perpendicularly at both their reference or "0" positions.   Each point in this system is located at the horizontal and vertical intersection of its defining co-ordinates.

Point (x , y)
   A (0 , 1)
   B (1 , 3)
   C (2 , 5)
   D (- 2 , - 3)

Once you have the co-ordinates of some sample points of and on the line, plot (display, draw or indicate) them, and then use a straight surface, such as a ruler's edge, as a helpful guide to draw and connect these points on the line.   [FIG 47]

The point where the line crosses the y-axis is called the (y) or vertical (axis) intercept. Here, the corresponding (x) co-ordinate of that point is always 0. To find this specific (y) co-ordinate, first set (x) to a value of 0, and then solve for (y):

y = 2x + 1
y = 2(0) + 1
y = 1   : this value is always equal to the (b) variable in the basic
                              expression of a linear equation: y = mx + b

Algebraically, and graphically, the coordinates of this point are: P(0,b)   : "y (axis) intercept point" of a linear equation,
                                                or "dependent axis intercept". This could be
                                                thought of as a "lead value" for (y) when (x)
                                                has no value yet and-or is equal to 0.

The point where the line crosses the (x) axis is called the (x) or horizontal intercept, or "root" of the equation. Here, the (y) co-ordinate of that point is always 0. To find the corresponding (x) co-ordinate, set y=0 and then solve for x:

y = 2x + 1   Let's rid non x termed variables to one side, or place x termed variables on one side to isolate x.
0 = 2x + 1   T. (+1), transpose +1 by adding the negative of that term to each side of the equation :
-1 = 2x   isolating x by dividing by its multiplying factor or coefficient, and then switching sides:

x = _-1_ = -0.5
       2

In general, the algebraic formula for this x coordinate can be created using the basic (slope-intercept form, of a) linear or line equation:

y = mx + b   setting y=0 and solving for x, we find:

x = _-b_   or = - _b_   : x coordinate of the x-axis intercept of a line
      m   m

Algebraically, the coordinates of this point is: P(-b/m , 0)   : "x intercept point"

Checking the slope (m) between points A and C, as indicated on the graph:

the polar co-ordinate system. Here, a point's (P) location is defined as its straight-line distance, known as the radius vector (R), from the origin (or "pole") to that point, and to distinguish it from all other possible points at this same distance (like the points on the circumference of a circle, all being the same distance since they all have the same (radius) distance from the center of a circle), its location includes the radius vector's angle from a reference line (called the polar (system) axis). The figure below shows polar co-ordinate system within a two dimensional rectangular system. It is very possible to have three-dimensional co-ordinate system for either or both methods. [FIG 48]Briefly, you can convert between polar and rectangular coordinates of a point when you consider the radius-vector line as the hypotenuse side or line (segment) of a right triangle and whose two other sides or lines are equal to the rectangular coordinates of that point. These two other sides can be used to determine the angle of the radius-vector line, and vice-versa.   A minimal amount of the knowledge of trigonometry, as discussed ahead in the TRIGONOMETRY section of this book, will be of assistance for these calculations. Polar coordinates will not be discussed further in this book, but further ahead, this book does include a basic discussion about vectors.

Given two lines, there are several methods to solve for the point of intersection of those lines. The lines cannot be parallel lines since parallel lines are defined as never intersecting. Here is a simple algebraic method.

$y1 = m1x1 + b1$: a line equation general format
$y2 = m2x2 + b2$: another line equation general format.
            Note, if m1=m2, and the value of (b) is different for each unique line, the lines are parallel.
                Parallel lines have the same slope both graphically, and numerically, which is the same rate of change of the dependent variable (y) with respect or in reference to the independent variable (x).

At the **point of intersection of two lines**, the difference between both the (y), and-or both the (x) co-ordinates of both points located at the point of intersection is 0 since they will have the same value for both points. This concept can also be used for many types of equations and curves.

$y1 - y2 = 0$ substituting their equivalent expressions:
$m1x1 + b1 - (+ m2x2 + b2) = 0$ distributing:
$m1x1 + b1 - m2x2 - b2 = 0$ at the intersection, x1 = x2, so let's express this value as just (x):
$m1x + b1 - m2x - b2 = 0$ factoring (x) out of the left side:
$x (m1 - m2) + b1 - b2 = 0$ transposing b terms:
$x (m1 - m2) = b2 - b1$ isolating x:

$x = \dfrac{b2 - b1}{m1 - m2}$: X CO-ORDINATE OF POINT OF INTERSECTION
                         OF TWO LINES

The corresponding (y) value of the point of intersection can be found by direct substitution of this value of (x) into one of the original linear equations.

Ex . Eq 1: $y1 = 2x1 + 1$ Eq 2: $y2 = 3x2 + 2$

         m1 = +2   m2 = +3
         b1 = +1   b2 = +2

At the point of intersection: $x = \dfrac{b2 - b1}{m1 - m2} = \dfrac{2 - 1}{2 - 3} = \dfrac{1}{-1} = -1$

By substituting this value of (x) into either of the original equations, we will have the corresponding (y) value of the point of intersection (Pi) of those two equations or lines:

   $y1 = 2(-1) + 1 = y2 = 3(-1) + 2$

y = mx + b

Ex. y = 2x + 3   : where m=2, and b=3

Given a set of points or data that appear (perhaps after plotting some points) to be linear or line-like, you can do the above process to find the equation of the corresponding line. Perhaps this equation can be used for predictions or estimates of data values, etc., however, given real-world data points, it is possible that several lines (although probably close together) could be created. There are advance methods of writing line equations that takes this into account and to create a single, "average" or "best fitting" line and equation to represent the given data. The appendix contains a simple or practical approach to finding a "reasonable equation" of a line given a set of points.

The following is a method of writing an equation of a line, however, it is one of the few discussions in this book that has a forward reference, particularly of the Pythagorean Theorem that is found in the TRIGONOMETRY section.

Notice in the drawing that if you let (c) represent the length of the line segment between the two points, that this is the (hypotenuse, longest side) side of a right triangle. The other two sides (a and b) of this triangle are the distances (here, calculated by difference) between the corresponding coordinate values of the points that define that line segment. By using the Pythagorean Theorem, we can write a general formula for the distance between any two points on a line, or anywhere within a coordinate system where both axis (reference of measurement lines) use the same units of measurement and scaled (indicated, successive increments of the units) distances.

$c^2 = a^2 + b^2$: **Pythagorean Theorem** for right triangles where (c) is the longest side

$c^2 = (\text{vertical change})^2 + (\text{horizontal change})^2$ taking the square-root of both sides :

$c = \sqrt{(\text{vertical change})^2 + (\text{horizontal change})^2}$ letting c = distance = d , this is calculated as:

$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$: **DISTANCE FORMULA BETWEEN TWO POINTS (2-D)**
                                                             (on a plane or two-dimensional "flat" surface).

For this formula, it makes no difference which point (with its coordinates) is chosen as "point 1" and which point is chosen as "point 2" since the square of the difference of the points corresponding coordinates, whether this difference (or length if you will) is positive or negative in sign (both differences would have the same absolute value), is always a positive value. In particular, a negative times a negative is still a positive value, and here, a positive value as required for any length to be physically real.

When a plane (2 dimensions) is moved perpendicularly, along another axis that is perpendicular to its' own two (reference or start of measurement) axis, a three-dimensional (3-D) space, or volume is defined. For example, if you take a part (ie. bounded and defined region such as a square) of a common two-dimensional (2-D), X and Y plane and move it perpendicularly above or below itself along a different axis (typically called the Z axis) or dimension, a three-dimensional space is defined. A common geometric example of three-dimensional space is a cube, and a cubic unit is naturally and commonly used as the reference unit of measurement for any volume or space. If you take a square (a region of a plane) which has 1 unit for its two (perpendicular) side dimensions and move it 1 unit perpendicularly to both of those side dimensions, a cube is created and the cubic unit (of measurement for space or volumes) is defined. Another common example of a three dimensional space is when a circle, or disk is moved a height, and this creates a cylinder or solid rod shape, "space" or volume.

The formula for the distance (D) between any two points within a three-dimensional space is very similar to that used for a two-dimensional system. The formula for the two-dimensional system is also used in the analysis for creating the formula to use for a three-dimensional (measuring and location) system. As shown in the drawing below of a three-dimensional system, (d1) is the distance between the two points when considering only the (X) and (Y), two-dimensional, coordinates of the point. This value of (d1) is also a side of a right-triangle within the three-dimension system, and the other side of

$D = \sqrt{X2\hat{}2 + Y2\hat{}2 + Z2\hat{}2}$

Ex.   Below is a simple "blueprint" (a drawing, often with a blue colored background, and which is a demagnified and proportional (often spoken as: "drawn to scale") image or drawing of the construction it represents. Here, the image is of a room as viewed from above it. In the room, you are to find the distance (D) between point A and point B. These points are located at different heights above the floor of the room.   The length of the room can be considered as the X dimension, the width as the Y dimension, and the (vertical) height or depth as the Z dimension. The lower left corner on the floor of the room is arbitrarily chosen as the origin (o) or point of reference for this three-dimensional analysis and co-ordinate system.   [FIG 51]

ROOM
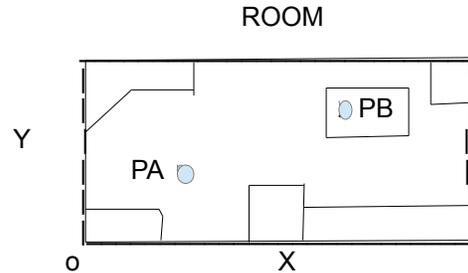
Let PA = P(X1 , Y1 , Z1)
Let PB = P(X2 , Y2 , Z2)

SCALE: 1/8 inch (on blueprint) = 1 foot (real)

$D = \sqrt{(X2 - X1)\hat{}2 + (Y2 - Y1)\hat{}2 + (Z2 - Z1)\hat{}2}$

Ex. Given a cube which is a 3-dimensional, volume shape with square sides (s) all of equal length, what is the length of the diagonal within it? We will be using the full side lengths where a side starts at 0 length, hence, for example:
Letting: X = (x2 - X1) = (x2 - 0) = X2 = s . If each dimension (length, width, and height) side has a length of 1:
side X = side Y = side Z = s

$D = \sqrt{s\hat{}2 + s\hat{}2 + s\hat{}2} = \sqrt{1\hat{}2 + 1\hat{}2 + 1\hat{}2} = \sqrt{1 + 1 + 1} = \sqrt{3} =$ **1.73205808** :**Cube Diagonal**
Constant

D $= \sqrt{3s\hat{}2} = \sqrt{3}\sqrt{s\hat{}2} =$ **1.7320808 s**   : **Diagonal formula of a cube with side (s)**

If the point of reference (ie., p(0,0,0) is considered as: p(X1, Y1, Z1) , the equation reduces to : $D = \sqrt{X2 + Y2 + Z2}$

Given the side of a cube, the diagonal length of that cube will be greater by the factor of 1.73205808

Ex. If s = 5 cm,   D = 1.7320808 (5 cm) **=** 8.660254038 cm

If the side length (s) increases by a factor of (n), the diagonal length of that cube will also increase by that same factor of n. For a square, the diagonal constant is 1.414213562 which is the square root of 2. The sum of the square root of 2 and the square root of 3 is 3.14626437... and it is very close to the value of (pi) = 3.141592654... , and with a difference of just 0.004671716...

$D = \sqrt{(ns)\hat{}2 + (ns)\hat{}2 + (ns)\hat{}2} = \sqrt{3(ns)\hat{}2} = \sqrt{3}\sqrt{n\hat{}2}\sqrt{s\hat{}2} = \sqrt{3}\,n\,s =$ 1.73205808 ns

A square also has a diagonal constant. $D = \sqrt{s\hat{}2 + s\hat{}2} = \sqrt{2\,s\hat{}2} = \sqrt{2}\ s =$ **1.414213562 s** :**Diagonal in a square**

        Besides the two endpoints of a line segment, or anything analyzed like a line, the point exactly between them is called the **midpoint of a line** (segment) and is often of importance for various constructions. Since midpoint means the "half-way" (of full length), half-length (length/2 = distance/2), "middle", "midway", or center point, simply divide the given

For the example given: the min. and max. points or locations are in reference to the 0 reference point of the entire system.   min = 100, and   max = 120. These points also define a line segment, and they are its boundary or defining points.

mid. = $\dfrac{length}{2}$ + min.   : length = length of the line segment, here its 10, using substitution:
This mid. point or location is the value that corresponds to the line segments center point's location on and in reference to the entire line.

Since the length of the line segment is the difference between its boundary points or locations: (max. value - min value):

mid. = $\dfrac{max. - min}{2}$ +   min.   : for this example, min=100, and max=120

This formula algebraically leads to a "simple average" formula:

mid. = $\dfrac{max. - min}{2}$ +   min.   combining fractions:

mid. = $\dfrac{max. - min}{2}$ + $\dfrac{2\,min}{2}$. = $\dfrac{max. - min. + 2\,min}{2}$

 Combining like terms in the numerator:

mid.   = $\dfrac{min + max.}{2}$: "middle or center" value between two other values.
: This is similar to an average of two values formula

Or, as indicated above, and will be shown further below:

mid. = min. + $\dfrac{change\ from\ min.}{2}$, or   = min. + 0.50 (change from min.)
: = "min. value plus half the total change"

For the last example:

$\dfrac{100 + 120}{2}$ = $\dfrac{220}{2}$ = 110   : the center point of 10 of the line segment corresponds to the point or location of 110 on the main reference line or system

|---|----------|----------------------->   : [FIG 54]
0   100   120   n units

110
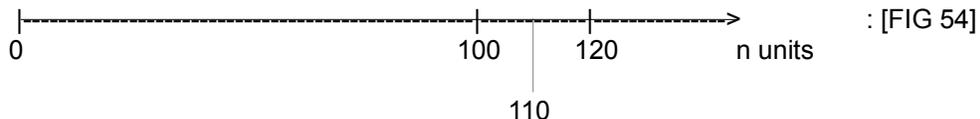
The co-ordinates of the midway point between two points in a two-dimensional co-ordinate system is analyzed as follows: [FIG 55]

, the mathematical relationship between the (y) and (x) variables is a direct relationship when (m), the slope, is positive in sign. If (x) increases, (y) increases and vice-versa. However, if you know of a linear like relationship where as one variable increases and then the other variable decreases, or vice-versa, the mathematical relationship between those two variables is said to be an **inverse relationship**. This would happen in a linear equation when (m), the slope, is negative in sign.

We know that the inverse or counterpart of a value is expressed by taking the negative, or "counterpart" of that value:

The inverse of +y is - (+y)   =   - y
The inverse of +x is - (+x)   =   - x

If (y) was equal to (x) as in the most basic of linear relationships:

y   =   x   : a linear equation, where m=1, b=0

and if the relationship between (y) and (x) is an inverse relationship, that is, if the inverse of (y) is (x), or vice-versa:

 -(y) =   x
 - y   =   x   by multiplying or dividing each side by (-1), this can be expressed as:

y   =   - x   : y equals the inverse or negative of x. Note that x, a variable, can still be either positive or negative
                         in value, and this expression essentially means the inverse of any value that variable may have.
                         Here, as the value of x increase, the value of y decreases, and this is an inverse relationship.

If the value of (y) is related to (x) (either directly or indirectly) by some constant factor (called m) other than 1 as seen above in: y=(-1)x = -x, this must be noted in the equation.   That is, if the ratio of (y) to (x) is always the same constant:

  __y__   =   m   therefore: y   =   m(-x)   : the variable name assigned for this constant is (m),
   -x   and mathematically, we then have:

y   =   - m x   : this indicates that y is always the same multiple of x. Adding in the arbitrary constant of (b):

y   =   - mx   +   b   : an "inverse form" of the basic linear equation of: y = mx + b

This form now implies the same linear equation but which now has the negative of a given slope. Linear equations where the (x) term is negative, such as when the coefficient (the slope, m) of (x) is negative, the corresponding (y) values for each (x) value get smaller as the (x) values increase. The effect on the line is that it will slope vertically (upward, in the (+y) direction) downward, rather than vertically upward, as (x) increases, as should be expected for an inverse relationship where as one value increases, the other value decreases.

Even though the line of a linear equation and the line of the inverse of that linear equation will cross, don't assume they cross perpendicularly. Lines that cross perpendicularly will and must have negative reciprocal slope values:

    m1 = - __1__   or =   m2   = - __1__   : **SLOPES OF PERPENDICULAR LINES**
              m2   m1   This is verified in the TRIGONOMETRY section.

Note also that (m1)(m2) = -1 always. For any two perpendicular lines (other than the axis or reference of measurement lines), one line will have a positive slope and the other line will have a negative slope, hence their product will always be a negative value (such as -1). As indicated in the equations above, these slopes are also inversely related. If one line has a very low absolute value for its slope, the other line will have a high absolute value for its slope.

        Note, if the (b) term is present in the equation of a line, the fundamental relationship between (x) and (y) is still linear, but the numerical relationship, as in the form of a constant ratio, between each corresponding pair of (y) and (x) is

An extra note, before this example is finished, is about using subscripts to indicate a particular instance of the same equation, and therefore, a particular instance of the variable(s). In this manner, the subscripts effectively identify (with mathematical notation) the corresponding results and values used in that equation. For example, given:

$y = 4x + 32$: $f(x) = 4x + 32$,   hence $y = f(x) = y(x) = $ "y is a function of x"
Again this is just notation, $f(x)$ and $y(x)$ does not mean multiplication.
Given $(4x + 32)$ is a stated function in or of x, and if we set y equal to this, it gives that function a specific name or identification. The function can now be called or indicate as y, and instead of using $f(x)$, we can use $y(x)$ to be more specific as to what specific function is a function of x. There could be other functions of x such as: $g(x)$, $h(x)$, etc., which generally would have different expressions containing the independent variable x.

y_1 would correspond to the value x_1 used in the equation.
y_1 is a specific instance of y and x_1 is a corresponding and specific instance of x.
x_1 and y_1 would be the coordinates of a point: $p(x_1 , y_1)$ or $p_1(x_1 , y_1)$

y_2 would correspond to the value x_2 used in the equation, and so on.
x_2 and y_2 would be the coordinates of a point: $p(x_2 , y_2)$ or $p_2(x_2 , y_2)$

$y = f(x) = y(x) = 4x + 32$: main function or relationship

$y_1 = f(x_1) = y(x_1) = 4(x_1) + 32$: instances of the function
$y_2 = f(x_2) = y(x_2) = 4(x_2) + 32$

LINE AND CIRCLE EXAMPLE

In this example, the circumference of a circle will be intercepted and passed through by a line. There will be two points of interception associated with this. Write a line (linear) equation of this intercepting line if the radius (r) of the circle is 3, and the (x) coordinate of the point of interception is 2. This line passes through the center of the circle which happens to be located at the center or origin (0,0) of a rectangular co-ordinate system.

The basic equations (including subscripts for this example) of a circle and line are:

$y_1 = mx_1 + b$: basic line equation
$r^2 = x_2^2 + y_2^2$: basic circle equation , and this also resembles the Pythagorean Theorem

(Note above, the subscript identified as 1 refers to the line, and the subscript identified as 2 refers to the circle).

Solving for and expressing the equations in terms of y:

$y_2^2 = r^2 - x_2^2$
$y_1 = mx_1 + b$

When x2 = 2, and evaluating the equation or function for y2 at this value; this can be noted with functional notation as f(2), or y2(2), which means to evaluate the expression or function when (x)=2: :

$y_2^2 = r^2 - x_2^2$
$y_2^2 = 3^2 - 2^2$
$y_2^2 = 9 - 4$
$y_2^2 = 5$ taking the square root of both sides:
$y_2 = 2.236067977$

At the point of interception, the two points (one on the line, and one on the circle) have the same location, and therefore have the same value: x1 = x2, and y1 = y2, hence:

$x_1 = x_2 = 2$
$y_1 = y_2 = 2.236067977$ substituting these values into the linear equation:

$y_1 = mx_1 + b$
$2.236067977 = m(2) + 0$: letting (b), the vertical or (y) intercept, equal 0 since this line passes
                                 through the y-axis at (0.0). The line also passes through the center
                                 of the circle which is located at the center or origin of the rectangular
                                 co-ordinate system. If the center of the circle is not at the origin, all
                                 points associated with the circle are effectively offset or "adjusted"
                                 by some (same) value from the origin, and therefore, the basic
                                 equation of the circle must be "adjusted" if the circle is to be in
                                 reference to this specific, "new" or local rectangular system located within
                                 the larger rectangular system. If the center of the circle is now at p(a,b),
                                 rather than (a=0, b=0) as in the larger or standard rectangular coordinate
                                 system, the (x) coordinate of the center and all other points associated with
                                 the circle's system must be incremented by (a) to so as to be in reference
                                 to the larger rectangular coordinate system. Likewise, the circles (y)
                                 coordinates must all be incremented by (b) so as to be in reference or with
                                 respect to the larger rectangular coordinate system. The equation of the
                                 circle therefore becomes this so as to be in reference to the main system:

               $r^2 = (x + a)^2 + (y + b)^2$: **GENERAL CIRCLE EQUATION**

There are several ways to solve for (x) when subtracting the equations, for example, you can square both sides of the linear equation and subtract it from the other equation, and with the difference in y values defined as being 0. Here is a method that uses algebraic substitution:

from $y_2{}^2 = 9 - x_2{}^2$ solving for y:

$$y_2 = \sqrt{9 - x_2{}^2}$$

Removing the subscripts since (x) and (y) are the same value at the intersection point, and substituting this into the linear equation:

$y_2 = y_1$: at the point of intersection of the two curves or equations

$\sqrt{9 - x^2} = 1.118033989\, x$ squaring both sides of the equation :

$9 - x^2 = 1.25\, x^2$ transposing ($-1x^2$) to solve for (x) :

$9 = 2.25\, x^2$ isolating the (x) variable, and then taking the square roots of both sides:

$x = \sqrt{\dfrac{9}{2.25}} = \sqrt{4} = 2.0$ solving for (y) using algebraic substitution:

$y = 1.118033989\, x$
$y = 1.118033989\,(2.0)$
$y = 2.236067977$: checks

By observing the above procedure carefully, a formula for the (x) coordinate of the point of interception can be derived:

$x = +- \sqrt{\dfrac{r^2}{m^2 + 1}}$: X COORDINATE OF CIRCLE AND LINE INTERCEPTION
                   (center at origin (0,0), and b=0 for the line)

This can also be expressed as:

$x = \dfrac{\sqrt{r^2}}{\sqrt{m^2 + 1}} = +- \dfrac{r}{\sqrt{m^2 + 1}}$

(y) can be found by substituting the value for (x) into one of the original equations, or from:

$m = +- \sqrt{\dfrac{r^2}{x^2} - 1}$: this was derived from the above expression for x

and since $m = TAN\ \phi = \dfrac{y}{x}$: TAN ϕ = tangent of the angle. More exactly, it is the ("tangent") ratio of two specific side lengths of a triangle. tan = opposite side/adjacent side. For this example and line, The angle is between the line and the x-axis.

When (change in y) = y, and (change in x) = x, (ie,. one point being at the origin, p(0,0)) we can algebraically get:

$y = mx$: the basic linear equation (when b=0). Since (m) is a initially an algebraic variable, it can have any value, including (1). If m = 1, the result is the simplest linear equation of: y = x.

LINEAR TEMPERATURE EXAMPLE

Here, we will derive the mathematical relationship between the Fahrenheit and Celsius temperature scales. Temperature is basically defined as the amount of energy something has. The more energy, the hotter (higher in temperature) it becomes. This is a direct relationship. The temperature measured is linear for both the Fahrenheit and Celsius temperature scales. If the objects (including the air) temperature doubled, the measured temperature indicated on a thermometer (a temperature metering or measuring device) will double in value. For both systems, the temperature with respect to the energy of an object, can be plotted as straight lines. If the energy of an object doubles (2), its temperature likewise increase by that same factor value and double. The Celsius temperature scale is named after Anders Celsius, from Sweden, who originally conceived of it in 1742. "Centigrade" (ie., 100 gradients) is an older word for the Celsius scale.

What is generally or commonly known about the Fahrenheit and-or Celsius scales is the temperature of the freezing point and boiling point of water:

	Freezing	Boiling
Fahrenheit:	32°	212°
Celsius:	0°	100°

: the Celsius scale appears to be more "metric-like". Some have even proposed that time and clocks should also be based on a "metric-like" system with 10 hours before noon, and 10 hours after noon. Each hour would have 100 minutes, and each minute would have 100 seconds.

Observe the drawing below: [FIG 58]
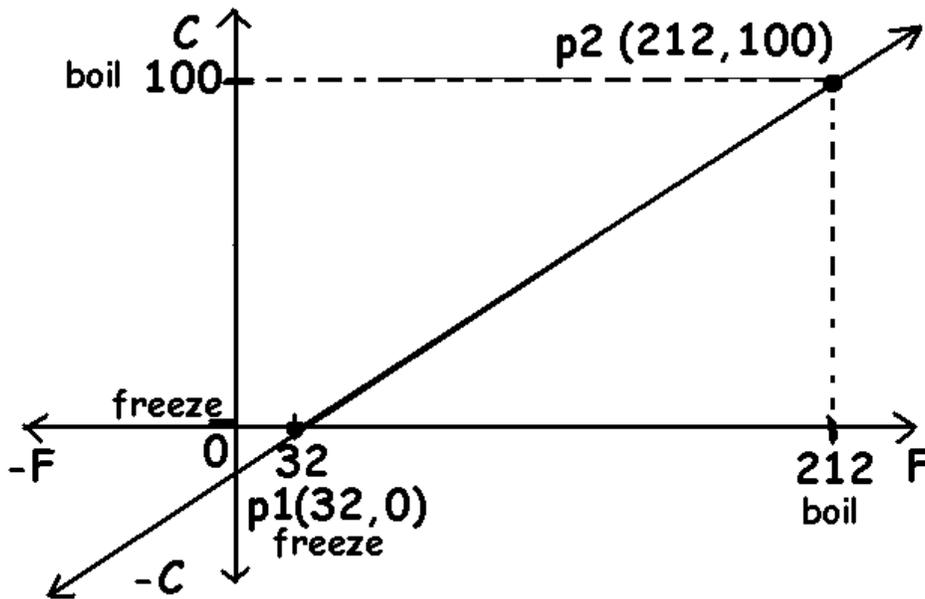
For finding the slope of this line shown, we will use these two known points on the line that have the format of:
p(x , y) = p(Fahrenheit , Celsius). The values of the two shown points are: p1(32,0) and p2(212,100):

slope = __vertical change between points__ = __change in Celsius_ = _(100 - 0) = _100_ = 0.5555... = 0.5555...
           horizontal change between points   change in Fahrenheit   (212 - 32)   180   1

Hence for each increment (1), or degree change in Fahrenheit temperature, the corresponding Celsius temperature will change by 0.5555... degrees which is a little over a half (1/2 = 0.5) a degree Celsius.

energy in your body tissues and blood will be wasted to the outside world. This initial effect on the surface of your skin is called **windchill** (ie., cooling due to the wind), and it is similar to the concept of evaporative cooling where water humidity (water vapor, gas) or moisture can effectively absorb and remove heat from an object, and then that moisture rises or is blown away by a wind and the object then has a reduced temperature. This is essentially what happens when a person sweats so as to help the body cool down. If the air temperature is at or below the freezing temperature of water, the wind can hasten frostbite (frozen, damaged body tissue) in those areas since it is removing what little amount of heat was still in those areas so as to keep them warm and unfrozen. The temperature that the bare unprotected skin will feel when wind is blowing upon it is called the windchill temperature. The windchill temperature is a calculated value using a formula that depends mainly on both the air temperature and the wind speed.   Obviously, for a constant air temperature value, the "windchill temperature" depends mainly upon the wind speed. The windchill temperature will be lower than the actual air temperature. When the air temperature is of a lower value, and for a constant wind speed, the difference between the actual air temperament and the windchill temperature increases. In wintertime with cool air temperatures, the wind has a significant impact on how cold you will feel, especially if not wearing layers of clothing (like insulation) to retain heat.   For a basic idea of the (subjected, human perceived) mathematical relationships, given a constant wind speed, for every 10 degrees colder in actual temperature, the windchill temperature feels about 12.6 degrees colder. The windchill temperature formula usually reflects only the data for temperatures 50° or lower where the effects are most noticeable. The windchill formula is not overly complex, but its derivation is beyond the space and scope of this book. Due to the average person only having a temperature reading or estimate, and maybe a windspeed estimate and no other data, here is a very simplified formula to obtain a rough value for the windchill or "feels like" temperature:

A windchill estimate temperature = F° - windspeed mph   : A simplified and practical windchill estimate formula

The author of this book has found that the wind is one of the worst parts of winter, and so therefore, you need to check the temperature and windchill and dress accordingly such as with several layers of shirts. The lower the air temperature, the more the wind has a cooling effect. Denim pants are not ideal for cool climates because they do not seem to keep the heat in and keep you warm. "Sweat pants" are much better than denim (ie. tightly woven cotton material) "blue-jeans" in the winter if you will be outside doing things, and without much of the worry of constantly trying staying warm. Wear a winter hat and gloves, and keep them immediately available and close by (ie. in a pocket or bag) for when you will need them.   If in a difficult situation with no warm shelter, brisk or fast walking can help raise your body temperature.

The concept of **heat-index** is essentially the opposite of the windchill concept. The heat-index is the temperature of how hot the air temperature feels like depending on both the actual current air temperature and the humidity level. Surely, whether hot or cold, the sunlight will always make the air temperature feel warmer, perhaps an additional 10°F warmer , and so this measurement, like the windchill temperature, also uses the measured air temperature in the shade. Humidity is moisture in the air which can carry much heat with it and make the air temperature feel hotter. It is essentially a "steam (heat) bath". **Danger:** If a persons core body temperature is 104°F = 40°C or greater, they are in severe health danger, and will most likely have symptoms of various mental confusion, especially if they are dehydrated and in need of plain drinking water. Overworking and in the sunlight and-or stress can quickly cause dehydration in the body. A person having these conditions needs to be cooled down fairly quickly. Place that person in the shade and remove any restrictive and warm clothing. Let them drink some water and sit in front of a fan and-or cool off in water with constant supervision by the persons(s) aiding them. A person can also be cooled off by immersion of their feet or hands into a container (such as a dishpan) of water that feels cool to them. The higher the humidity, the higher the heat-index temperature because evaporation to cool the body has slowed down. The transfer of heat energy from the body to the air is less or slower in higher humidity or moisture climate conditions. Occasionally, cool moist air can actually make a person feel cooler, and the heat-index temperature will be slightly lower than the actual air temperature.

Heat Index estimate temperature = F° + (%humidity / 10) :**A simplified and practical, heat-index estimate formula This is the "feels like temperature".**

The average or normal, internal body temperature is: 98.6°F = 37°C =~ 310.15°K :**Normal Human Body Temperature**

Since a meter is 3.28 times longer or higher than a foot, the above rate corresponds to: -0.01148°F / meter

For every 1000ft higher the predicted, average change in temperature would be; according to the above equation:

(change in temperature) = (change in altitude)(-0.0035°F / 1ft)

change in temperature = (1000ft)(-0.0035°F /1ft) = -3.5°F

A general formula for the global average temperature at an altitude above sea level is:

temperature at altitude = 57° - (altitude)(change in temperature / change in altitude)

temperature at altitude = 57° - (altitude in feet)(-0.0035°F / 1ft)

For conversions, and as shown in the book, a change of 1°F corresponds to a change of 0.55555°C, therefore, using proportions or equivalent fractions, a change of -0.0035°F corresponds to a change of -0.0019444°C =~ -0.002°C, and the above formula can be expressed as:

temperature at altitude = 57° - (altitude in feet)(-0.002°C / 1ft)

The change in temperature per 1000ft ~= 305m =~ 0.305km is about -3.5°F ~= -1.9444°C ~= -2°C

The change in temperature per 1km ~= 3280ft is about -11.48°F ~= -6.38°C

For some other conversions (approximate): 1m = 3.28ft , 1 ft = 0.3048m , 1km = 3280ft = 0.6214 miles ,
1mile = 5280ft = 1.60934 km = 1609.34m
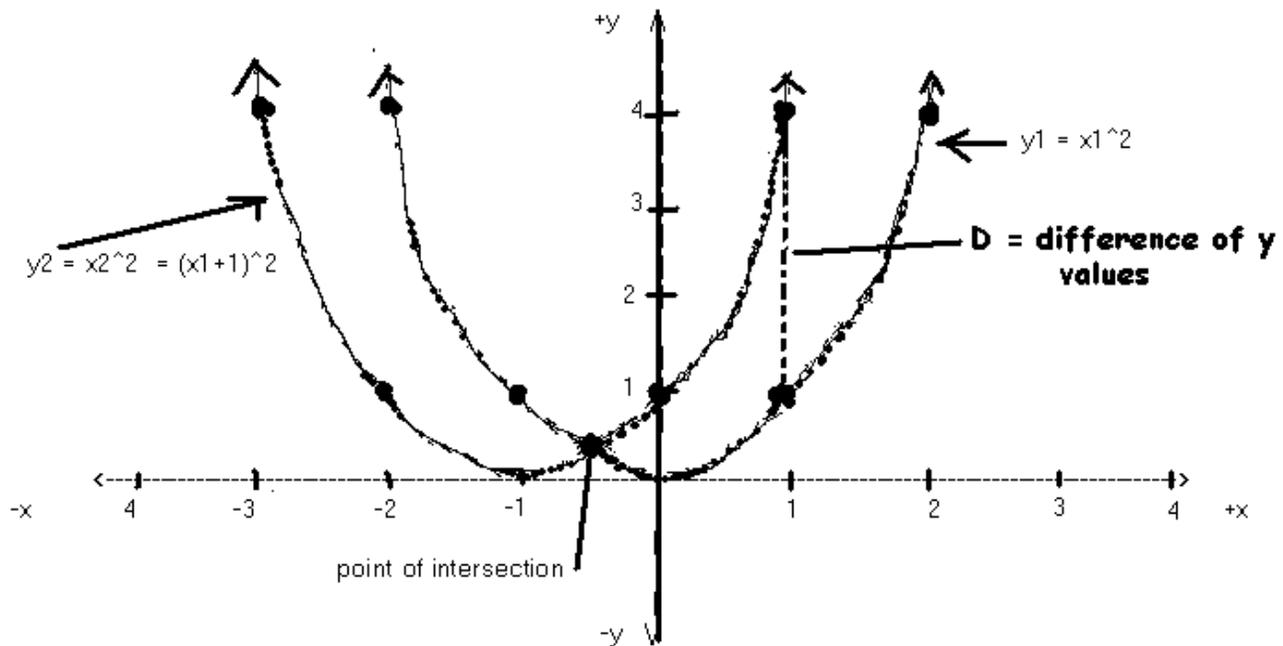
As an extra note; clearly, the curves cross or intersect at some point. The difference between their x values (and y values) is 0 since the x values are identical at this point. You can find this (x) value by setting the y values equal, that is, set the two equations as equal:

$$y = y1 = y2$$: at the point of intersection, the values are equal. Using substitution:
$$x1^2 = (x1 + 1)^2$$ extending the right side:
$$x1^2 = x1^2 + 2x1 + 1$$ transposing the left side:
$$0 = 2x1 + 1$$: essentially setting the difference D = (y2 - y1) = 0.

After solving for x1 (or the general x variable at the point of intersection):

$$x = x1 = x2 = \frac{-1}{2} = -0.5$$: the x coordinate of the point of intersection

Even though -0.5 is not an integer number, it goes to show that it is possible that the expressions of x^2 and $(x+1)^2$ can and do have the same values at one point. The y value of this intersection point can be found by the substitution of this value of x into the any or both of the original equations:

$$y = y1 = x1^2 = x^2 = -0.5^2 = 0.25 = y2 = (x1 + 1)^2 = (-0.5 + 1)^2 = (0.5)^2 = 0.5^2 = 0.25$$

The point of intersection is therefore: p(x, y) = p(-0.5, 0.25)

Now, let's discuss the value of the difference (D) a bit more.

The difference between $(x + 1)^2$ and x^2 is:

$$(x+1)^2 - x^2 = 2x + 1$$ substituting a value for x:

Ex. let x = 2: $(2+1)^2 - 2^2 = 2(2) + 1 = 4 + 1 = 5$

HALF-LIFE EXAMPLE

Half-life is the period (time to complete a repeating cycle or event) it takes a decaying radioactive ("giving off", releasing, radiating energy) mass to effectively equal one-half (1/2) of it's original or "starting" radioactive "mass" and-or measured value. Mass is essentially the quantity or amount of the substance (ie. it is number of atoms which occupies a specific volume [a three-dimensional measure of space]). Remember, mass is not the same as the weight (a force) of something, but a mass under the influence of-and being constantly accelerated by gravity (a force) will produce a weight (force). The same mass or amount of matter on one planet will weigh differently on another planet where gravity is stronger or weaker than that of Earths gravity. A good example of the use of the concepts of half-life is in the dating old objects such as by using a process called "carbon-14 dating". For example, a reference mass of 1 gram of a certain type of living tree contains a certain mass of carbon-14 while it is living. After the tree dies, the carbon-14 will start to decay (decrease). To determine how long ago a similar tree died (which is roughly how old the artifact is), the carbon-14 mass remaining in a size equivalent to the reference size (one gram for this example) is determined from the artifact (a tree or piece of wood in this example), and that value is then used in an algebraic formula to determine how old it is.

Let: Mo = Original or starting radioactive mass of the decaying substance.
       Ml = The remaining radioactive mass of the decaying substance left after a period of time (such as the
           half- life time). "Mass left".
       P = Half-life time, or time the period (length of time) of the half-life of a certain radioactive, decaying material
       N = (Number of the) Period in question.
       T = Time elapsed.

Clearly, the total time elapse (T) is equal to the time of each period (P) times the number (N) of periods gone by. That is:

  T = 1 period of time + 1 period of time + 1 period of time . . .

Since all terms are the same, and there are N terms, this repeated addition can be expressed with multiplication:

  T = N x period of time = N P

  T = NP :   0P ------------ 1P ------------ 2P ------------- 3P ------------ 4P ---- . . . (and so on)

$$Ml = Mo \qquad \frac{Mo}{2} \qquad \frac{(Mo)}{\frac{(2)}{\frac{2}{1}}} = \frac{Mo}{2^2} \qquad \frac{Mo}{2^3} \qquad \frac{Mo}{2^4}$$

Hence, to find the value of Ml after the next period of time, the value of Ml is divided by 2, and we find that the denominator to Mo is equal to an indicated power with a base of 2 which has an exponent that corresponds to, and is equal to the period in question, that is, the denominator after the end of each period in question (N) and-or start of the next period of time is: (2^N).

from: T = NP, $N = \dfrac{T}{P}$

hence: $2^N = 2^{(T/P)} = 2^{(\text{total Time elapsed / time of Period})}$: for this example, time of period = half-life time

By using this expression for N, we can account for the mass left at all times, even when the time is in-between periods (ie. portions or fractions of a period gone by).   Expressing all this data into an algebraic formula for the mass left after any period or amount of elapsed time in question:

Given 1 gram of this particular substance, it will take 111.67 days till 0.750 grams of it remains. Likewise, it will take this long for (1g - 0.750g) = 0.250 grams of it to disappear or change to another state or substance.

Frequently, the period of half-life of a substance, when not being calculated, is found from a "look-up" or reference table. For example, the half-life of carbon-14 is noted as being about 5,700 years.

If you like, you can use "relative" fractional or percentage values for Mo and Ml. To do so, place these percent values in numerical (decimal) form. Usually, the original mass (Mo), or any mass considered as the original or "starting" mass of the system (where T is considered as 0), is considered as 100% since it has not decayed yet, and therefore Mo = (100/100) = 1.0 will be used in the formulas. For example, if the mass left over (Ml) is 50%, this will be numerically represented as Ml = 50% = fifty parts per hindered parts = (50/100) = 0.50 in the formulas. The mathematical reason that these percent or fractional values can be used is that the ratio of the masses is also equivalent to the ratio of their corresponding percentages.

Below, another method of calculating half-life is derived, and some more generalized formulas will be developed.

After one period (T) of half-life time, the ratio of the mass or substance left (Ml) to that of the original mass (Mo) is always found to be 50% (or 0.50) in half-life situations.   Since mass left or remaining is in question, this value could be called the percent remaining after each period of time.

$\dfrac{Ml}{Mo}$ = PR = percent remaining (PR) after any one period. Mo = mass at start of the period in question, Ml=mass at end of the period in question.

Algebraically: $Ml = Mo(PR)$

Regardless of the period in question chosen to find PR, PR will always be a constant as long as the physical decay rate is always a constant value.

Period Number =   0   1   2   3   . . .

Ml = Mo   Mo(PR)   (Mo(PR))(PR)   ((Mo(PR))(PR))(PR)   . . . these can be simplified to:

OR: Ml = Mo $Mo(PR)^1$ $Mo(PR)^2$ $Mo(PR)^3$ . . .

Notice that Mo and PR are common factors of the remaining mass left (Ml) at the end of each period. Also notice that the exponent of PR is equal to the number of the period (N) in question.   Expressing this data into a general algebraic formula:

$Ml = Mo(PR)^N$ or= $Mo \left[\dfrac{Ml}{Mo}\right]^N$:Mass Left Formula
(PR corresponds to the specific decay rate used, and what remains)

Notice that the physical decayed or lost value (a change or difference = mass at start of period - mass at end of period) becomes smaller and smaller as the periods increase since the same (constant) percentage (PR), when used a decimal value, is multiplied to a smaller value (the mass starting each new period = Ml). Overall, in decay situations, PR is less than 1, and any value less than 1, a proper fraction, raised to a power greater than 1, or multiplied by itself, is smaller than itself. A fraction of a fraction results in a smaller value. Therefore, in the first few periods, a relatively large decay (loss of the substance in question) will be produced, and as the periods increase, the actual physical decay values will be less and less and it will eventually appear as a very slow, prolonged or endless decay where there is always some minimal or minuscule value still left (Ml).

The actual value of the decay between the starting mass and the mass at the end of a period in question can be calculated as:

N = $\frac{t}{T}$ = $\frac{time\ elapsed}{time\ of\ period}$: Determining The Period In Question Formula

The (time) units of measurement of (t) and (T) must be the same, such as seconds, minutes, hours. days, or years.

since: Ml = Mo(PR)^N   ,we can derive a more generalized formula:

Ml = Mo(PR)^(t/T)   : Mass Left Formula
PR = percent remaining = 50% = 0.50 for half-life problems.
t=total time, T=half-life period time, Mo=starting mass

Ex. If you began with 270g, (Mo), of substance, and after a period (T) of 3 years, 50% of it or 135g (Ml) remains. How much will you have after a total of 12 years of elapsed time (t).

In this example the percent remaining (PR) is given as 50%, it can be calculated from:

PR = $\frac{Ml}{Mo}$ = $\frac{135g}{270g}$ = 0.50   : PR after a 3 year time period, and in its usable decimal form for calculations.
Since half of the substance is remaining after this length of time, this length of time, 3 years, is also the half-life time or period.

Ml = Mo(PR)^(t /T)

Ml = 270g(0.50)^(12yrs./3yrs.)

Ml = 270g(0.50)^4   : the half life period is 3 years, and 4 of these (3 year) periods is 3 x 4 = 12 years

Ml = 270g(0.0625)

Ml = 16.875g   or =   16g + 875(10^-3)g   =   16g + 875mg

Note, the ratio (PR) of Ml to Mo per period can be greater than 1. This happens when there is a growth or gain such as an account that has gained interest, growth of living things, growth of crystals, etc.

Since PR = percent remaining, assigning a variable name of PC (Percent Changed) to the percent that was decayed or lost from the 100% value:

$$PC = \frac{Change}{Initial\ value} = \frac{Mo - Ml}{Mo} = \frac{Mlossed}{Mo} \quad : (per\ period), \quad therefore:$$

(Note, for purposes other than this immediate discussion, if you want to keep PC with a sign indicating if the change was greater or less in value, use (Ml - Mo) for the numerator above.)

Ml = Mo - Mlossed
Ml = Mo - Mo(PC)

The sum of PR and PC is equal to 100% (or 1.0). This will now be verified:

$$PR + PC = 1.0$$

$$\frac{Ml}{Mo} + \frac{(Mo - Ml)}{Mo} = 1 \quad\quad combining\ fractions:$$

$$\frac{Ml + Mo - Ml}{Mo} = 1 \quad\quad combining\ terms\ in\ the\ numerator:$$

$$\frac{Mo}{Mo} = 1 \quad\quad : checks$$

From: 1 = PR + PC   : Here, 1 is the decimal equivalent of 100%,   therefore:
PC = 1 - PR   and:
PR = 1 - PC

Ex.   When PR = 90%:
PC = 1 - PR
PC = 100% - 90%
PC = 10% = 0.10

Ml = Mo - Mo(PC)   factoring out the HCF of Mo:

Ml = Mo(1 - PC) = Mo(PR)   : Formulas For Money or Mass Left For A Single Period

Therefore, the formulas for the money (such as in an interest account) or mass left (such as of a decaying mass), can be expressed in terms of either the percent remaining or the percent changed.

From: $Ml = Mo(PR)^{(t/T)}$
       $Ml = Mo(1-PC)^{(t/T)}$: Money or Mass Left Formulas (decays)

If the change was an increase, such as from a growth or interest account, this value is algebraically noted with a positive sign.

Ml = Mo + (change of M)
Ml = Mo + (+ Mgained)
Ml = Mo + Mgained

From the concepts of logarithms:

If $e^x = 2$,   then: $x = \log_e 2 = \ln 2$,   therefore:

  $e^x = e^{(\ln 2)} = 2$ and:

  $M1 = \dfrac{Mo}{2} = \dfrac{Mo}{e^{(\ln 2)}} = Mo\, e^{(-\ln 2)}$: Often, a negative exponent indicates a decaying function or value rather than a growth. Here, $e^{(-\ln 2)}$ is actually a numerical coefficient (a multiplying factor to) of Mo. It will cause the value of Mo to decrease or decay.

By the end of the second half-life period:

  $M2 = \dfrac{M1}{2} = \dfrac{\frac{Mo}{2}}{2} = \dfrac{Mo}{2^2} =$

$\dfrac{Mo}{(2)(2)} = \dfrac{Mo}{(e^{\ln 2})(e^{\ln 2})} = \dfrac{Mo}{(e^{\ln 2})^2} = \dfrac{Mo}{e^{(2\ln 2)}} = Mo\, e^{(-2\ln 2)}$

Notice that at the end of each half-life period in question (n), that the numerical coefficient of the exponent of (e) is the same as the period. Expressing this in a formula:

  $Mn = Mo\, e^{(-n\ln 2)}$: Formula For Mass Left After A Decay , (expressed with e), where the decay is 50% for each period. (n) is the period number.

Here's how to express the formula for times in between periods, and hence, at any time in question:

Since total time (t) is the number of periods (n) gone by times the time-length or duration of each period (T), as shown previously:

  $t = nT$ therefore, $n = \dfrac{t}{T}$

Substituting this for (n) in the formula we get:

  $Ml = Mo\, e^{(-t/T\,\ln 2)} = Mo\, e^{(-0.69314718\, t/T)}$: Mass Left After A Decay Formula (for 50%, "half-life" decays)

Notice that the formula can be written using the associative law as:

  $Ml = Mo\, e^{(-t\,\ln 2/T)}$

The value ($\ln 2/T$) is a constant for the given problem, and therefore it only needs to be calculated only once. For example, if the half-life period is 10,000 years, perhaps for a half-life analysis using carbon-14 dating:

  $\dfrac{\ln 2}{T} = \dfrac{0.69314718}{10000} = 0.000069314$ and the resultant formula is simply:

  $Ml = Mo\, e^{(-0.000069314\, t)}$

TOTAL   =   M (1 + I)^t/T   : total savings in an account with compound interest of I
M=starting balance, t=total time elapsed, T=time between interest payments or payment interval time

Ex. If the initial deposit into a savings account was $25, and with 5% compounded interest every 4 months, how much will the account grow to after 12 months of time?

Here,   t = total time elapsed = 1 year = 12 months
        T = period of calculating the interest rate = 4 months
        TOTAL   =   M (1 + I)^t/T

TOTAL = $25(1+ 0.05)^(12months/4months) = $25(1.05)^3 = 25(1.15$\underline{8}$)   = $28.95

A gain of: ($28.95 - $25) = $3.95

    Next, a general discussion about paying interest on a loan will be included in the example given. Again, observe how algebraic formulas can be created.

The remainder or amount of the loan left to be paid (without regard to any interest) can be equated or expressed as the total amount of the loan less the amount already paid back:

From: Mo = loan paid back + MI
           MI = Mo - loan paid back

Since the amount of loan paid back can be calculated in terms of the base payment (B) and the number of intervals (N) or times it was paid, payment (P) can be calculated in terms of an interval in question:

from: loan paid back = (base payment)(intervals)   : = Mo - MI
           loan paid back = BN   : (without adjustment, see below)

When calculating the loan paid back at the start of a new interval, N must be reduced by 1. For example, starting when the loan was first taken out, this is considered interval 0 since no (0) payment or interest is yet due. When the first payment is due, interval 1 is entered and the next payment will be due at the end of this interval or start of interval 2. In other words, the intervals and-or time actually start at 0, and not 1. Therefore, when the first (1) payment or "interval 1's" payment is due, no amount of the loan has yet been paid back. To correct the loan paid back formula, an adjustment must be made, and that is to subtract 1 from N:

   loan paid back = B(N-1)   :loan paid back at the interval (N) in question

   MI = Mo - B(N-1)   :this will be substituted into the Payment Formula, but first:

Here is another derivation of the remainder (MI) of the loan to be paid:

Payment N   MI in which interest for a payment is to be calculated with.
    0
    1   Mo   : payment due at the start of the interval
    2   Mo -B   =   Mo - B   =   Mo-1B
    3   (Mo -B) -B   =   Mo - B -B   =   Mo-2B
    4   ((Mo-B)-B) -B =   Mo-B-B-B   =   Mo-3B
 (and so on)

Observe that for each interval or payment, B, the base payment, is to be subtracted [algebraically adding (-B) since the subtraction of a value really represents an addition, or combining with, of a reduction or loss, (ie. a negative value)] from the remaining part of the loan (MI). We also observe that this is essentially adding (-B) to Mo, (N-1) times. In short, the coefficient of the B term is 1 less than the interval in question. Expressing these facts algebraically for MI, we see that this checks with the equation previously shown.

   MI = Mo - (N-1)B   or   MI = Mo - B(N-1)   : by using the commutative law

   P = B + Interest
   P = B +   MI I
   Pn = B + (Mo - B(n-1)) I   : n=Nth (interval) Payment Formula

Since Mo = NtB ,   we can get an alternate payment formula:

   Pn = B + (NtB - B(N-1))I   factoring out the common factor of B:

   Pn = B + I(B(Nt - (N-1)))
   Pn = B + BI(Nt - (N-1))   : alternate Nth Payment Formula
                                         : Here, the value of the loan, Mo, need not even be known or used

formula for the sum of integers from 1 to (Nt-1):

The general formula for the sum of the first N integers from 1 to N is:

$$Sum = \frac{N^2 + N}{2} \quad or = 0.5N^2 + 0.5N = 0.5(N^2 + N)$$: **sum of the first N integers from 1 to N**

Here is a simple verification/derivation of the above formula:

Ex. We are to sum (S) up (from an initial sum of 0) the first 4 integers (1, 2, 3 and 4), hence letting N=4 in the above expression.

```
    S  =  1 + 2 + 3 + 4          : this is N=4 terms,  adding the sum to itself, we will have S + S  =  2S:
+   S  =  4 + 3 + 2 + 1          : remember, addition is commutative, so the sum should still be
   2S  =  5 + 5 + 5 + 5            the same regardless of order, and here, it's a descending order of values.
```

The right hand side can be represented with multiplication since multiplication is basically a repeated addition of the same value. Notice that each term is algebraically $(N + 1) = (4 + 1) = 5$, and that there are N terms, and expressing this with multiplication:

$$2S = (N+1)(N) \quad \text{solving for S:}$$

$$S = \frac{(N+1)(N)}{2} \quad \text{clearing grouping symbols by distributing N:}$$

$$S = \frac{N^2 + N}{2} \quad \text{substituting the given value of N:}$$

$$S = \frac{4^2 + 4}{2} = \frac{16 + 4}{2} = \frac{20}{2} = 10 = 1 + 2 + 3 + 4$$

Checking:
```
        2S   =  5 + 5 + 5 + 5
      2(10)  =  20
        20   =  20            : checks
```

Extra: Here's the sum of the first 100 integers from 1 through 100:

$$S = \frac{100^2 + 100}{2} = \frac{10000 + 100}{2} = \frac{10100}{2} = 5050$$

For terms which are not necessarily integers, and-or don't have a difference between them of 1, a similar derivation and more generalized formula can be found in the study of mathematical progressions and series.

In the interest example, the highest coefficient of B is (Nt -1) = (12 - 1) = 11. The sum of the first eleven integers is:

$$\frac{11^2 + 11}{2} = \frac{121 + 11}{2} = \frac{132}{2} = 66$$

checking: 1+2+3+4+5+6+7+8+9+10+11 = 66

Substituting the algebraic expression for the coefficient of B into the total interest formula:

Pav = $1,250 + 162.50 = $1,412.50

Since there are Nt average payments, the total you will have to pay back to the bank can be found by:

Total = Mt = Loan + Total Interest = NtPav   : Total Payment Formulas

Mt = $15,000 + $1,950 = (12)($1,412.50) = $16,950

REFLECTION EXAMPLE

A certain type of mirror reflects 90% (or 0.90) of the incident (incoming) light.   If a beam of light is reflected to and off of a series of five (5) of these mirrors, what percent of the original beam will be reflected off of the last (5th) mirror?

Letting Mo = Starting light beam value, quantity or intensity
$\quad\quad\quad$ Mn = Value, quantity, or intensity of the beam after reflecting off of the nth mirror.
$\quad\quad\quad$ I = Percentage of incident beam reflected reflected off each mirror and available for the next mirror.

$\quad\quad\quad$ Here, I = PR = $\dfrac{Mn+1}{Mn}$ = 0.90 $\quad\quad$: 10% of the light is lossed due to absorption and scattering
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ at the mirror (reflective surface).
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ PR = percent remaining, as used in a previous example

After reflecting off of the first mirror (n=1), the remaining light beam is:

M1 = (Mo)I

After reflecting off of the second mirror, the remaining light beam is only 90% of the fist reflection:

M2 = (M1)I = (MoI)I = MoI^2

By observing the correlation (relationship, likeness, similarity) between the value of the stage (n) or interval (physically each mirror reflection) of the reflection series, and the exponent of (I), we can generalize this and express it into an equation:

Mn = Mo I^n

Since the numerical value of Mo was not given we can consider it the whole (numerically as 1) starting value, or 100% (= 100/100 = 1) of the original beam.   This is sometimes referred to as doing calculations in percentage, portion, or relative terms. For this example, the ratio of Mn to Mo will always be the same for a given value of (n) since the loss is a "simple loss" and not compounded like interest.

Mn = MoI^n $\quad\quad\quad$ for the 5th mirror:
M5 = (1)(0.90^5)
M5 = 0.59049 = 59.049%

Checking: If Mo has a value of 34 units, the value of M5 is:

M5 = (34 units)(0.90^5)
M5 = (34 units)(0.59049)
M5 = 20.07666 units $\quad\quad\quad$ and:

GROWTH OF ORGANISMS EXAMPLE

If a certain type of organism divides into two organisms after about every 20 minutes, how many organisms will be present after 4 hours if given just 1 organism to start with?

Let's work out the basic data first and then try to derive a formula from that data. First let each growth interval (n) have a period (T) of time of 20 minutes long. Also, for this analysis, we do not know the average life span of the organism, so therefore, we will not consider any possible organisms that have died into the equations. The analysis is therefore more correctly for how many organisms have been born.

The basic formula between successive intervals or periods is:

organisms at n+1   =   (organisms at n) (2)   : after each interval, the number of organisms is twice
                                                                            the previous interval

n	0	1	2	3	. . .
organisms	1	1 x 2	(1 x 2) x 2	((1 x 2) x 2) x 2	. . . which could be expressed as:
organisms	2^0	2^1	2^2	2^3	. . .

Mathematically, we see that the number of organisms is actually a power of 2 whose exponent is equal to the interval or period number. Expressing this we have:

organisms n   =   2^n   : basic growth formula in terms of the interval elapsed

Since each interval is a 20 minute long time period (T), the total time (t) for this growth process can be expressed as the number of intervals times the time of each interval:

t = n T   therefore, algebraically:

$n = \dfrac{t}{T}$ expressing this into the basic growth formula we have:

organisms n   =   2^(t/T)   : basic growth formula in terms of time

Substituting the values given we have:

organisms   =   2^(4 hr./20 min.)

Since the units of time must be the same here, we must convert one to the other. We will convert 4 hours to an equivalent time of minutes:

$\dfrac{60 \text{ min.}}{1 \text{ hr.}}$ as= $\dfrac{x \text{ min.}}{4 \text{ hr.}}$ after solving for x min. :

x min. = $\dfrac{(4 \text{ hr.})(60 \text{ min.})}{1 \text{ hr.}}$ =   4 (60 min.)   =   240 minutes

organisms   =   2^(240 min. / 20 min.)   =   2^12 = 4096

If you want to express the formula for the number of organisms using (e), you can convert the value of 2 to its' equivalent power value of (e):

F1   : 1st note
F2 = F1(R) =   F1R^1   : 2nd note
F3 = F2(R) = (F1R^1)R = F1R^2   :3rd note
F4 = F3(R) = (F2R)(R) = (F1R^2)(R) = F1R^3   :4th note

Notice the correlation of the indicated power of R to the n_{th} note in question, and the indicated power of R is one less. Writing an algebraic formula to express this:

frequency of note n   =   (previous notes frequency)(R)   =   (first notes frequency)(R^(n-1))

F_n = F(n-1)R = F1R^(n-1)   : Frequency of the nth note of a common (chromatic, or 12 steps or notes)
                                                      musical scale

At the 13th note, or octave note, the frequency is always twice the first note:

2F1 = F1R^(13-1) =   F1R^12

Now, R can be completely solved for since the F variable can be eliminated. After dividing both sides of the equation by F1, and switching sides:

    R^12 = 2

Here, we can use the concepts of logarithms to solve for the base (here R) of the (indicated) power. Taking the logarithm of both sides of the equation using any base you choose:

    Log R^12 = Log 2   Using the "log power rule":
     12 Log R = Log 2

        Log R = $\frac{Log 2}{12}$ Hence, by the definition of logarithms:

    R = 10^((Log 2)/12)   again, using a log base of 10:
    R = 10^(0.301029995/12)
    R = 10^0.025085832

    **R = 1.059463094   : (R), Ratio Of The Frequencies Of Successive Notes (of the common 12 note,
                                          "chromatic" musical scale)**

Checking: If F1=1000hz, the 13th note, or "octave note", should be twice this frequency or 2000hz:

F13 = F1R^12 = 1000hz(1.059463094^12) = 1000hz(2) = 2000hz   : checks

Given a frequency of 500hz, the frequency of the next (or second "step" in the chromatic scale) note is:

Since: R = $\frac{next note}{note}$ = $\frac{Fn+1}{Fn}$ algebraically:

   Fn+1 = FnR^(n-1) = FnR^(2-1) = FnR^1

    F = Fn+1 = Fn(1.059463094)
    F = 500hz(1.059463094)
    F = 529.7hz   : (roughly 530 hz)

example, this will allow a song to be easily transposed to another key. For a major scale:

```
        1,   3 ,   5 ,   6,   8 ,   10,   12,  13        : tone numbers of the chromatic scale
        1,   2 ,   3 ,   4,   5,    6,    7,   8         : tone numbers of "major scale".
Fundamental or "Root" Tone,   W,   W,   H,   W,   W,   W,   H    : W = whole-tone or step
                                                          : H = half-tone or step
```

Knowing how to calculate the frequencies of the notes of a scale can be very handy for people such as computer programmers and some musical instrument designers and makers. Stringed instrument makers are often called luthiers, and the etymology (word source) of this word is derived from the stringed instrument called the lute.

From the study of physical science (physics), the "fundamental" or "natural" frequency of a vibrating string or wire can be calculated from the formula:

$$Fhz = \frac{1}{2Lm} \sqrt{\frac{Tn}{Mkg/m}}$$

: Formula by (Marin) Mersenne (1588-1648) from France, a part of 'Mersene's Laws'. Similar types of equations were later developed by Christian Huygens for pendulum motion, and Isaac Newton for the laws of gravitation. He was also an associate of Galileo Galilei.   Ts = 1 / Fhz = 1 / cps

L = Length of the string with meters (m) as the units of measurement
T = Tension (Force) applied to the string, and here it is essentially a "stretching" force.
    Force = (mass of an object)(acceleration of an object) = ma. In simple terms, a force is the application of energy.
    The units of force are commonly newtons (N), 1N = (1 kg)(1 m/s^2) = (mass)(acceleration) = force
    A force of 1N constantly applied to a 1K mass will accelerate (an increase in speed) it by 1 meter faster per second.
    Note that when a force being applied, there must be an acceleration or change, otherwise the force is not being
    applied. This force applied to a musical instrument string is usually adjusted by turning a tuning knob, and
    therefore adjusting the tension and natural (ie., "resonant", inherent) frequency of that string. Strings are subjected
    to some increased tension when pressing down on it to play a note, and the frequency heard could be slightly
    higher than expected. This is more noticeable on the thicker strings, and the design of the instrument may
    increase those string lengths so as to effectively reduce the frequency to the correct frequency. This concept is
    called something such as "bridge compensation" and "intonation". This may then create a problem that the "open"
    (unfretted) or full length string will have a note that is then slightly in error, and some instruments do consider this.
M = Mass of the string in kilograms per unit (meter) length. This is also called the linear or length density of the string.
    "Deep", "low" or "base" sounding stringed instruments usually have thicker and heavier (ie. more mass/length)
    strings so as to produce lower frequency waves and-or sound more efficiently. F is inversely related to L and M and
    directly related to T.   The units of mass = M used here are (kg / meter of length = kg/m)

As a basic verification to the above formula, here is a derivation of a similar formula: Work = **Energy** = (force)(distance) = (f)(v)(t) = (Mass)(acceleration)(distance) = (M)(a)(d) = (M)(d/t^2)(vt) = (M)(d/t)(v) = (m)(v)(v) = mv^2 = m (d^2/t^2) = md^2 / t^2, and after dividing both sides by (d), we have: Force = md / t^2 , if we use M=m, and Tension for force we have: T = Md / t^2 ,   solving for t, we have: t = sqrt (Md / T) , and taking the reciprocal of time, we have frequency = = f = sqrt (T / Md) = sqrt (T / M Length)

Right now, we will be mainly interested in the relationship between the length of the vibrating string and frequency produced. These variables in the radical, for all practical purposes, will remain constant for a given situation, such as of a particular string. The radical factor (which resolves to a constant for our particular study here) can be completely eliminated since it is theoretically possible to have a tension of 1 newton applied to a string which has a mass of 1 kilogram per meter, and the square root of 1 is always 1. The formula is then reduced to just the fundamental (inverse) relationship of the frequency (f) and length (L) variables:

$$Fhz = \frac{1}{2Lm} = \frac{1}{t} \quad \text{solving for L:} \quad Lm = \frac{1}{2Fhz} = \frac{1}{2(1)(t)} = \frac{t}{2} \quad \text{and} \quad t = \frac{1}{Fhz} \quad : = 2L \text{ for this string where } T=1N \text{ and } M=1kg/m$$

Length of string at Fret 1   = 0.943874313   algebraically:
          25inches

     Length of string at Fret 1 = 0.943874313(25in) = 23.59685783in

Hence the difference in string length, or the fret size (length or space) for the next note on that 25in long string is:
     25.0 in - 23.59685783 in = 1.403142175 in   :first fret size

Due to the constant ratio of successive string lengths of successive notes, their is a constant ratio between successive note or fret positions. The length of string between those two successive notes is also known as the "fret size" to musical instrument makers. Given just a fret size, the distance to the next successive fret, and so on, can be calculated without knowing or calculating the frequencies of notes and their string lengths. For example, once the first fret size has been calculated for a length of string, the next fret size and others can be calculated from that.

     second fret size   = 0.943874313
        first fret size

     second fret size = 0.943874313 (first fret size)   Using the above value for the first fret size for a 25.0 in string:
     second fret size = 0.943874313 (1.403142175in) = 1.324389856 in

Likewise, the size of the third fret can be calculated using the size of the second fret:

     third fret size = 0.943874313 (second fret size)
     third fret size = 0.943874313 (0.943874313 (first fret size))
     third fret size = 0.943874313^2 (1.324389856 in.) = 1.250057565 in.

Finding all these fret sizes can be done quickly with either a computer with this equation programmed into it, and-or with a calculator. Since many calculators retain the last input "function" or operation (here, multiplication), it is simply a matter of pressing the "equals" or "enter" key so as to find the next successive result (here, fret size) of using the last displayed result or "output value" as the operand of that same function.

Notice that in order to calculate the (n)th fret size, that the exponent of the FRET SIZE RATIO (FSR) is one less than the (n)th fret size in question, and that the first fret size for a given string length or "(complete, possible) scale length") can be used as a constant factor for successive fret sizes. Expressing this in a formula:

Fret Size n = (1st fret size) $FSR^{(n-1)}$ = (1st fret Size) $0.943874313^{(n-1)}$: **Fret Length (size, space) Formula.**
                                                                      1st fret size = (FSR)(string length)

Incidentally, and very conveniently, note that FRET SIZE RATIO = 1/(Note Ratio) = 1/NR , and this is verified below where NR is the ratio of the frequencies of successive notes in the standard chromatic 12-note (per octave, or frequency doubling) scale:

Letting L2 be the string length of the next higher note in the chromatic scale, and produced on a string of length L1:

$$\text{FRET SIZE RATIO} = \frac{L2}{L1} = \frac{\frac{1}{2(F2)}}{\frac{1}{2(F1)}} = \frac{2F1}{2F2} = \frac{F1}{F2} = \frac{F1}{F1(R)} = \frac{1}{(NR)}$$

FRET SIZE RATIO = FSR = 0.943874313 = $\frac{1}{1.059463094}$ = $\frac{1}{NR}$: FSR and NR are reciprocals of each other.
                                                                      : NR = ratio of frequencies of successive notes

NR = Note Ratio = 1.059463094 = the ratio of frequencies of successive ascending notes in the chromatic scale.

short scale bass guitar, such as the Hofner co., violin shaped bass is said to have a 30" scale length, and the neck feels somewhat like a standard guitar neck that he was originally used to. A Fender co. Precision Bass and-or Jazz bass has a 34" scale length. A recent and helpful discussion about playing guitar and music notation was placed near the end of this book, and for luthiers, it includes the fret spacings of the common scale sizes.

CHANGING THE PERCENTAGE OF A SUBSTANCE IN A MIXTURE

Ex. If a bleach solution for household use is noted as being 6% chlorine bleach per volume (say 1 cup of volume = 8 fluid ounces (fl. oz) of that solution is being used for this example) and it is diluted (combined, lessened) with 1 cup of water, what percent of that total mixture or solution is chlorine bleach?

   Since plain water substance is being added into this volume of solution, it will reduce or dilute the percentage of any other substance (here chlorine bleach) in that total volume.

   6% chlorine bleach of 1 cup or 8 ounce is: 8oz (0.06) = 0.48 oz of each cup is chlorine bleach

   Total volume of the mixture or solution = 1 cup of bleach solution + 1 cup of water solution = 2 cups =
                                             = 8oz + 8oz = 16oz

   This 16 oz of solution still contains only 0.48 oz of chlorine bleach, and the percentage of bleach in this total solution
    or mixture is:

   $\dfrac{\text{0.48 oz chlorine bleach}}{\text{Total mixture or solution}}$ = $\dfrac{\text{0.48 oz}}{\text{2 cups}}$ = $\dfrac{\text{0.48 oz}}{\text{16 oz}}$ = 0.03 = 3% of this mixture is chlorine bleach, regardless of
                                                                                                 the volume used. "a 3% solution of chlorine bleach".
                                                                                                 (Total mixture) must be > 1 unit volume.

   Note, the total mixture contains: 2 cups = 1 cup of bleach solution + 1 added cup of water   .

   This formula will also work if the reference volume, here cups, is some other reference volume unit such as a for example 1 to several fluid ounce measuring cup, and where the bleach solution will still be 6% per that volume.

   We can create an equation for the number of total number of (generic, non specific in size) volume units of solution needed so as to create a solution with a desired percentage (or dilution) of the chlorine bleach:

   $\dfrac{\text{6% chlorine bleach per unit volume}}{\text{Total mixture solution in volume units}}$ = Desired percentage in total volume ,   from this, we mathematically
                                                                                                                 have:

   Total mixture solution in volume units   = $\dfrac{\text{6% chlorine bleach per unit volume}}{\text{Desired percentage in total volume}}$

      Here, 1 volume unit will from the original 6% chlorine bleach mixture, and the amount of water to add into this solution will be = (Total mixture solution in volume units) - (mixture solution of 1 unit volume).   The desired percentage in the total volume must also be 6% or less, since all the solutions will be a diluted ("thinned", less of a percentage of chlorine bleach) solution.

      Ex.   6 units of total volume = 6% solution / 1% desired solution , : hence add (6-1)units = 5 units of water so
                                                                                          as to create a 1% chlorine bleach solution

      Rather than always use 6%, the above formula can even be made more generic or useful by using a variable such as P for the % of the substance in the original or given undiluted volume. Pharmacists and chemists use similar formulas.   For the formula use entire and generic volume units, rather than a specific volume such as 8 fluid ounces. Later, you can then convert each volume unit to its actual associated units.

mixture.

1000cc + 666.67cc = 1666.67cc   : total volume of final mixture

5% of 1000cc = 50 cc   : amount of substance A in the original or starting mixture
3% of 1666.67cc = 50cc   : A's percentage or portion of the entire new mixture is now reduced,
                                          yet the physical amount of A in the new mixture remained the same.

95% of 1000cc = 0.95 (1000cc) = 950cc   : amount of substance B (water) in the starting mixture
97% of 1666.67cc = 0.97 (1666.67cc) = 1616.67cc   : amount of substance B (water) in the final mixture

Also: starting B + added B = ending B

        950cc + 666.67cc = 1616.67cc   : total volume amt. of water in final mixture

New amt. of substance A + New amt. substance B = New amt. of total mixture
        50cc   +   1616.67cc   = 1666.67cc

Now we will look at some various other ideas and solutions, in particular, using fractions for the percentages:

A's volume in starting mixture = A' volume in ending mixture

        $100\%A = 100\%A$: A's physical volume or amount is the same in both the
                                            starting and ending mixtures.

        $(0.05)(v1) = (0.03)(v1 + x)$: v1 is total starting volume, x is amt. to add to total volume.
                                            Notice that in order to keep this equation in balance, as one
                                            factor decreases (here 0.05 decreased to 0.03), the other
                                            factor to that total decreased value must increase (here,
                                            v1 increases to (v1 + x)).

                                            We could assign or let v2 = (v1 + x) = ending volume:

        $(0.05)(v1) = (0.03)(v2)$ Solving for v2 we would get: v2 = 1.667v1.
                                            Note here the "reverse ratios": (v1 / v2) = (0.03 / 0.05)
                                            If we let v1=1=100% of the whole or initial starting volume,
                                            then x essentially becomes the percent increase in volume:

        $(0.05)(1) = (0.03)(v2)$ solving for v2:

   $v2 = \dfrac{(0.05)(1)}{0.03} = 1.66\underline{7}$: hence the starting volume must be increased by this
                                            factor, where: v2/v1 = 1.66$\underline{7}$/1 = 1.667 = 166.7%, also:

        $(0.05)(1) = (0.03)(1 + x)$ solving for x:

        0.05 = 0.03 + 0.03x
        0.05 - 0.03 = 0.03x

        0.02 = 0.03x

        $x = \dfrac{0.02}{0.03} = 0.666\underline{7}$

mixture or construction, and only 3% in the ending mixture or construction being made. The percentage of water in both mixtures is also a different value. Whereas one percentage (here, of water) will increase, another percentage (here, substance A) will effectively decrease and this is not a direct mathematical relationship but an inverse mathematical relationship. Also, whereas the physical amount (here, volume) of water increases, the total physical amount (volume) of substance A remains the same in both (non-mathematically similar, or non-proportional) mixtures or constructions.

This below was briefly mentioned above, and will be mathematically explained further here:

Consider for example that: 2 is 2% of 100   and   2 is 1% of x

The solution to this considers that to keep the same value (here 2) but only reduce the percentage that is of the whole or total, then the whole or total must be increased. This is an inverse mathematical relationship.

2 = 2% of 100 = 1% of x   : here 100 is the starting quantity or "whole entire value". This can be expressed as:

   $(0.02)(100) = (0.01) x$ this can be expressed as two equivalent and inverse ratios: solving for x:

   $\dfrac{0.01}{0.02} = \dfrac{100}{x}$ in pseudo: $\dfrac{(new\ percent)}{(old\ percent)} = \dfrac{(old\ total)}{(new\ total)}$: notice the "inverse" ratios

   $x = \dfrac{(0.02)}{(0.01)} 100 = (2)100 = 200$: the percent ratio decreased by a factor of (0.02/0.01) = 2, and the whole value increased by that same factor of 2. Or in other words:
(0.01/0.02) = 0.5   and the reciprocal of 0.5 is: (1/0.5) = 2
The percent was multiplied or "increased" by a factor of 0.5, and the whole value was multiplied or "increased" by the reciprocal of that factor = 1/0.5 = 2

Here are some common cooking examples involving percentages:

Ex. A recipe for a certain type of bread requires 1/4 = 0.25 = 25% of the total flour mixture to be whole wheat. This can be easily done by using 3 measurements (scoops, cups, bags, etc) of white flour, and 1 of the same measurement of whole wheat. This will create a flour mixture with a total of 4 measurements, and with 1/4 being whole wheat flour.   1/4 could be thought of as 1 part being whole wheat of 4 total parts. If the recipe was for 1/3 = 0.33 = 33% of the flour mixture being whole wheat, for each 2 measurements of white flour, use 1 measurement of whole wheat flour.   1/3 could be thought of as 1 part being whole wheat of 3 total parts. For a 1/5 = 0.20 = 20% whole wheat mixture, use 4 measurements of white flour, and 1 measurement of whole wheat flour which makes 1 part of the 5 total parts. For a 1/10 = 0.10 = 10% of the whole mixture, use 9 measurements of the main ingredient(s) and 1 measurement of the ingredient that will compose a 10% of that entire mixture.

Ex. A recipe states to use 2 and 1/4 cup flour = 2.25 cups of flour. If you decide to use 25% = 1/4 of the total flour mixture with whole wheat flour, how much whole wheat flour will you need, and how much regular white flour will you need?

   whole wheat flour = 25% of the total flour mixture
   whole wheat flour = (0.25)(2.25) cups
   whole wheat flour = 0.5625 cups

   total flour = white flour + whole wheat flour
   2.25 cups = white flour + 0.5625 cups
   white flour = 2.25 cups - 0.5625 cups
   white flour = 1.6875 cups

GEARS EXAMPLE

This example and explanation will show how a simple gear system of two or more connected rotating gears or wheels of different sizes can be used to change the resulting rotational speed or the torque (a force that cause a turning or rotation about a point, hence torque is a rotating force or force of rotation) of one of those wheels or gears. The appendix also contains a few defined terms relevant to this example that you may optionally review if needed. [FIG 61]

A WHEEL OR GEAR

Rod Or Axle

Front View   Side View

BASIC GEAR SYSTEMS OF TWO OR MORE GEARS

Output

Input

Input

Output

Speed Increase
Torque Decrease

Speed Decrease
Torque Increase
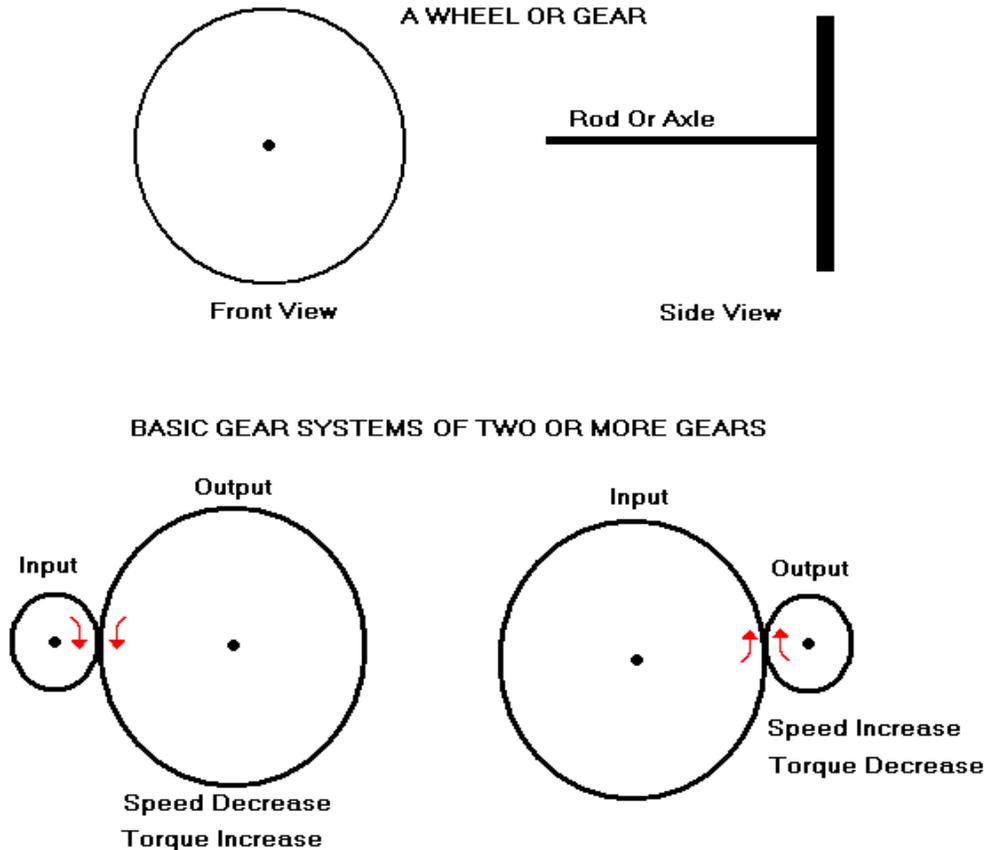
The rod ("drive shaft") or axle is placed at the center of the wheel and can be used to hold the wheel in position for it to rotate with stability and efficiently, and-or to apply or transfer power (from elsewhere, perhaps a hand-crank or motor) to the wheel or gear so as it can rotate about it's center ("hub"). For a given rotational speed per unit of time, the larger the wheel and its circumference (ie., distance, length), the greater the distance a vehicle would travel per unit of time. It's also possible that the wheel can be used to apply power to the axle, such as in a water powered wheat grinding or saw mill. A large water wheel does not rotate very fast but it collects the high input force of the water to turn it, and this speed or revolutions per minute (rpm) of this can be changed later by using gears if needed.

Many vehicles have gearing to change the speed or torque of its wheel(s) so as to improve the vehicles movement in certain driving situations or needs. Many other machines also use some type of gearing to alter (change) the speed or torque of some moving part so as to improve it's usefulness and efficiency in some way.

A gear is much like a wheel, however the rim or edge of it has what is called the teeth of the gear, and these are somewhat triangular in shape and allow efficient contact with another gear with the same size (but generally not the same number) of teeth, and so as to transfer power from one to the other without slipping past each other like a wheel might.

Gears or wheels of the same or different sizes (ie. due to their radius and-or circumference values) can be used to

of the equation in balance and also have conservation (ie., same net sum of input and output values, no loss, maintain) of energy. Mathematically, from the equation we have these ratios:

$$\frac{\text{input speed}}{\text{output speed}} = \frac{\text{output torque}}{\text{input torque}}$$

: = same "magnification or gain factor" or
ratio value of these inverse ratios or
proportions

Given two fixed (having a constant circumference) sized wheels or gears, the speed or torque can be altered (changed, increased or decreased) in the output gear when it's size is not the same size as the input gear. If the size of the output gear is less or is decreased, it's speed (revolutions per second, rotation or angular speed, or velocity) will increase.

Without knowing the specific value of the applied forces (torque) or speeds of any gear, we can calculate the corresponding magnification or "gain factor" in either the speed or the torque by knowing the size of the two gears and calculating their ratio (ie., the mathematical relative, fractional, or portion of one with respect to the other). For gears and wheels, and considering speed and torque, the size value is either the circumference, diameter or radius of the gears. Given two gears or wheels, the ratio (r) of the corresponding values chosen will be the same as any other corresponding values chosen:

$$\frac{\text{circumferenceA}}{\text{circumferenceB}} = \frac{(2)(\text{pi})(\text{radius A})}{(2)(\text{pi})(\text{radius B})} = \frac{\text{radius A}}{\text{radius B}} = \frac{(\text{pi})(\text{diameterA})}{(\text{pi})(\text{diameterB})} = \frac{\text{diameterA}}{\text{diameterB}} = \text{size ratio of two circles, wheels and-or gears}$$

When a larger sized input gear is applying power (ie. force, torque) to a smaller sized output gear, the smaller gear will then rotate more (per unit time) and to help understand this, consider that a point on its circumference must and will travel and equal the same circumference or distance length as that of a point that travels on the larger input gear when it rotates just once.   Expressed another way, due to the "junction" or "connection" point at both wheels, both wheels will actually rotate the same amount of distance along their circumference, however with the smaller wheel or gear, that distance corresponds to a much larger central angle and amount of degrees. In short, when a larger input gear spins once, a smaller output gear will spin more than once. This also means that the smaller gear will rotate "faster" or more times per amount of a given time, hence its rotational or "angular" speed is more. [For a potential future reference, you may consider this equation: ¢2 = ¢1 (factor), and that this factor is equal to (C1/C2) or= (r1/r2) , and mathematically: C1 ¢1 = C2 ¢2 , and: (¢1 / ¢2) = (C2 / C1) , which are "reverse ratios".].

Circumference = (pi) Diameter = 2 (pi) Radius , and when given two wheels or gears:

larger wheel size / smaller wheel size = (x) = wheel size magnification factor of the larger wheel or gear
This value will be >1 since the numerator is larger.

therefore: (smaller wheel size) (x) = larger wheel size
        or: (smaller circumference distance)(x) = larger circumference distance

Consider that 1 rotation is equal to the circumference length or distance, therefore:

(1 smaller distance)(x) = 1 larger distance

These distance values correspond to (whole or fractional) rotations of each wheel or gear:
(1 rotation of small wheel)(x) = many rotations of small wheel = 1 rotation of large wheel

Therefore, a smaller output wheel or gear will correspondingly rotate (x) times more or faster so that a point on it's circumference will travel a distance equal to a point traveled on the circumferences of the input gear or wheel. It will then be said that the speed (number of rotations or revolutions per second), also known as "angular speed or velocity" of the smaller gear (in this two wheel or gear system) is more (greater), and is sometimes, somewhat incorrectly noted as being "faster" (even though it, and-or a point on its circumference would actually travel the same (linear) distance as the larger gear or wheel would in the same amt. of time).

From: gain in torque = output gear size / input gear size

output gear size = (input gear size)(gain in torque)   and for this specific example:

output gear size = (input gear size) (2)   : the output gear size should be twice (2) the size of the input gear

Note that since torque in this specific system will be increased, that the associated speed will be decreased by the reciprocal of this gain (increase, magnification factor) in torque:

gain in speed = (1 / gain in torque)   and for this specific example:

gain in speed = 1 / 2 = 0.5   : the rotation speed of the output gear will be half (1/2) of that of the input gear

If for example the speed of the input gear was 100rpm = 100 revolutions per minute of time, the output gears speed would be:

From: output speed / input speed = gain in speed   mathematically:
output speed = (input speed) (gain in speed)   or:
output speed = (input speed) (1 / gain in torque)   and for this specific example:

50rpm = (100rpm)(0.5)

It is possible that two gears or wheels of different size can be on the same rotating shaft or axle. They will both rotate at the same speed or angular velocity (ie., rotatonal velocity, ex. degrees per second) but a point on the circumference of the larger gear will have a greater linear velocity.   This type of simple machine is much like a **pulley** described below, and is essentially the reverse of a wheel and axle system and where here, the input force or power is applied to the larger wheel rather than the smaller diameter shaft or axle. This system is sometimes often seen in some simple hand powered (ie. cranked, rotated, spun, turned [torqued]) machines that get water from a deep well more easily with less input force needed, but may also be seen in pulleys for raising objects up to a height, such as into a barn. This machine may be sometimes called a compound wheel system.   The gain in input force is equal to the ratio of the radius of each wheel and is essentially like the ratio of two lever arms. The larger "input wheel" applies a larger torque upon the axle and-or upon any other smaller wheel on that axle. This system is often called a "wheel and axle" system.

 A common example where torque (force) will be increased mechanically is by pulleys (a rope or chain, and wheel system), and electric winches which contain an internal gear system so as to pull (apply force) heavy loads. A **pulley**, like a lever, is considered as a simple mechanical (force driven) machine that provides a "mechanical" or force advantage (ie., a greater output force than the input force) . A basic ("fixed") pulley consisting of just a wheel affixed to an above height and rope to apply ("pulled, tension) forces does not offer any mechanical advantage, but essentially changes the directions of the input and output forces, and the object is still essentially pulled, but in an upward direction instead of along the ground. See the figure below. Once an object is raised to a height, it can be set fixed at that height by "anchoring" the rope to the ground or to a heavy object.   If the pulley wheel is not affixed to a height, and the object is affixed to the pulley on the rope, this pulley can still be used to lift the object to a height. One end of the rope is affixed to a height, and the other end is where the user can apply force. The advantage of this system is that anchored or affixed end of the pulley will support half the weight of the object being raised to a height. With a "compound pulley" or "movable pulley system" [also contains another pulley(s) non-fixed or movable along the rope or chain] with many pulleys working together, a larger portion of the object weight will be held by the pulley system and its fixed support, and therefor creating an improved mechanical advantage and making it even easier to lift an object, but requiring more distance and-and time by the user. The total work required to lift an object a height will still be the same as that without using a pulley, but it could be said that the work is easier and-or less stress on the body with the pulley:

As indicated above, a pulley is an example of a "simple (mechanical, with motion and force) machine" or mechanism to assist (called a "mechanical advantage") in doing something, but as we can see by the numbers, that there is also a "trade off" or "cost" of having an increase in one variable, and that there is a decrease in another variable(s). For an electric winch or pulley, the advantage is a gain in force, but the disadvantage (ie. a mechanical disadvantage) is that it will take a corresponding gain in time to go (ie. lift up to the same unaided height, or pull the same unaided distance) the same distance because speed has decreased by the same factor:

 From: distance = speed x time

 time = distance / speed   We see that time and speed are inversely related. For a fixed distance, if speed is decreased by some factor, then the corresponding time value necessary to go that same distance will increase by that same factor.

This note is mentioned further ahead in this book when discussing power and torque, and it is also included here due to its importance:

In terms of torque: power = (Pw) = (torque) x (rotation speed) = (force)(Lever arm)(rotation speed)

The units of force are usually called Newtons, and the units of power are Newton-meters (Nm). We see in this equation of power, that torque and speed are mathematically inversely related for a given (fixed, constant) amount of power as such for the input and output power of gears or wheels. Note that even though they are inversely related, they are generally not mathematical reciprocal in value to each other, however it is important to note that in a gear system, that the specific corresponding gains (increases, multiplying factors) or losses in torque and speed are mathematical reciprocals of each other. Please also see the article: **An extra note about torque and power** in the late entries section of this book.

Closely related to gear analysis is lever analysis of which an article is presented further ahead in this book. Consider a smaller output wheel, it has a smaller radius, and therefore, it essentially has a smaller leverarm. Since Torque = (force)(leverarm), if leverarm decreases, the torque (T) decreases, but its speed (ie., angular velocity = degrees per time) will increase. Consider this mathematical relationship: (Tin)(AngularVelocity in) = (Tout)(AngularVelocity out)

When a wheels or gears increases its rotational or angular speed, the effective linear velocity of a point on its circumference also increases by the same factor.

For an example of where gearing is used to increase torque is that when a vehicle is going up a hill and it will need more force to do that than as compared to a flat or level road. This can be accomplished by using a larger power output gear so as to increase the torque (ie. [rotating] force) of the wheels, but of course it will be at a lower speed, but still, is will make it much more possible for that vehicle to reach the top of the hill. A vehicle at rest will initially need more torque to get it moving to overcome friction and to get it to its desired velocity. It takes a force (ie., torque) to alter the speed of an object. Once the vehicle is at the desired traveling speed, it will slow a little due to the wheel friction (ie., resistance, drag) with the roadway, and it will take only a relatively small applied input power and force to maintain that desired speed.

With any two or more wheel or gear system, given a system or motor that is turning just one output wheel such as for a vehicle, if that wheel size is increased, the speed of the vehicle will increase. If that wheel size is decreased, the torque of that wheel will increase.

The maximum amount of torque can be measured by using weights suspended on a string around the rotating shaft, and the weight can be be lifted vertically upward. It is also possible to use a common weight or mass scale to measure torque in a mechanical type of linkage and-or manner. According to Power = (torque)(rotational speed) , mathematically: torque = Power / (rotational speed).   Theoretically, if the input power to a very efficient motor is:

Pw = (V)(A) = Pmotor,   the max. torque of an electric motor can be calculated by: torque = (V)(A) / (rotational speed)

through a resistor, and that power is then unavailable to be used by the rest of the circuit. Though unavailable to the rest of an electric circuit, the energy or power (a measure of available or energy used. Power = voltage x current = P=VI, and units of power are called watts) wasted in and through a resistor is not actually lost but was rather converted to other forms of energy such as: heat from a resistor, motion from a motor, sound from a speaker, and-or light from a light bulb. From this concept, it is said that energy is "conserved", "not lost", or "is maintained" and that no amount of energy actually disappears. The sum or total output energy is equal to the sum of total input energy, and this is called "the conservation of energy".

So that a formula can be developed for the total (effective) resistance of two or more resistors in parallel to each other, the topic of conductance needs to be considered. Conductance (G) is a measure of the ability to pass or allow current, and it is therefore, the exact opposite to the concept of resistance, and it is mathematically expressed as the inverse or reciprocal of resistance:

$$G = \frac{1}{R}$$

: the units of conductance are often called "mhos" which is derived from the word "ohm" spelled backwards, and that resistance and conductance are electrical opposites or inverses of each other.

If you consider a wire as like a pipe to carry or allow the conduction or flow (ie. current) of water, and if you had two pipes in parallel (along the side length of each other), you will have a greater ability or more "room", "space" or area to allow more flow, perhaps for filling up some container with water. In relation to a wire, you would have more current ability, and hence this is described as having more conductance. Total conductance is simply expressed mathematically as the sum of each (parallel) resistors conductance ability. When conductance increases, resistance decreases. When resistors are placed in parallel, the effective or total conductance of that system will increase, and the resistance will decrease. When resistors are placed in series, the total resistance increases and the conductance will decrease.

Note, it may now seem that the total resistance of parallel conductors or resistors is: Rt = 1/G1 + 1/G2 + 1/G3 + . . . , but this is actually just another expression for the total resistance of series resistances or resistors. To find the right expression for parallel resistors, first consider this expression for total conductance of parallel resistors:

$$Gt = G1 + G2 + G3 + ... = \frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} ... = \frac{1}{Rt}$$

: conductance of a system of parallel conductors or resistors. Mathematically:

$$Rt = \frac{1}{Gt} = \frac{1}{G1 + G2 + G3 + ...} = \frac{1}{\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} ...}$$

For only two parallel resistors, the formula reduces to a simple expression to memorize:

$$Rt = \frac{1}{\frac{1}{R1} + \frac{1}{R2}} = \frac{1}{\frac{R1 + R2}{R1 \times R2}} = \frac{R1 \times R2}{R1 + R2}$$

: Formula for two parallel resistors. Their product, divided by their sum.

A common example is using two of the same valued resistors in parallel, and the total effective resistance is simply half of that common value. Note that parallel resistors always create a lesser total or effective resistance because the total or effective conductance increases when two resistors are in parallel.

If R1 = R2:

$$Rt = \frac{R1 \times R1}{R1 + R1} = \frac{R1^2}{2R1} = \frac{R1}{2} = 0.5\,R1$$

: Rt is half of the common resistance value

Some considerations on the value of the effective or total resistance of two parallel resistors:
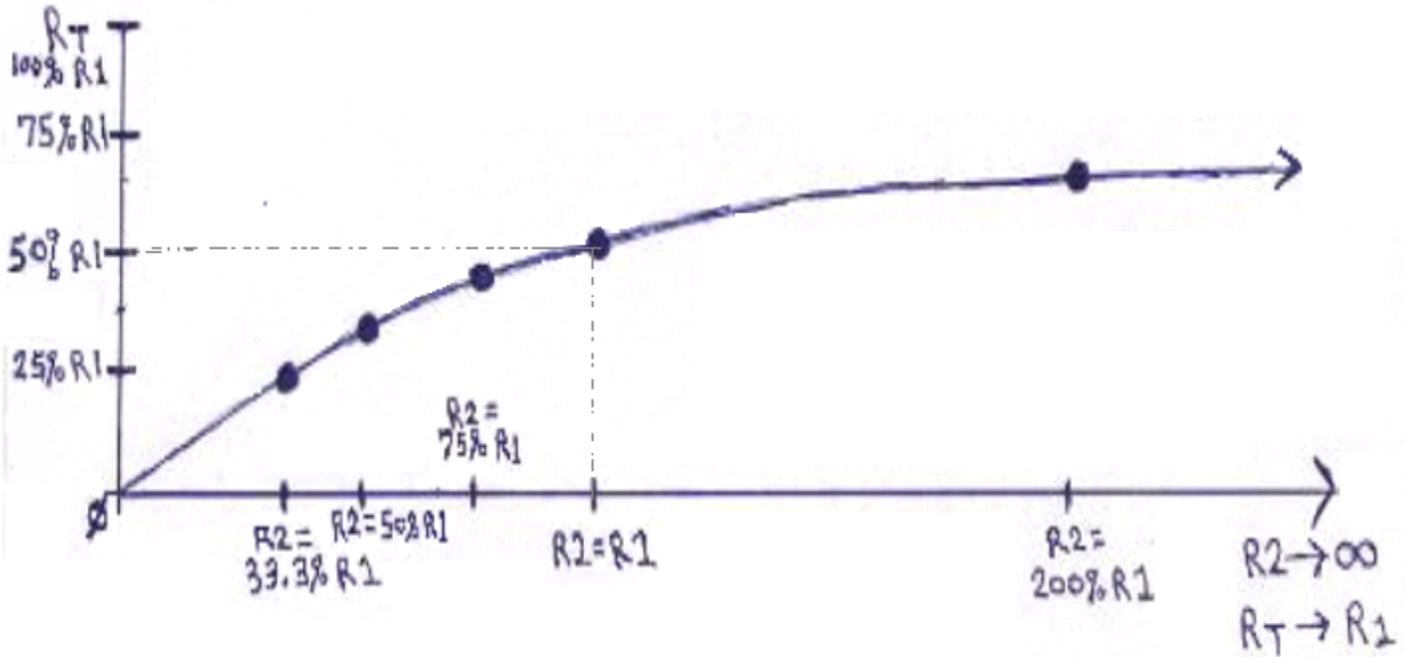
As R2 increases the value of Rt (the effective parallel resistance of two or more resistances or resistors) does not change much. The slope of the curve is becoming near horizontal or level. This is due to that the rate of the changes in the corresponding values of each variable is getting less and less, and is approaching a value of 0, hence Rt is not changing by much in value with respect to the changes in R2.

Rate or ratio of changes of the dependent variable with respect to the independent variable:

$$\frac{\text{change in dependent variable}}{\text{change in independent variable}} \ = \ \text{ratio or rate of changes} = \text{slope}$$

This slope value is how the dependent variable will change when the independent variable changes.

$$\frac{\text{change in Rt}}{\text{change in R2}} \ = \ \frac{(Rt.2 \ - \ Rt.1)}{(R2.2 \ - \ R2.1)} \ = \ \text{slope}$$

Since a curve does not have a constant slope value like any line would, we can consider the curve as composed of just many small line segments, and therefore, we will make the change in the independent variable to be as small (or "close") as possible so that points chosen for this equation are as if they are on the same line or line segment, and that point or section of the curve can be analyzed like a small line segment for its slope value.

$$\frac{\text{the corresponding small change in Rt as R2 changes}}{\text{a very small change in R2}} \ = \ \text{slope of the curve at one point} \ = p(x,y) = p(R2,Rt)$$

The parallel resistance effect creates the condition that as Rt approaches a certain limiting value that is determined by and equal to R1, that Rt will approach (become "nearer and nearer to", or "closer and closer to") the value of R1, and that the changes in Rt, as R2 changes, get smaller and smaller until there is practically 0 change in Rt. The curve becomes near horizontal, like a horizontal line, as R2 increases.

involved is huge, and the frequency of the electro-magnetic radiation depends on the amount of energy applied to the electrons.   Consider a wire that is warm (ie., heat, thermal energy and radiation) due to its resistance and with some electricity going through it. If the amount of electric energy applied gets higher, that wire could radiate more heat and reddish light such as is the case for an electric heater. If the amount of energy applied gets even higher, heat and white light will be radiated, such as is the case for an incandescent light bulb. White light has also been found to be a mix (ie., a "rainbow") of colors such as red, green and blue, and with each color due to the specific frequency (ie., of a wave or pulses) of that electro-magnetic radiation, with each individual pulse or packet of it being called a photon particle. The electric and magnetic fields of each photon are at right angles to each other and are sinusoidal in amplitude (ie., intensity). The intensity of the light perceived depends on the amount of photons received per unit of area. In theory, once cycle of the electromagnetic wave corresponds to one photon of electromagnetic energy.

EQUATIONS AND SHAPES OF COMMON CURVES

For basic familiarity, below are some basic equations and a representative shape of the resulting curves when those equations are plotted on a graph. Some, such as the line, have already been discussed and others will be presented. [FIG 64]

LINEAR (LINE) : $y = mx + b$
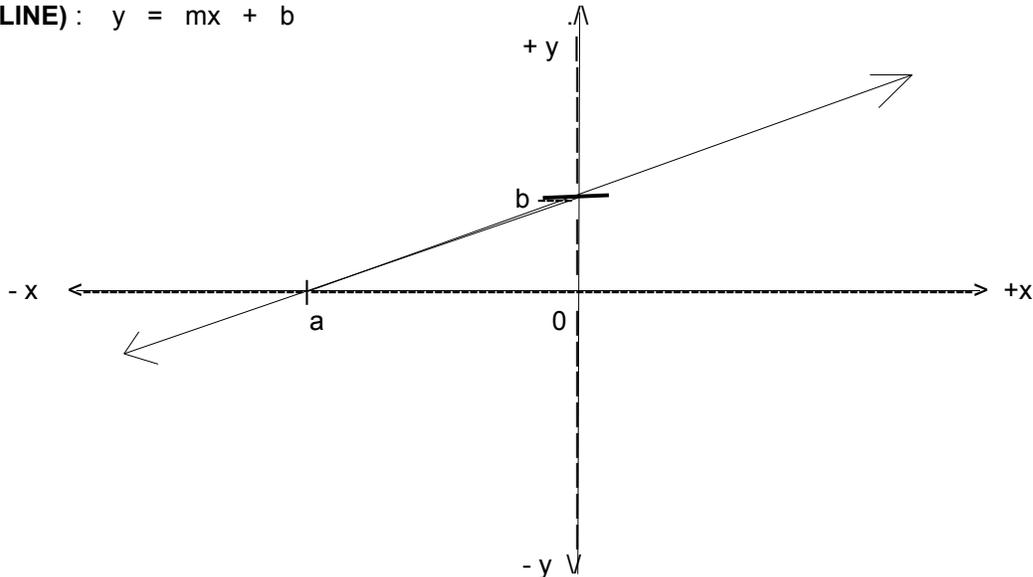
A line is technically considered a special instance of all possible curves. As a simple verification, consider the curve or circumference of a circle being so huge that a relatively small portion of it, perhaps just a portion or arc of the circumference that corresponds to of just 1° (one degree) of rotation or movement of a point on it as it moves and rotates about the center of the circle, and is practically a line with the "bend" or curve in it being ever so slight that it is imperceptible (barely perceived or seen, very slight, small) when you are close to it. In more advanced reasoning, a very short or small segment of any curve will also resemble a line segment (a part of a line) and could be (locally, at that specific location, section, or point on the curve) analyzed as such, as for the slope of the curve at that point or location. A common example that considers this is with a concept called interpolation where an unknown value that is between two known values of an equation and-or a curve can be calculated like a point on a line or line segment.

The (y) or vertical intercept point, where x=0, is P(0, b). This can be found by substituting 0 for (x) in the linear equation and then solving for (y). To find the (x) or horizontal intercept point, where y=0, set (y) equal to 0 and solve for its corresponding (x) value. This (x) value is generally noted as variable (a) for a given line, and like (b), it is a constant for a given or certain line. The result is P(x,y) = P(xi, 0) = P(a, 0), where:

From: $y = mx + b$: setting y=0 so as to find the x-axis intercept point:

$0 = mx + b$ solving for the corresponding value of x for this point:

$mx = -b$

$x = \dfrac{-b}{m}$

$xi = a = -\dfrac{b}{m}$: x-coordinate of the **x-axis intercept (i) point**, or simply the "x-intercept".
Algebraically, we also can find: **m = - b/a** ,   b= - ma ,   and

POWERS : y = x^n   : a power equation is when the variable is the base value, and the exponent is constant

x is a variable, and x >= 0, n is a constant, and if n >= 1 , that is, the expression is not actually a root.
If 0<n<1 (ie., the expression actually represents a fraction (ie., < x) amount or root of x, and with this mathematical condition, the value of y will get smaller as x increases.)
[FIG 65]
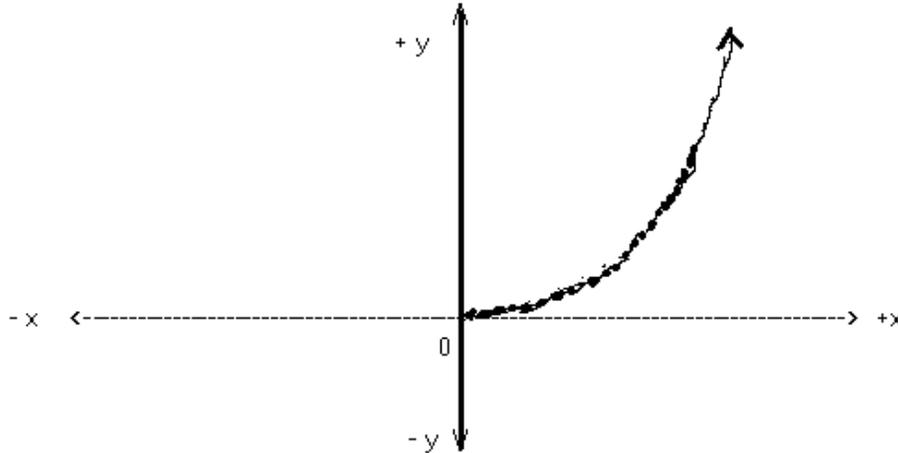
For powers, when (x), the independent variable is greater than 1 and gets larger, even small increases in it will cause a huge increase or change in the value of (y)., the dependent variable.

ROOTS :   y = n$\sqrt{x}$ = x^(1/n)   : n is the index or indicated root of the radicand x

x is a variable, and x >= 0 (ie., x is not a negative radicand), n is a constant.   If n<1, the expression is actually a power of x.   [FIG 66]
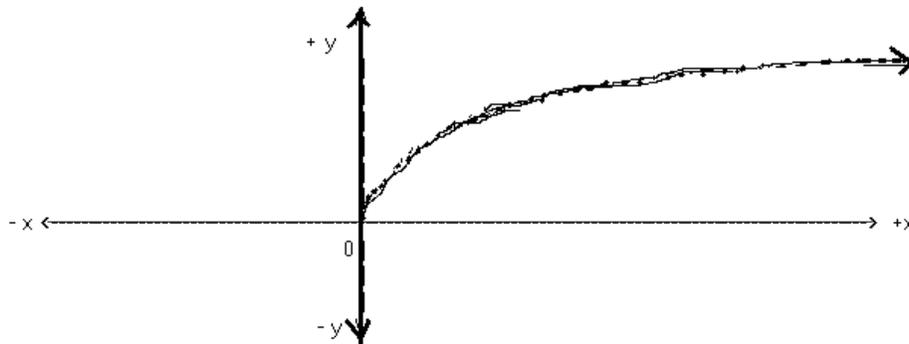
As x grows larger in value, the curve resembles and will approaches that of a horizontal line which has a slope of 0. This means that the relationship of y to x, or the root with respect to the radicand, is not growing as much or quickly (rate) as when the radicands are lower in value. Consider x=radicands with the low values of 16 and 100. Their difference is 84, and their y=square roots differ by $\sqrt{100}$ - $\sqrt{16}$ = 10-4.0 = 6. Now consider higher value radicands of 10000 and 20000. Their difference is 10000, and yet their square roots differ only by $\sqrt{11000}$ - $\sqrt{10000}$ = 104.88 - 100 = 4.88

EXPONENTIAL :   y = n^x   : an exponential equation is when a variable is in the exponent, and the base is constant

Here, x, the independent variable, is an exponent, and this is why the equation is called an exponential equation.   If you were to solve for (x), it is equal to: x = logn y = log y = log y / log n
                                                                                                      n

For an exponential equation, (n) is a constant. When n >= 1 this is the typical curve shape: [FIG 67]
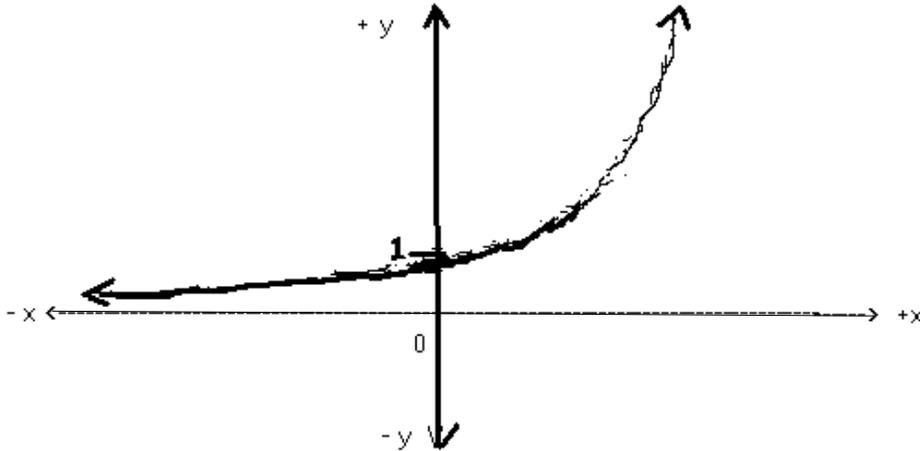
When x = 0, y = n^x = n^0 = 1, and this is where the curve crosses the y-axis.

Like other equations, an exponential equation may contain a multiplying constant (c):

    y = cn^x   : c != 0 so as the indicated power actually exists

The (y) or vertical intercept can be easily be found by setting (x) equal to 0:

    y(0) = cn^0   since n^0 = 1:
    y(0) = c (1)
    y(0) = c   : hence (c), the multiplying constant, is the vertical axis intercept; where x=0.

If the exponential equation is not known, (c) can be found by reading a graph of the corresponding data (points) of the unknown equation. If the equation is known, (c) can be easily found by:

$$c = \frac{y}{n^x}$$

Given a (y) value, the corresponding (x) value can be found by observing the graph of the equation, or mathematically by using logarithms:

      y = cn^x   taking the logarithm of both sides:
    log y = log cn^x
    log y = log c + log n^x
    log y = log c + x log n

$$x = \frac{\log y - \log c}{\log n} = \frac{\log (y/c)}{\log n}$$

If (y), (c), and (x) are known, perhaps from a graph, (n) can be solved for:

0.707317073 $= \dfrac{1}{n^{1.5}}$ taking the 1.5 root of both sides:

$1.5\sqrt{0.707317073}$ = 0.707317073^(1/1.5) = 0.707317073^0.66666666 = 0.793857881 = $\dfrac{1}{n}$

n = 1.259671314   : checks, as shown above

LOGARITHMIC :   y = log $_b$ x   [FIG 68]

When x > 0, and b > 0 :
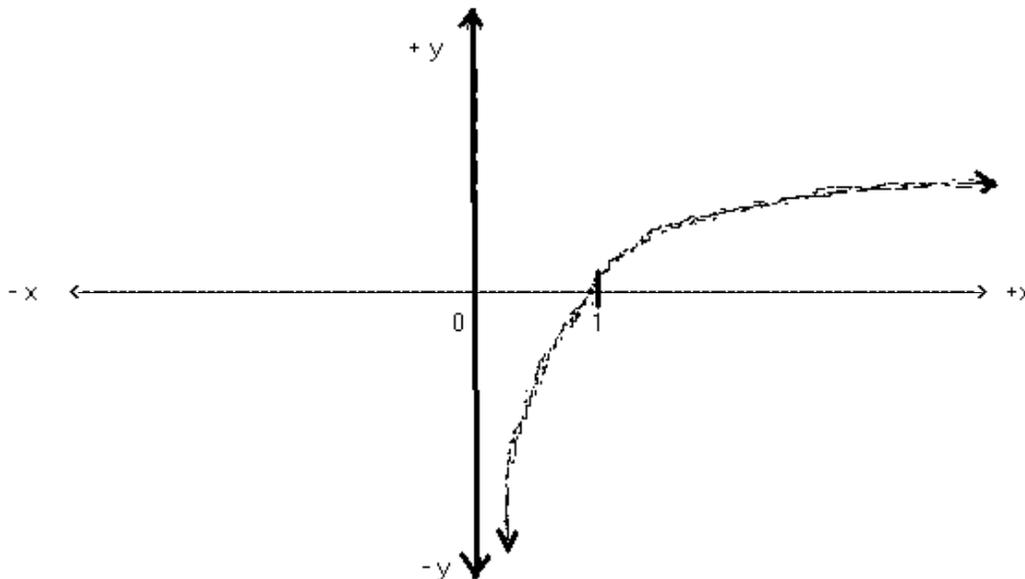
Note as (x) approaches infinity, the curve approaches the shape of a line. If you were to exchange the (x) and (y) axis of either the logarithm curve or the exponential curve, you would have each other curve. This is so since exponential and logarithms are inverse mathematical operations.

Here is a discussion and graph of the logarithm of a power of N with respect to the indicated (x, the exponent) power of N.

Given: y = log N^x   using the log rules, this can be expressed as:

y = x log N   or:

y = log N x   both forms are of a basic linear equation of:
(In short, (x) the exponent, causes a multiple of log N.)

y = m x   : here, m = log N , and is a constant and the slope of the line

between the two variables:

$$2y \ = \ 6x \ - \ 2$$

$$y \ = \ \frac{6x \ - \ 2}{2} \ = \ \frac{6x}{2} \ - \ \frac{2}{2}$$

$$y \ = \ 3x \ - \ 1 \qquad : \text{this equation has the basic linear equation form of:} \quad y \ = \ mx \ + \ b \quad : \text{here b = -1}$$

Given the general linear equation shown previously, and for the sake of simplicity, we will assume that the numerical coefficients ((a) and (b)) are 1, and letting f = c, a placeholder for an arbitrary constant, we now have:

$$(x \ + \ y \ + \ f)^1 \ = \ 0$$

By squaring the general first degree or linear equation, we will have the general second degree or quadratic equation:

$$(x \ + \ y \ + \ f)^1 \ (x \ + \ y \ + \ f)^1 \ = \ (x \ + \ y \ + \ f)^2 \ = \ 0$$

By using distribution and combining like terms on the left side, we have:

$$x^2 \ + \ 2xy \ + \ y^2 \ + \ 2fx \ + \ 2fy \ + \ f^2 \ = \ 0$$

Dividing each term by (f) and assigning variables to these newly created numerical coefficients of the variables, we have the general or formally defined second degree or quadratic equation of:

$$ax^2 \ + \ bxy \ + \ cy^2 \ + \ dx^1 \ + \ ey^1 \ + \ f \ = \ 0 \qquad : \text{General Or Formal, Second Degree or}$$
$$\text{Quadratic Equation}$$

CIRCULAR (CIRCLE) : $\ x^2 \ + \ y^2 \ = \ r^2$

Notice that for this simple equation of a circle, being (part) of a conic section, that it contains only some terms of the general form of a second degree or quadratic equation, that is, it need not have all the terms of that form, especially if those terms have a value of 0. This circle equation here is a simplified or unique form of the general or full formal circle equation which would have (x^1) and (y^1) terms. More specifically, this "reduced" or "partial" equation here is for when the center of the circle is at the origin (0.0) of the rectangular coordinate (rectangular, planer, two-dimensional measuring and addressing) system when it is graphed. The equation of the circle is actually a form of the Pythagorean Theorem for right triangles, where (r), the radius of the circle is essentially the hypotenuse side, and the corresponding "legs" or leg sides of this right triangle construction are the (x) and (y) values from the origin of this measurement system and center of this circle.

Algebraically arranging the equation into a form for plotting (graphing the relationship of corresponding values as points) a graphical or visible relationship between the variables:

$$y^2 \ = \ r^2 \ - \ x^2 \qquad \text{taking the square root of both sides of this equation:}$$

$$y \ = \ \pm \ \sqrt{r^2 \ - \ x^2}$$

Note, for a given or unique circle, r (radius) is actually a constant value.

Ex. Place the equation of: $3x^2 \ + \ 3y^2 \ = \ 9$ into standard form.

Translation Of Axis And Coordinates

What if a point(s) is to be in reference to a another location or axis?

If a point(s), p(x,y) is to be centered about another location, say p(h, k), you must translate (ie., shift, adjust) the coordinates of each point (x, y) so as to be centered about p(h, k). The x location of each point must be increased by h, and the y location of each point must be increased by k so as to be still in reference to p(0,0).

The x coordinate value of each point in reference this new system centered around p(h, k) will be called x' = "x-prime", and the y coordinate value will be called y' = "y-prime".

In relationship to the standard reference system with p(x=h=0 , y=k=0) as the origin, a translated point will have the coordinates of:

p(x , y) = p(h + x' , k + y') or= p(x' + h , y' + k)   , from this we mathematically have:

p(x' , y') = p(x - h, y - k);

If the coordinates of a point are x = 3, and y = 2 and the center or point of reference is the standard p(0,0), and if this point is now to be in reference to, or centered about point p(10, 5), we will let h=10, and k=5:
p(x' , y') = (3, 2) and p(x , y) = p(x' + h, y' + k) = p(3+10 , 2+5) = p(13,7);

Let's write the equation of a circle that has a radius of 10, and its center point (0,0) translated to be located a p(4,6).

x^2 + y^2 = r^2   is re-expressed or translated to:

x'^2 + y'^2 = r^2 =

(x - h)^2 + (y - k)^2 = r^2   substituting the known values , if r = 10,   then r^2 = 10^2 = 100 :

(x - 4)^2 + (y - 6)^2 = 100   and: (y - 6)^2 = 100 - (x - 4)^2   and   y = $\sqrt{100 - (x - 4)^2}$ + 6

PARABOLIC (PARABOLA) :

Observing the drawing below, a parabola curve is formed when the distance (d1) from any point on the curve to the (fixed or constant) focus point is equal to the (perpendicular) distance (d2) from that same point to an imaginary line (called the directrix) that is perpendicular to the axis of the parabola. Note, that this distance (d1 and d2) of either or both of these lines (to those points mentioned) is not constant since it changes from point to point. The focus of a parabola is located on the axis of the parabola. The parabola curve is symmetrical (ie, similar or the same in shape) about this axis and extends endlessly and appears to get wider and wider as the distance to its axis increases. The directrix line is located at the same relative distance that the focus point is from the vertex (ie. the "narrow point" and where the curve and slope changes direction and value) of the parabola, but on the opposite side of the vertex and perpendicular to the axis of the parabola. Often, the equation and shape of a parabola are constructed around an initially specified focus point.   [FIG 75]

$$y = \frac{x^2}{c} = \frac{(1)(x^2)}{(c)(1)}$$ expressing this reciprocal value of (c) as a new constant (a) :

$y = a\,x^2$: This is the most basic parabola equation, which is actually a (simple) quadratic equation, and also a "power (here it's 2) of x" equation. Note that if F is negative, then (a) and the (y) values will be negative and the curve will expand or open downwards.

Note: $a = (1/c) = 1/(4F) = 0.25 / F$ and therefore, $F = 0.25/a = 1/4a$:the focus point for this equation and parabola would be: $F(0 , 1/4a)$

Ex.   Given : $y = 0.025\,x^2$, find the focus point of this parabola.

This has the expressed form of: $y = ax^2$, where $0.025 = a$, and b=0 which is essentially the vertical offset of the curve.

Since: $a = \frac{1}{c} = \frac{1}{4F}$, (c) is the reciprocal of (a):
Also of note: $F = (1 / 4a) = (0.25 / a) = (c /4)$

$$c = \frac{1}{a}$$

Taking the reciprocal of the numerical coefficient (a) of the squared variable (x):

$$c = \frac{1}{a} = \frac{1}{0.025} = 40$$

Hence $y = a\,x^2 = \frac{(1)}{(c)}\frac{x^2}{1} = \frac{(1)}{(4F)}x^2 = \frac{x^2}{40}$: here, expressed in the general parabola form.
Note also that: $a = (1/40) = 0.025$

Here, $c = 4F = 40$,   solving for F:

$$F = \frac{c}{4} = \frac{40}{4} = 10$$: note also that $F = 0.25 / a = 0.25 / 0.025 = 10$

$F = +10$: the focus of this parabola is at point: $P(0,F) = P(0,+10)$

Given the original equation of a parabola: $x^2 = 4Fy$, and taking the square root of both sides:

$$x = +,- \sqrt{4\,F\,y}$$

This shows that for each (y) or vertical value, there are two (x) or horizontal values, one positive and one negative in sign, but both are the same in absolute value. These two points with the same (y) value or y-coordinate are on opposite sides of the axis of the parabola. Clearly then, the axis of this parabola is the vertical or (y) axis, and the parabola will be symmetrical about it.

The equation of a parabola when the vertex is at the origin, and the axis of the parabola is along the x-axis is of an identical form that has the variables switched:

$$y^2 = 4Fx   \text{ and }   y = \sqrt{4Fx}$$

great distances from the vertex, the two "arms" or sides of the parabola will be practically parallel to each other as they get nearly vertical when the slope of the curve becomes a very high value.

If a plane parabola shape or curve is rotated about its (symmetry or symmetrical) axis line, the three dimensional solid or volume created is called a paraboloid. Cross sections or planar "slices" parallel to the directrix will be circular.

Here is a short discussion about the different basic forms of a quadratic or parabola equations and their curves.

$y = x^2$: This parabola curve will have a point, here the extreme or minimum point, at (0,0). When x is less than one, it's square will always be a smaller value since this is essentially a fraction of a fraction. The curve is "slow growing" for these values. When x is greater than one, the "exponential effect" of squaring of a value, is easily seen as the change in y values grows larger and larger for each change in x. The curve becomes "fast growing". Since the (rate of) change is not a constant as for linear or line equations, the equation and the relationship of the variables of a parabola equation is said to be a non-linear. When the x^2 term is negative: $y = -(x^2) = -x^2$, the curve will open downwards and have a maximum point. The value of x^2 without the -1 multiplier is always positive in value for all values (both pos. and neg.) of x. $y=-(x^2)$ can also be expressed as $y=(-1)(x^2)$, and we see its the same as the equation of: $y=(x^2)$ but each value of y is essentially converted to its corresponding negative value by the (-1) factor. As the values of x increase more and more , the corresponding values of y will decrease more and more, and produce the inverted or "upside down" parabola curve.

$y = x^2 + c$: Here, a constant is added into the most basic quadratic equation. The effect. like as in a line equation, will shift the curve vertically. The vertex or minimum point of the parabola will no longer be at (0,0), and any part of that curve will not cross at that point either since for this equation, when x=0, y=c. This constant (c) is much like a "lead value".

$y = x^2 + x^1$: Here, with the addition of the variable to the first power will shift the axis of the parabola horizontally (left and right). This will change the origin and vertex of the parabola and it will no longer be: (h=x=0, k=y=0). Still, the curve will cross through this point since when x=0, y=0.

$y = x^2 + x^1 + c$: This will be similar to the equation above, however the curve will be shifted vertically along the y-axis, and it will not cross through point (0,0) since for this equation, when x=0, y=c. (c) will not change the shape (and therefore, it also wont change the slope) of the curve, and it will have the same shape as that of: $y = x^2 + x^1$

If the sides of the curve were not a parabola shape, but flat or straight shape, there would not be a common single focus point of the incoming or collected rays of light as an example. A common line or plane placed between the two reflective sides could be called the focus line or focus plane instead of a focus point. The incoming rays need not even be parallel to strike this line. In a "corner reflector" (ie., right angle or 90° reflector) often used for indicating objects in dim or dark conditions, the two reflective surfaces or sides have a 90 degree angle of separation. Each reflected ray will be reflected back in the same direction or parallel to the incoming ray such as from a light source. A similar reflector is called a retroreflector. [FIG 76]

ELLIPTICAL (ELLIPSE) :

The shape of an ellipse is like that of a stretched or elongated circle, hence its shape is much like an oval or an egg. If you look at a circular ring shape and tilt it away from you, it will appear very similar to an elliptical shape. Numerically, a measure of this elongation, non-circular, or "unbalance" is called the eccentricity (e) of the ellipse. Don't confuse the variable (e) used here with the ("natural") base value used in the context of natural logarithms. The longer the elongation, the greater the eccentricity (e) of the ellipse. The eccentricity of a circle is obviously 0 since the length of the horizontal and vertical axis within a circle are the same and therefore produce no elongation in any one direction. The horizontal and vertical axis lengths within a circle are obviously equal to the diameter or (half-diameter) radius value of that circle. A circle is actually a special instance of an infinite number of ellipses. The eccentricity of a circle is 0. When the eccentricity of an ellipse is 1, the ellipse is completely elongated into a line. Most planets have a near circular orbit, hence they have a low value for orbit eccentricity. Comets usually have an orbit having a high eccentricity value. Both the circle and ellipse are examples of what are known as "closed curves" since they completely define and enclose (contain, surround) a bounded region of a plane.   All circles are similar, but may vary in sizes, hence they are not identical.   Ellipses that have the same eccentricity value are similar, and they may even vary in size, and are therefore not identical ellipses. [FIG 77]
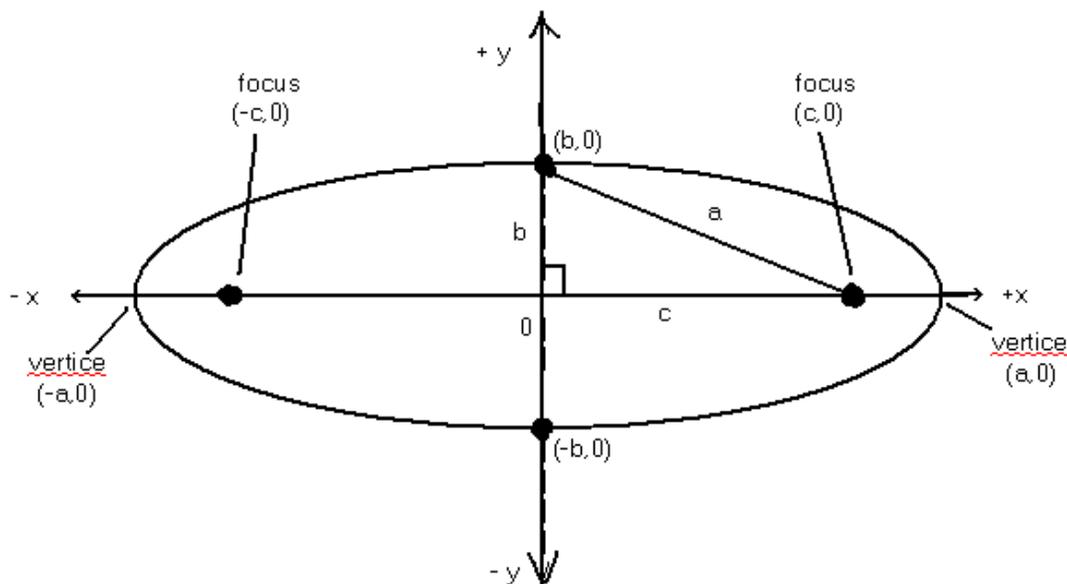
Orbits of outer-space objects, such as planets and comets going around the sun are actually elliptical in shape. Here, the sun is effectively at one of the two focus points of the ellipse or elliptical orbit. This orbit shape of a planet is caused by the constant pull of the force of gravity from the Sun, and will cause the object to slowly change direction and speed.

If energy (ex, sound, light) was emitted directly outward from any focus point of a (solid, physical) ellipse shape, the energy would be reflected from the inner curve edge of the ellipse to the other focus point. An egg is somewhat of an ellipsoid (three dimensional elliptical) shape where an ellipse is fully rotated about its major or minor axis.

An ellipse ("closed") curve is defined or formed when the total distance from any point on the ellipse to both foci (plural of focus) is always the same, and that distance has a value of (2a). This distance is also equal to the length of the major (ie. longest) axis of the ellipse which passes through both foci and the center or origin of the ellipse. For a circle, it's center point is effectively both focus points being in the same place. Here (r=radius or 2r) is the common distance. The more eccentric the ellipse, the more elongated it is and the farther apart the foci are, and vice-versa, however the maximum eccentricity (not a length value, but practically a ratio value) is defined as a value of 1. The focus points on an ellipse are equidistant from it's center point along the major axis. Before the standard equation of an ellipse is shown, the ellipse will first be discussed using the common circle and its equation since a circle is a special instance of

there is still a focus along the longest axis of it (the curve shape) due to the symmetry of the curve on both sides of that axis. It is also fair to guess that there are two focus or "foci" points due to the symmetry of the curve about the other axis. This leads to a conclusion that there are two foci points which are the same distance from the center of the curve along the major axis. If a circle was only slightly "squashed" or elongated (ie., stretched) to an ellipse shape, it is fair to guess that the foci points are still near the center of that circle as before it was "squashed" or elongated very slightly. As the circle or ellipse gets more elongated, the farther the foci points will be from the center.

A ray of energy (perhaps light, radio waves, or sound) emitted from one focus point will be received by the other focus point after reflecting from the ellipse curve shape to be collected or concentrated at that receiving focus. This is obvious for a circle. For an ellipse, this can be easily verified when the point of incidence and reflection is at the y-axis interception point of the curve. By drawing a tangent line to the curve at that point, and by using the symmetry of the angles of incidence and reflection with the y-axis, and even the angle of transmission and reception of the rays at the foci, then this fact (and the constant distance of 2a) is verified. It must be noted that all the rays emitted (in various directions) from the center (not the foci points) of an ellipse will generally not pass through both foci or even back through the center.

When the (y) values are divided by a high value (this is equivalent to multiplying each (y) value by a very small constant value = [1/high-value]), they approach a value of 0, and the square of those (y) values (being less than 1) will approach a value even closer to 0:

$r^2 = x^2 + y^2$ if y --> 0 (read as: "if, or as, y approaches a value of 0"), then:
r^2 --> x^2 r^2 will approach a value of x^2, and r will approach a value of x.
               After taking the square root of both sides:
  r --> x   With substitution:

$y = \sqrt{r^2 - x^2}$

$y = \sqrt{r^2 - r^2}$

$y = \sqrt{0} = 0$

So regardless of the value of (x), all the (y) values will be 0 and the curve of the ellipse approaches the shape of a (horizontal) line, hence it's the line: y=0. We then also have some verification that a line (segment) is a special instance of a conic section, and in particular, an ellipse.

Given: $y = \pm \sqrt{1 - x^2}$,

If each (y) value is multiplied by a numerical coefficient, say variable (b), as discussed above for making an ellipse, the equation for each (y) value is now:

$y = \pm b\sqrt{1 - x^2}$ squaring both sides, we have:

$y^2 = b^2 (1 - x^2)$ after dividing both sides by b^2 :

$\dfrac{y^2}{b^2} = 1 - x^2$ and:

$x^2 + \dfrac{y^2}{b^2} = 1$ after solving for x :

$x = \pm \sqrt{1 - \dfrac{y^2}{b^2}}$

If each (x) value is multiplied by a numerical coefficient, say variable (a), and after squaring both sides of this equation to rid the radical, we will have:

Using the general equation and expressing (y) in terms of (x) for plotting purposes:

$$y = \pm\, b \sqrt{1 - \frac{x^2}{a^2}}$$: Note that when x=0, y= +-b as should be expected, and this point (0,b) is the y-axis intercept.

You can find the focus point: F(c, 0) from a Pythagorean, right triangle, type of relationship of the distances:

$$a^2 = b^2 + c^2$$: where (c) equals the ± x coordinate of a focus point

To verify that the hypotenuse is equal to (a) = half the major axis, consider the point (b,0). According to the definition of an ellipse, the total distance to both foci points is 2a after a ray is emitted from one focus and reflected off the ellipse to the other focus. Half of this distance is equal to (a).

$$c^2 = a^2 - b^2$$,   the focus point is: F(c , 0)

Due to the elongation (of an axis) of an ellipse, you might suggest that the formula for eccentricity (e) should be the ratio of the minor and major axis, and have resulting ratios less than or equal to 1, and then and adjustment made by subtracting this ratio from one (1) since longer elongations, where the major axis is larger, should have higher (rather than lower) eccentricity values rather than lower eccentricity values. Consider the following:

$$\frac{minor\ axis}{major\ axis} = \frac{2\ (semi\text{-}minor\ axis)}{2\ (semi\text{-}major\ axis)} = \frac{b}{a}$$: this is sometimes called the "compression factor" or "aspect ratio"

A "proposed formula" for (e) $= 1 - \frac{b}{a}$: This specific value is associated with the compression factor is sometimes called the "flattening factor", but it is not the formal formula for eccentricity.

Combining fractions, this can also be expressed as:

$$1 - \frac{b}{a} = \frac{a}{a} - \frac{b}{a} = \frac{a - b}{a}$$ = ratio of: (difference in length values) to the longest length value

As a check on the "proposed formula" for (e), a circle is defined as having no eccentricity (e=0), the length of the major and minor axis are the same, therefore (a=b):

$$1 - \frac{a}{a} = 1 - 1 = 0$$: (e) of a circle

And for a line being defined with the highest eccentricity possible, the ratio of the minor (2b) axis to the major axis (2a) is very small (since b approaches 0), and is practically 0:

$$1 - \frac{b}{a} = 1 - 0 = 1$$: (e) of a line

This "proposed formula" seems to be a very logical mathematical definition of eccentricity (e), however, it is not the one used. (e) is actually defined in such a way as to be applicable to all conic sections. A mathematical definition of the eccentricity of a conic section is:

$$\frac{distance\ from\ any\ point\ on\ the\ curve\ to\ a\ fixed\ point\ known\ as\ the\ focus}{distance\ from\ the\ same\ point\ on\ the\ curve\ to\ a\ fixed\ line\ known\ as\ the\ directrix} = \begin{array}{l}eccentricity\ (e)\\ of\ a\ conic\ section\end{array}$$

HYPERBOLIC (HYPERBOLA) :

The word "hyperbolic" is an old word used to describe things that are excessive or extreme, and is rooted in the meaning such as "an excessive bole [throw]". The equation and formal definition of a hyperbola are somewhat similar to that of an ellipse, however with a hyperbola, the difference between the distances from any point on the hyperbola curve to the two foci points is always the same. Unlike the ellipse and circle, the hyperbola is not a closed curve. It has two symmetrical (similar or mirrored) branches that extend endlessly. The two branches are due to a planar section of two identical cones connected at their vertex or "top points". Each branch is considered hyperbolic. An asteroid or comet from outside our solar system may get attracted and then have a trajectory towards near to the Sun due its gravity, and yet not orbit the Sun like objects in our solar system do with an elliptical orbit, but it will rather just pass through and out of our solar system due to its high kinetic energy. That asteroid is then said to have a hyperbolic trajectory or path.

When the center of the hyperbola is at the origin, and its axis (passes between the two vertices of the hyperbola, and is similar to the major axis of an ellipse, and where the foci of the hyperbola are located upon) is along the x-axis, the equation of a hyperbola has, or can be reduced to the basic form of:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$: (a > b) when the x-axis is the axis of the hyperbola.
This equation is similar to an ellipse equation, but the
coefficients of the x and y terms have opposite signs.

There is another and simpler equation of a hyperbola (though with different axis) that is often encountered, and this is:

$\frac{x\,y}{c} = 1$: For this equation to be true, (c) must equal (x)(y). (c) is a constant.
Mathematically:

$x\,y = c$: Both (x) and (y) are inversely related to each other. After solving for
(y) for plotting, this resembles a common "division equation", where
the divisor, and therefore the quotient change in an inverse manner.

$y = \frac{c}{x}$: (y) is a some multiple (c) of the reciprocal of x = (1/x).   y=(c)(1/x).
The equation's curve shape is shown below. Since the graph shows the
relationship of how (y) changes with respect to (x), (c) must remain constant.
In general, due to the form of this equation, it could be called a "division equation"
and the graph of it could be called a "division curve". If we let c=N=numerator,
or dividend, and x=D=denominator, or divisor, and y=Q=quotient, we have:

$Q = \frac{N}{D}$: when N=D, Q=1
: when N>D, Q>1
: when N<D, Q<1 , a result less than one is a fraction of 1

In dividing values such as successive integers: N / (N+1) is less than 1, and
approaches closer to 1 as N approaches infinity:

Ex. 1/2 = 0.5   ,   2/3 = 0.666<u>7</u>   ,   3/4 = 0.75 , 9/10 = 0.9   ,   542/543 = 0.998<u>2</u>

In an opposite way, (N +1) / N is greater than 1, and approaches closer to 1
as N approaches infinity.

Ex. 2/1 = 2   , 3/2 = 1.5   , 4/3 = 1.3333...   , 10/9 = 1.111...   ,   543 / 542 =   1.0018...
These quotients are reciprocal in value to the above results, since the values used were reciprocals.
(N+1) / N   and N / (N+1) are reciprocals of each other.

Mathematically drawing a conic section or curve without a conic or second order equation:

You can mathematically draw a conic section on a piece of paper. Draw a line on the paper and choose a point nearby, perhaps an inch away. This line will become the directrix of the conic section or curve, and the point will be the focus (F) of the that curve being drawn.

For each point (p) on the curve being created, it will be a certain perpendicular distance from the directrix line, and a certain distance to the focus point, and the mathematical relationship (here, as a ratio) that the distance each point is from the directrix line and focus point is a constant of (e):

$\dfrac{\text{(distance to focus)}}{\text{(distance to directrix)}} = e$: e is a constant and is called the **eccentricity** of the conic section or curve

When (distance to focus) = (distance to directrix) , a constant, the curve is a parabola
When (distance to focus) < (distance to directrix), e<1, and the curve is an ellipse.
When (distance to focus) > (distance to directrix), e>1, and the curve is a hyperbola.

It could be said that circles have no eccentricity (e=0) or stretching, ellipses are like a circle stretched a little and have a little eccentricity (0<e<1), parabolas have moderate eccentricity (e=1), and hyperbolas have high eccentricity (e>1).

5^3 ft = 125 ft having units of feet = ft^1, is not the same as 5 cubic-feet having units of: ft^3 = (cubic feet)

A unit of measurement: unit = unit^1   : for example, the unit of measurement or reference could be a
                                                                          length unit such as feet (ft), miles (mi), or meters (m), etc.
                                                                          Ex: 5.23 feet

A squared unit, or a square unit: (unit) (unit) = unit^2   : Ex.   5.23 ft^2 = (5.23ft)(5.23ft) = "5.23 feet, squared"
                                                                                                    = "5.23 square-feet"
                                                                                                    = "5.23 feet=squared"

                                                                   Ex. 2ft long, by 2ft high = (2)(2)(ft)(ft) = 4 ft^2 = "4 square-feet"
                                                                   Ex. 2ft long, by 3 ft high = (2)(3)(ft)(ft) = 6 ft^2

A cubed unit, or cubic unit: (unit) (unit) (unit) or = (unit^2) (unit^1) = unit^3   : Ex. 5.23 ft^3 = "5.23 cubic-feet"
                                                                                Ex. 2ft long, by 3ft wide, by 4ft high =
                                                                                (2ft)(3ft)(4ft) = 24 ft^3 = "24 cubic-feet"

Ex. If considered in the shape of a cube, a million cubic feet of material would have a side or dimension length of:

   $3\sqrt{1000000 \text{ ft}^3}$ = $3\sqrt{1000000}$ $3\sqrt{1 \text{ ft}^3}$ = (100)(1 ft)   or = 1 (10^(6/3) ft) = 10^2 ft = 100 ft

     That is, this cube or cubic shape of material would have a length of 100 ft, a width of 100 ft, and a height of 100 ft.

GOLDEN RATIO (In a geometric series, with a basic curve example.)

Though not often encountered as much as the other common curves, the golden ratio (GR) is still important.   For
simplicity, we will use GR =~ 0.618.

If GR was the first term of a series, and if you took 0.618 of it and of each successive term, the sum would equal:
   1+GR = 1 + 0.618 = 1.618 = 1/GR = 1/0.618 :

Term :	1	2	3	4	5	6	7	. . .	
Value :	0.618	0.382	0.236	0.146	0.0902	0.056	0.034	. . .	Algebraically:
Expression:	GR	GR(GR)=GR^2	GR^3	GR^4	GR^5	GR^6	GR^7	. . .	

The value of each term (n) is: GR^n   : n=term number

The sum of all the terms approaches: 1.618**...** = (1+GR) = 1/GR   : as (n), the number of terms, approaches infinity

A peculiarity is that if you were to subtract each term (GR^n) from its preceding term, you would have the next term. This
is similar to a (natural, basic) Fibonacci Series, if those terms above were reversed into an ascending series, and the next
term was the sum of its two previous terms.   Here, the next term of this series is the difference between the last two
terms.

(GR)^(n+1) = (GR)^(n-1) - (GR)^n   :ex. 0.236 = 0.618 - 0.382 = 0.236   ,   Mathematically:
(GR)^(n-1) = (GR)^n + (GR)^(n+1)   Also considering just the first two terms:
1.0 = (GR^1) + (GR)^2

[This space for book edits.]

rotate near to, or very far from the center (of rotation), and yet each angle would still be the same value regardless of the distances or size from the (reference) center. Any circle has a 360 degree angle for a full rotation about it center, and a (360/2) = 180 degree angle for half of a full rotation, and a (360/4) = (180/2) = 90 degree angle for a quarter or one-fourth of a full rotation.   A common symbol for an angle is: ¢ , and may be sometimes noted with the degree units symbol: ¢°
 [FIG 81]
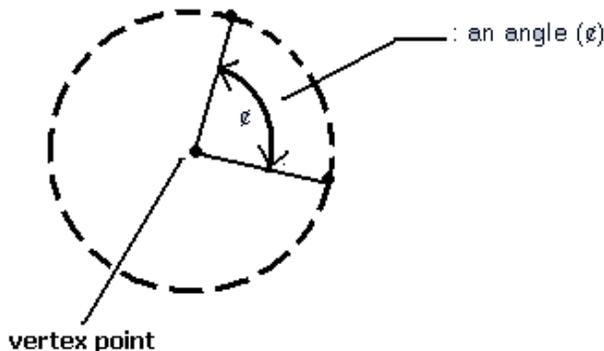
vertex point

        The point where two sides (considered as lines) of a angle or triangle meet or intersect is formally called the vertex ("point", "location" , "meeting", "joining", or "intersection") of the angle. The word of "vertex" is rooted in the words such as "vertical peak", "apex" (vertex is a short and combined form for the words: "vertical apex"), or "summit" where the sides of a hill or mountain meet (intersect). The word "vertical" basically means in an upward or downward direction, as opposed to the word "horizontal" (ie. "horizon") direction which means "level" (no upward direction or angle) or in the left or right directions. The curved arrows are often used to indicate that some movement, separation or rotation has happened, hence an angle and-or its shape was created or is present.

A right angle is defined as one-quarter (1/4) of an angle that would give a full rotation of 360°, hence a right angle is (360°/4 =) 90° ("ninety degrees"). A triangle that has two sides that intersect at a right angle (perpendicularly, like two sides of a square or rectangle) is called a right triangle. The sum of any triangles interior angles is always 180°. [FIG 82]
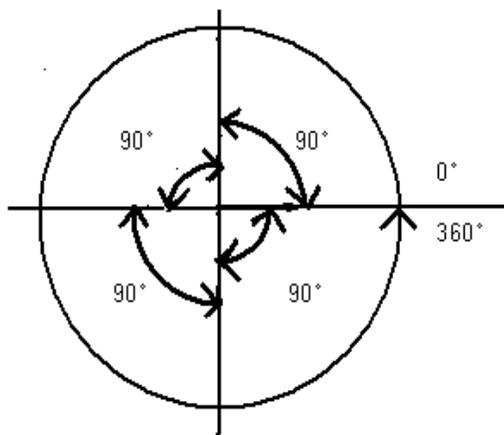
$$\frac{360°}{4} = 90°$$

$$90° + 90° + 90° + 90° = 90° (4) = 360°$$

The above illustration also shows that when a line is completely rotated (a 0° to 360° angle) once (1) about the fixed center (point of rotation), that an angle of 360 degrees is created.   Clearly a rectangular coordinate system or a circle can be divided up into 4 equal sections and each is called a quadrant (meaning one-fourth or a quarter of something). Each quadrant corresponds to, or has a maximum angle of (360 degrees / 4) = 90 degrees.

hypotenuse side:

        c > a   and   c > b   : c, the hypotenuse, is always the longest of the three sides of a right triangle

        (a + b) > c   : this can also be expressed as: c < (a + b)

To determine if an angle is indeed a right, 90° angle, you can first measure the same distance <u>from</u> the angles vertex and along on each of the sides or "legs". The distance from one point to the other will be about $\sqrt{2}$, (= about 1.414) times that of the same distance along each leg.   This can be derived and checked from the Pythagorean Theorem.

opposite sides of that ("transversal") line are called alternate angles, hence they are also called alternate interior angles.   Here, c and a, and b and d are alternate interior angles.

$$\frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1} = n \qquad \text{: the reciprocal of each ratio, or "inverse reciprocal", will also}$$
have the same value, and that is of the reciprocal of n or= (1/n).

Given that triangle 1 and triangle 2, as shown in the last figure, are similar triangles, here are some examples of all the possible equivalent fractions or proportions of their parts:

ratio of parts of a triangle = corresponding ratio of parts of any similar triangle

Ex. $\frac{a1}{c1} = \frac{a2}{c2} = r1$: the ratio of sides of similar triangle 1 = the ratio of corresponding sides of similar triangle 2.
If the angles, and-or ratios are not the same, the triangles are not similar.

Ex. $\frac{a1}{b1} = \frac{a2}{b2} = r2$: these ratios (r1, r2, r3, etc.) of the sides of similar triangles are generally not the same value

Ex. $\frac{b1}{c1} = \frac{b2}{c2} = r3$

When dealing with similar triangles, there are two different types of ratios to consider. One type of side ratio is a ratio of the parts of just one triangle. For another type of side ratio, it is a ratio of corresponding parts of two similar triangles.

As indicated previously, for similar triangles, the relationship among the sides is that the ratio of any side of triangle1 to the corresponding side of triangle2, is equal to the ratio of another side of triangle1 to its corresponding side of triangle2:

Ex. $\frac{a1}{a2} = \frac{b1}{b2} = m$: the ratio of a corresponding sides of two similar constructions, such as similar triangles, is the common magnification ((m), ie., multiplication) factor for all corresponding sides of the similar triangle

For the verification of this, notice that for the previous examples, that these two different types of side ratios for similar triangles can be mathematically derived from each other:

Given: $\frac{a1}{a2} = \frac{b1}{b2} = m$: the ratios of corresponding sides of two similar triangles are equivalent
after multiplying each side by (a2/b1), and canceling common factors in the num. and den.:

$\frac{a1}{b1} = \frac{a2}{b2} = r$: the corresponding ratio of sides within each similar triangle are equivalent.
In general, (m) and (r) are not equal in value.

You might ask: "What is it that makes these ratio values equal to each other?" "Why is the ratio of a1 to a2 equal to the ratio that b1 is to b2?" There's nothing intuitive that the ratio values should be the same, other than the sides are now known to be mathematically proportional, a simple answer is that you are actually viewing the same triangle either closer (where its apparently bigger and measured as such) or farther away (where its apparently smaller and measured as such), hence the triangles are "visually proportional". For similar triangles, corresponding sides will be different, but the corresponding angles always remain the same. Hence in a fundamental sense, similar triangles are initially due to identical angles. These identical angles define the similar shape of all similar triangles. Although the concept of magnification and its value to create a similar triangle or construction were mentioned, a verification to this equivalent magnification ratio (m) value of all corresponding parts or sides of a similar triangles or other planar constructions is presented further ahead after the discussion about trigonometric ratios.

With the information presented above, when given two similar triangles, the length of a (unknown) side can now be solved for if there is enough information to place into a proportion (ie., equivalent fraction) problem as shown above:

Ex. Using the triangles shown previously in [FIG 85], solve for c2 given: a1=3, b1=4, c1=5, a2=5.25, and b2=7.

CONSTRUCTING SIMILAR TRIANGLES

Constructing similar triangles, whether actual constructions, or drawings, is quite easy. Most of the facts and details to do this were were previously indicated or mentioned, and a few others will be mentioned.

The simplest method to create a similar triangle is to use a line(s) that is parallel (and possibly even on the same line or an "extended line") to each corresponding side. Of course, to construct a similar triangle, the side lengths will be changed in value so that all its sides can intersect at some point, creating the triangle. In general, to make a parallel line to any side of a triangle, extend or reduce both (adjoining) nearby sides by the same factor or percentage value, and then connect these new end points with a (parallel) line. The length of this parallel line will correspondingly be the same portion, factor or percent longer or shorter than the corresponding side of the original similar triangle.

Ex. Below, triangle1, with sides: a1, b1, and c1, is a general representation of any triangle. Sides b1 and c1 were extended (here, partially indicated with the dotted lines, and arrow) to create a larger similar triangle with sides a2, b2, and c2. Sides b2 and c2 were extended (lengthened) by the same percentage or multiplier, and then side a2 of triangle2 will be parallel to a1. After the construction below, triangle1 will becomes an "internal" similar triangle to triangle2. [FIG 86]
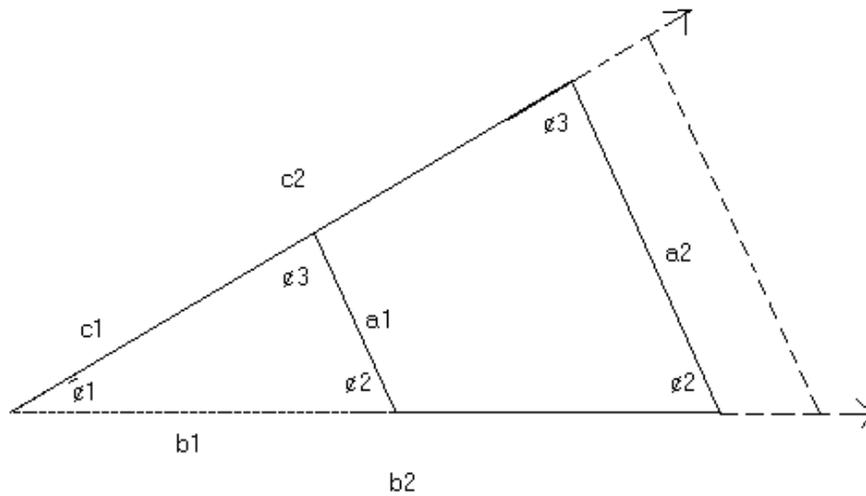
Obviously, ¢1 for each of the similar triangle's above is the same. The verification that ¢2, and ¢3 are the same for both triangles, as expected for similar triangles, is discussed ahead in the topic of: Corresponding Angles Created By A Transversal Line.

In the figure below, a line parallel to side a1 of a given triangle construction (a1, b1, c1) will be created and it will be called a2, and will be used to construct a small internal similar (right) triangle: [FIG 87]

Ex.   At a tenth (or 10%) of b1, the height (a2) is also a tenth of a1.

$$\frac{b1}{10} \;=\; b1(0.10) \;=\; b2, \quad \text{and} \quad \frac{a1}{10} \;=\; a1(0.10) \;=\; a2$$

In short, to construct a similar triangle, you can reduce or increase each side by the same factor or percentage value.

If given just a single triangle, solving for the value of a side using proportions would not be possible since there is no other similar triangle to relate or be in reference to. There is however, a mathematical relationship among the sides of a single right triangle which will allow us to solve for a side if enough parts of that triangle are known. This method uses the Pythagorean Theorem, and it is discussed next. There are also some generalized methods that are discussed later that can solve for a part of any triangle if enough of its other parts are known.

a^2 + 2ab + b^2 - c^2 = 2ab   after solving for c^2, the hypotenuse of the triangles:

c^2 = a^2 + b^2   : **PYTHAGOREAN THEOREM** (verified for right-angled triangles with (c) as the longest side)

$c \ = \ \sqrt{a^2 \ + \ b^2}$

Ex.   A right triangle has sides: a=3 and b=4. What is side c, the hypotenuse side ?

c^2 = a^2 + b^2   isolating c by taking square root of both sides:
c = $\sqrt{a^2 + b^2}$ substituting the given values for the variables:

c = $\sqrt{3^2 \ + \ 4^2}$ simplifying the radicand:
c = $\sqrt{9 +16}$: ORD 1, order of operations step 1; powers and roots, here, in the radicand of the radical
c = $\sqrt{25}$: ORD 4, order of operations step 4; combine
c = 5   : ORD 1, order of operations step 1; powers and roots

When the two sides or legs of a triangle are equal in value, the hypotenuse side is equal to the square root of 2 times that side length. This would also equal the diagonal length within a square shape:

c = $\sqrt{side^2 \ + \ side^2}$ = $\sqrt{2 \ side^2}$ = $\sqrt{2}$ $\sqrt{side^2}$ = $\sqrt{2}$ side   : note,   side = c / $\sqrt{2}$ = ~ (0.707) c
c = 1.414213562373096 side   = ~ about 1.414 side

Ex. If each side is 3, the hypotenuse side is equal to the 3rd multiple of the square root of 2, which is: 4.242640687

When one side or leg of a triangle is equal to twice (2 times) that of the other side or leg, the hypotenuse side is equal to the square root of 5 times that side length:

c = $\sqrt{side^2 \ + \ (2 \ side)^2}$ = $\sqrt{1 \ side^2 \ + 4 \ side^2}$ = $\sqrt{5 \ side^2}$ = $\sqrt{5}$ $\sqrt{side^2}$ = $\sqrt{5}$ side
c = 2.23606797749979 side

When one side or leg of a triangle is x (a variable factor which can even be a fractional value less than 1) times longer than that of the other side or leg, the hypotenuse side is equal to:

c = $\sqrt{side^2 \ + \ (x \ side)^2}$ = $\sqrt{1 \ side^2 \ + x^2 \ side^2}$ = $\sqrt{(x^2 + 1) \ side^2}$ = $\sqrt{(x^2 + 1)}$ $\sqrt{side^2}$

c = $\sqrt{(x^2+1)}$ side

As the value of x gets larger and larger, the added 1 term has less of an effect to both x^2 and the result value, which is (c), and the expression approaches or becomes very close to the value of just:

c = $\sqrt{x^2}$ side   which simplifies to:

c = x side

What about other triangles that are non-right triangles, does the Pythagorean Theorem hold true for them? Lets consider that if the right angle is changed to just slightly over or less than 90°. The triangle is then a non-right triangle but still appears to be as one.   It is not to difficult to imagine that all the sides of this non-right triangle are nearly equivalent in length to that of a right triangle, and therefore, the Pythagorean Theorem is still practical for estimating and-or checking the results, however we will eventually need to use general (ie., for any) triangle analysis methods to find the sides and-or angles of a general triangle.

Three other, but less used trigonometric ratios are effectively the reciprocal of the three ratios just given:

cosecant ¢ = csc ¢ = $\dfrac{\text{hypotenuse}}{\text{opposite}}$ = $\dfrac{1}{\sin ¢}$

secant ¢ = sec ¢ = $\dfrac{\text{hypotenuse}}{\text{adjacent}}$ = $\dfrac{1}{\cos ¢}$

cotangent ¢ = cot ¢ = $\dfrac{\text{adjacent}}{\text{opposite}}$ = $\dfrac{1}{\tan ¢}$

Ex.   For the triangle below, evaluate the sin, cos, and tan of angles A and B.   [FIG 92]
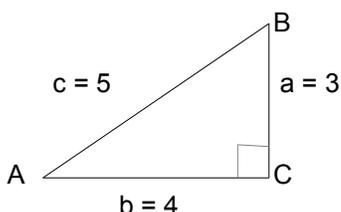
SIN ¢A = SIN A = $\dfrac{\text{opposite A}}{\text{hypotenuse}}$ = $\dfrac{a}{c}$ = $\dfrac{3}{5}$ = 0.6

COS ¢A = COS A = $\dfrac{\text{adjacent A}}{\text{hypotenuse}}$ = $\dfrac{b}{c}$ = $\dfrac{4}{5}$ = 0.8

TAN ¢A = TAN A = $\dfrac{\text{opposite A}}{\text{adjacent A}}$ = $\dfrac{a}{b}$ = $\dfrac{3}{4}$ = 0.75

SIN ¢B = SIN B = $\dfrac{\text{opposite B}}{\text{hypotenuse}}$ = $\dfrac{b}{c}$ = $\dfrac{4}{5}$ = 0.8

COS ¢B = COS B = $\dfrac{\text{adjacent B}}{\text{hypotenuse}}$ = $\dfrac{a}{c}$ = $\dfrac{3}{5}$ = 0.6

TAN ¢B = TAN B = $\dfrac{\text{opposite B}}{\text{adjacent B}}$ = $\dfrac{b}{a}$ = $\dfrac{4}{3}$ = 1.333333333. . .

Usually, some of the intermediate steps would be eliminated:

Ex.   SIN A = $\dfrac{a}{c}$ = $\dfrac{3}{5}$ = 0.6

Notice that the values of: sin A = cos B,
cos A = sin B,
tan A and tan B   are reciprocals of each other: tan A = 1/tan B   and   tan B = 1/tan A

If given a trigonometric (ratio of two sides) value of an angle, you can mathematically solve for one of the corresponding sides of that trigonometric value if the other corresponding side is known:

VERIFICATION OF THE SIDE LENGTHS OF SIMILAR TRIANGLES

Here is a verification that the ratio of corresponding sides of similar triangles is equal. Observer the drawing below of a line that effectively creates a general angle (¢) -*for this analysis, and for simplicity, side (c) of each constructed (right) triangle is not indicated: [FIG 94]
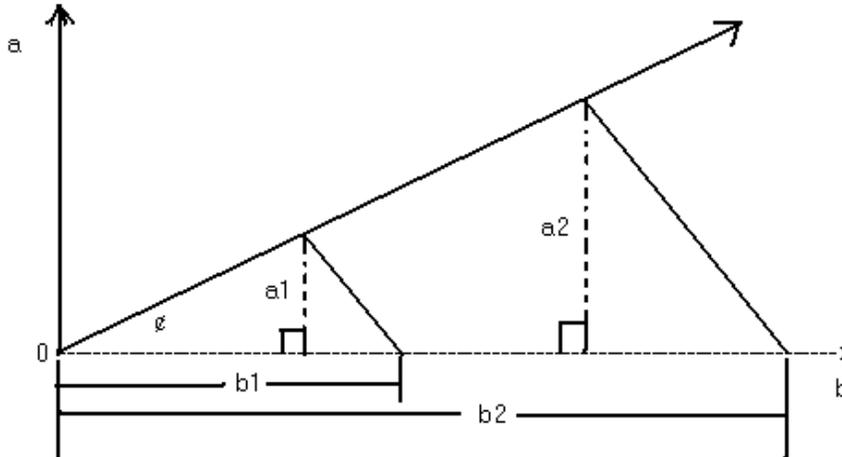
If we divide the value of a1 by its' corresponding value of b1 on this line, we obtain a ratio (r) value:

$$r = \frac{a1}{b1}$$

This particular ratio value of the particular points (or sides of a triangle) is also equal to the slope (m=rise/run) of the line, and it is also equal to the tangent of the angle (TAN ¢) = opp./adj.

Since for similar triangles which have different side lengths, to retain their similar shape, the angle(s) is constant, therefore the trigonometric ratio (of sides) values (such as TAN ¢, and therefore the ratio=r) remain constant. In simple terms, the ratio of corresponding sides remains constant.

Taking another set of corresponding values of (a) and (b) on the line we get:

$$TAN \; ¢ \;\; = \; r \; = \; \frac{a2}{b2} \qquad\qquad \text{hence:}$$

$$TAN \; ¢ \;\; = \; \frac{a1}{b1} \; = \; \frac{a2}{b2} \qquad \text{and algebraically:}$$

$$\frac{a1}{a2} \; = \; \frac{b1}{b2} \qquad \text{or:} \qquad \frac{a2}{a1} \; = \; \frac{b2}{b1} \qquad\qquad \text{: verification}$$

If you want to find other corresponding points (or sides of the similar triangle) on this line you can solve for a relationship between the (a) and (b) coordinates, and this relationship is the ratio value (r):

From: $r \; = \; \dfrac{an}{bn}$: n is some multiplying factor such as used when creating an equivalent fraction.

Other useful facts can be derived here. Observe the drawing below of two general (non-right) similar triangles where the (a) value indicated within each triangle is equal to the (perpendicular) height (h, or "altitude") with respect to the base side (b) that it intersects with. These (a) values are also the (proportional) sides of similar right triangles. We can conclude from this that the ratio of (corresponding) height values of all similar triangles, and not just similar right-triangles, is equal to the ratio of any corresponding sides of the similar triangles. Remember that for a construction to be similar, each and every part, including the height or altitude sides, are magnified by the same value. For example: [FIG 96]
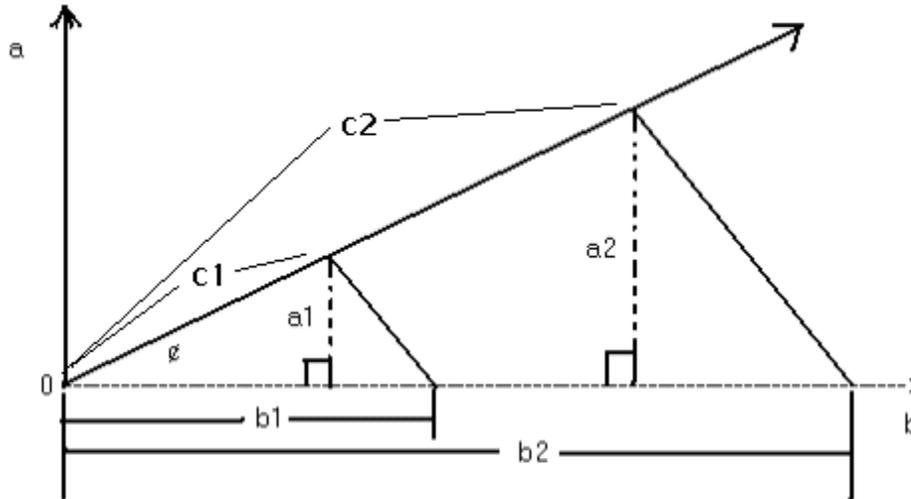
$$\frac{h2}{h1} = \frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1} = r$$

: h is used to indicate height or altitude
: r is essentially the "magnification factor" of the similar construction that is a triangle.

The above graph shows that all sides, including the "height sides" of similar triangles are proportional (same portions or fractional values). The heights of similar right triangles are obviously proportional since any side, other than the hypotenuse, is actually the height with respect to the other side.

If corresponding sides of similar triangles are related by a ("magnification") ratio, is the perimeter (the outside edge length) of the corresponding similar triangle related by the same ratio? The answer is yes, and this is algebraically verified below where P1 and P2 are the perimeters of similar triangles:

P1 = a1 + b1 + c1   and   P2 = a2 + b2 + c2

Since for example a2 = r a1, by algebraic substitution:

P2 = r a1 + r b1 + r c1   factoring (r) from each term:
P2 = r (a1 + b1 + c1)   dividing P2 by P1:

$$\frac{P2}{P1} = \frac{r(a1 + b1 + c1)}{(a1 + b1 + c1)}$$ canceling common factors of the numerator and denominator:

$$\frac{P2}{P1} = r$$

$$\frac{P2}{P1} = \frac{h2}{h1} = \frac{a2}{a1} = \frac{b2}{b1} = \frac{c2}{c1} = r$$

A SIMPLE DEVICE TO MEASURE SMALL DISTANCES

Using the concepts of similar triangles and vertical angles that are identical angles on both sides of the intersection of two lines, a device can be created to measure small distances and-or to obtain a distance measurement which will have greater precision (precise, the fineness, more least significant values, smallest capability) possible than without using it. The device presented below is only a very simplified form of such a device which can eventually be improved upon. Typically, a finely scaled ruler can only measure (precisely) to within as small as one sixty-fourth = 1/64 = 0.015625 of an inch. The device shown here is designed to measure to within about one thousandth = 1/1000 = 0.001 of an inch which is a more precise measurement. [FIG 97]
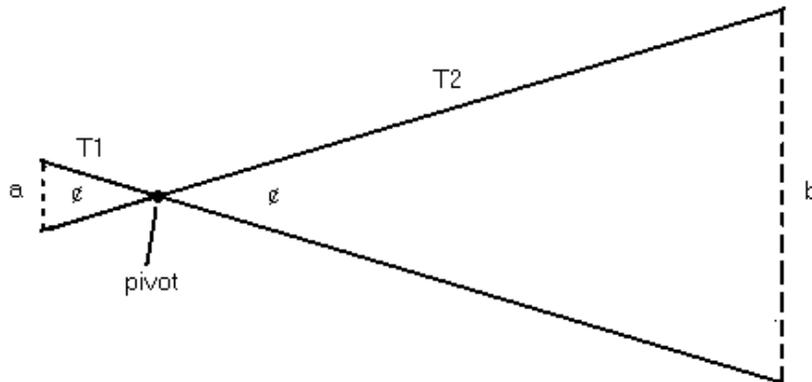
The device above is designed to measure a max. distance (or thickness) of 1 inch at side (a). That is, side (a), the "input location", of T1 (triangle 1) opens to 1 inch. Side (b) of T2, the "output or result (location or measurement)", is 20 inches when the distance being measured at side (a) is 1 inch. Hence, for any given distance measurement at side (a), side (b) will be 20 times longer or larger. Therefore, given this longer or magnified representative distance, to find out what it does actually represent (the true or actual measurement) we will divide (to reduce or demagnify) this output value by the known magnification factor or 20, hence essentially making the value of (1/magnification factor) = (1/20) as a demagnification factor.

What is the verification that side (b) is always 20 times longer? To begin with, both angles indicated are the same due to the concept of vertical angles. A vertical angle is the (equal) angle created on the opposite side of an angles vertex after extending the sides of the angle. You can recall the concepts of creating similar triangles to help verify this. The sides of T2 are extensions of the sides of T1, and side (b) is parallel to side (a), therefore, T2 is a similar triangle to T1. All the sides (and heights) of T2 are constructed to be 20 times longer than the corresponding sides of T1, hence the ("magnification") ratio (r) value is 20:

magnification ratio = $\dfrac{output}{input}$ = $\dfrac{b}{a}$ = r   mathematically:

output = input (r)   and   input = $\dfrac{output}{r}$ = output (1/r)   : input = (output)(demagnification factor)

b = a(r)   if (r) = 20:
b = a(20)   or:

b = 20a   : the "magnified value of (a)"
                   : this is of the form of a linear equation of: y =mx
a = $\dfrac{b}{20}$: the actual, true or "unmagnified" value of (a) .   or:

and likewise, the larger opening is where the (applied) force is less or decreased. The force gain or multiplier is determined by the ratio of the length of the sides or arms (or "lever arms") on opposite sides of the vertex or (rotational, "axis") pivot point of the construction. For premade mechanical devices that can measure small distances, such as the thickness of a piece of metal or some material, research the words "micrometer" and-or "veneer calipers". Many of these devices are relatively inexpensive (perhaps under $20 USD) as of the year 2022. Some of these devices have mechanical or printed scales, and some may have a digital-electronic display of the measurement taken.

With the device shown in the previous figure, it is even possible to plan and construct a similar device to find a fractional portion of any length positioned at the "output end". The input end would be scaled in fractions of its length, and it is even possible to used similar shaped and length (here, diameter) objects for this such as similar coins. For example, using 3 coins, each would then indicated (1/3) of the total input length.   Using 5 coins, each would then indicate (1/5) of the input length. The output length needs to be perpendicular to the input length, and the (opposite) endpoints of each length need to be connected by an imaginary line, hence showing a complete magnification of the full input length. .Using a string(s) or rod(s) with a fine pointer at the center of its end, say (1/3) or (1/5) of the input length can then be effectively projected and-or magnified onto a corresponding (1/3) or (1/5) portion of the entire output length.

CONVERTING BETWEEN AN ANGLE AND ITS CORRESPONDING TRIGONOMETRIC RATIO VALUE

The easiest way to convert between an angle and one of its corresponding trigonometric ratio values, or vice-versa (in reversal), is to use a pre-written reference table or a ("scientific") calculator that has these ((automatic) calculating or conversion) functions or abilities. Section 4, the ADVANCED TOPICS section of this book, will show you some formulas for how to make these calculations, especially when a scientific calculator or reference table is not available. These calculations can be greatly assisted by using a simple ("5 function") calculator, and if that is not available, performing the calculations by with pen and paper can always be used.

Ex.   What is the sin of 60° ?

Using a scientific calculator, first enter 60 and then press the SIN function key or button. The result displayed should be about 0.866 when the calculator is in standard degrees angle input and output (processing, conversion) mode.

SIN 60° = 0.866

Ex. If the sin value of an angle is 0.866, what is the angle ?

Using a scientific calculator, first enter 0.866 and then press the inverse sine key. Sometimes this key is labeled INV SIN (INVERSE SINE), SIN^-1 (this is just a certain notation or indication, so do not use the mathematical reciprocal since the notation here rather means the inverse function of, rather than an inverse or reciprocal value), ARC SIN or ARCSIN (as in the arc or angle of a circle). This specific calculator key needed could be located on the calculator where some other common key (such as the SIN key itself) can be reused as another or second function, such as INV SIN if you first press the calculator's "2nd function key" first, so as to activate all the "2nd function keys". The specific "dual or second function key" here is often for the SIN key itself. The result displayed should be 60 which is to be interpreted as 60 degrees = 60°. Expressing this mathematically:

ARCSIN 0.866 = 60°

The formal or standard mathematical expression for the corresponding trigonometric function of a given angle,
and the formal or standard mathematical expression for the inverse trigonometric function to find the corresponding angle given its trigonometric value have this form:

To find the corresponding trigonometric value of a given angle:

Actually, this table can be reduced to a single trigonometric value (such as SIN ¢) per angle since the other two values, COS and TAN, associated or corresponding with that specific angle, can actually be calculated from that just the SIN value as will be shown further in this book. Many formulas for the conversion of trigonometric values will be shown ahead. With the table above, you can find the SIN, COS, or TAN trigonometric value of an angle. On the other hand, if given the trigonometric value, you can work in a reverse manner so as to find the corresponding angle.

Notice that TAN 45° = 1, this happens when the side opposite the angle, and the side adjacent the angle are equal in value and their ratio is 1. Ex. TAN ¢ = (side opp ¢/side adj ¢) = 5/5 = 1. Some other helpful values to consider memorizing are:

SIN 0° = COS 90° =   0
SIN 30° = COS 60° =   0.5
SIN 45° = COS 45° = ~ 0.707
SIN 60° = COS 30° = ~ 0.866
SIN 90° = COS 0° =   1   : the maximum SIN or COS trigonometric value

You can plot the above values to show the relationships graphically. For example, letting x = ¢, and y = sin x = sin ¢, so as to view the sin ¢ trigonometric curve or waveform.

a = 5m SIN (36.87°)
a = 5m(0.6)
a = 3m

We can solve for side (b) since it is used in angles A's trigonometric cosine ratio:

COS A = b/c:

b = c (COS A)
b = 5m COS (36.87°)
b = 5m(0.8)
b = 4m

Checking, using the Pythagorean Theorem:

$c^2 = a^2 + b^2$
$5^2 = 3^2 + 4^2$
$25 = 9 + 16$
$25 = 25$: checks

It is very helpful to remember that the side opposite an angle (formally) corresponds to that angle. For example, side a is opposite from angle A. Side b is opposite from angle B.   Side c is opposite from angle C.   It is also very helpful to remember that the two sides of an angle correspond to the other two sides. For example, angle A is composed of sides b and c.   Considering the above example:

Side (a) can also be found and-or checked using angle B's trigonometric COS ratio:
COS B = adjacent / hypotenuse = a / c
a = (COS B) c

Side (b) can also be found and-or checked using angle B's trigonometric SIN ratio:
SIN B = opposite / hypotenuse = b / c
b = (SIN B) c

If the airplane is farther away, and yet at the same altitude or height, the angle it appears to be above or with respect to the horizon would be less and perhaps giving the appearance that it is at a lesser height.

A SMALL ANGLE ERROR CAN MEAN A LOT OF ERROR

The following describes the type of situation when a small, seeming insignificant angle error can grow and become a large and significant (meaningful, large) error. First, a common or practical example is when a rod or something like a tree branch moves (ie., rotates, like on a pivot) just a small angle and distance one end, and it will correspondingly move a much larger distance at its other, more distant end.

Observe the drawing below, the shortest distance between two fixed points is the straight-line distance between those points. [FIG 101]

If something is to go from point A to point B, whether it is to move from point A to point B, or be some type of construction that connects the (A and B) points, and is "off" ("the mark", exactness, an error) by only a small angle, say a 1° error with respect to the straight-line direct course from A to B, how much will it actually be or result in the amount of (distance) error from point B? This type of question can be analyzed without unnecessary complexity by using the concepts of a right-triangle: [FIG 102]
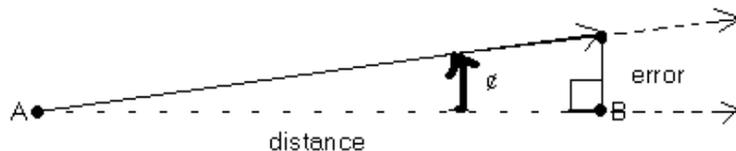
Considering the angle remains constant, the amount "off" or error depends upon the distance between those two points (A and B). The smaller the distance, the smaller the error, and vice-versa. In many non-critical applications, small errors are typically accepted without being a problem. Notice in the drawing above that the error value (the distance between the expected location and the error location (due to the angle error) can be considered as a side of a right triangle, and the distance between the (start and ending) points can be considered and analyzed as another side of that right triangle, particularly, the hypotenuse side.

From: TAN ¢   = $\frac{opposite}{adjacent}$ = $\frac{error}{distance}$: ¢ = "error angle", solving for error:

            error =   (TAN ¢)(distance)   : the farther (greater) the distance from the vertex ("start or origin") of the angle,
                                                                the greater the error. This is expressed in the equation as a direct relationship.
                                                                Also, since TAN ¢ is directly related to the angle, the larger the amount of (error)
                                                                angle from the (reference) angle, the larger the "error" distance.

If the distance between the points is 1,000 feet, and the "error angle" is only 1°, the (perpendicular) distance error is:

error = (TAN 1°)(1,000 feet) = (0.017455064)(1,000 ft.) = 17.455 ft.

THE APPARENT SIZE OF AN OBJECT

You or someone else might want to know the actual size of a distant object that you have seen. The discussion below initially assumes that there are no other nearby objects to it with their known (as a reference for measurement and comparison) size.

A object having a certain constant actual size will appear bigger when closer, and smaller when more distant. But why? The reason is that the apparent (visible with your eye) angle of the object is changing, and that the (observed, observation) angle size or value is related to the objects distance from you. The specific mathematical relationship between the observed angle and the objects distance from you is an inverse relationship. In more technical words or terms, it is said that when the distance between you and the object changes the angle that the object will subtend (be directly across from, and corresponding to) will change. When the object is closer to you, the apparent or observed angle of its size or width will be greater, yet the true or actual size of the object is still the same. As the object gets farther away from you, its apparent or observed angle and size will be less.

Consider these illustrations of the same object being farther, and then closer: [FIG 105]
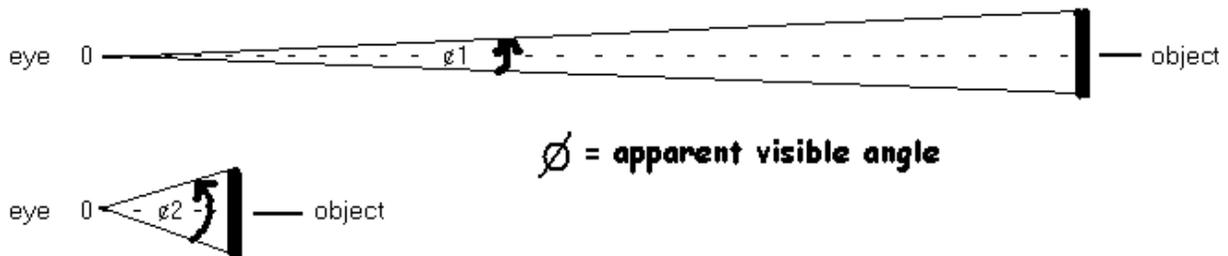
Notice that when the same object is farther away from you, that the ("height" or "width") angle is smaller. Clearly, the distance to the object and the angle are inversely related. However, you still cannot determine the actual size of an object just by this angle alone. Two or more objects of various sizes may subtend the same observed angle and visually look the same size when the distance to each particular object is such that this same angle condition exists. For example: [FIG 106]
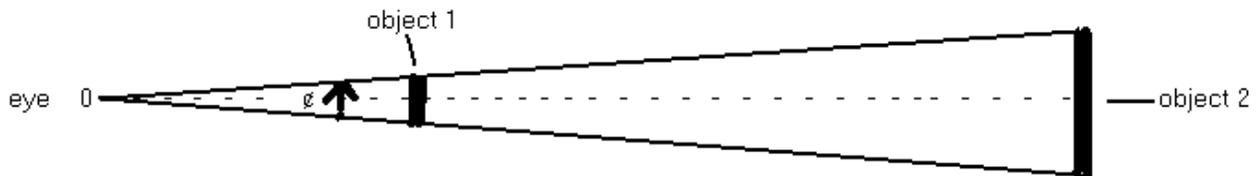
As seen above, even though the angle is the same for both objects, object 2 is clearly farther away and bigger. These object will appear as the same size even though they are not. This is an example of an "optical illusion". In this situation both the actual size of the objects and distances to them are not known and can only be estimated if there is not other data or a reference size known.

Basic Lens Or Refractor Telescope

objective lens

light in

eyepiece lens

concentrated light out

Fo

Fe

focus

light

objective lens
focal length = Fo

eyepiece lens
focal length = Fe
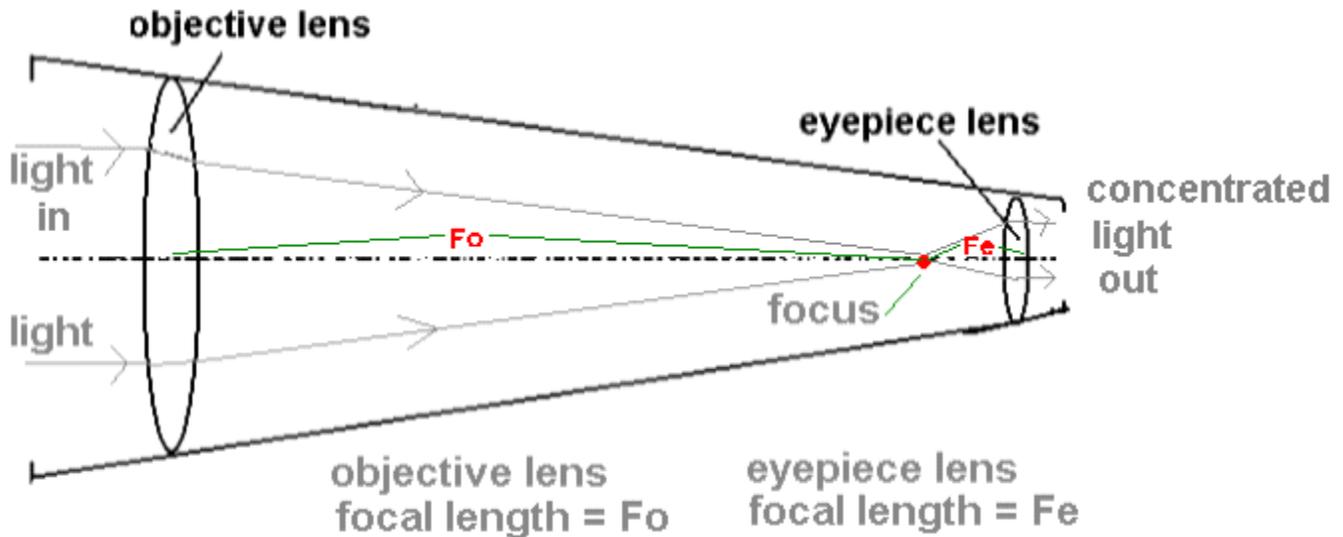
Today, the initial magnification of an image by using a telescope is still the most important factor, but with modern image sensors (ie., a digital camera) with many pixels (picture elements or parts) being able to produce a higher **resolution** (smallest discernible distance or portion on or of the object and-or its image) than that possible with the human eye, it increases the effective telescope system magnification and resolution, perhaps by two or three times more than without it. Rather than state the optical magnification of a telescope, it is perhaps better to state the final image resolution of the entire telescope system including the image sensor and-or photograph. For example, 1 pixel of the image or photograph is equal to a certain value of degrees of arc (ie., viewing angle) in the sky, and-or some minimal (resolvable, resolution) distance on the subject itself. A smaller arc and-or image resolution of the entire telescope system is a good comparison of the abilities of a particular telescope and-or telescope system. Many modern telescopes, such as those having the "(Isaac) Newtonian" design, use silver or aluminum coated glass mirrors, and in such as way and combination so as to reduce the overall size of the telescope and yet still increase the focal length of the telescope to have a high amount of magnification.   It is rather better to state the minimal angle that a telescope can resolve (make clear, distinguish), because for example, if the (linear) resolution or minimal part of an object is say 1 meter of length of an object and-or in the (photographic, or screen) image of it, if the distance to that object changes then that value of linear resolution is not constant and will change, but still, the (minimal) resolution angle of the telescope system will remain the same,   A (maximum) image or resolution angle is sometimes called the (total, focal, "viewing angle") "**field of view**" or ("viewing field" of the width seen at a distance) of the telescope, binoculars or camera system, and may be stated as a ratio of distances such as 100 meters wide (across. left to right) at 1000 meters distance, or a specific total viewing angle, say 35°. This angle is then found from: Tan ¢ = (side opposite) / (side adjacent) = 100m/1000m = 0.1 , and ¢ = arc-tan (tan ¢) = arc-tan 0.1 = 5.71° This field of view angle is usually not the (smallest) resolution angle, and for example, the resolution angle may be just a fraction of this value, say (1/100th) of 5.71° and that value is: 5.71°/100 =   0.0571° Note that the resolution angle by a human eye vision may be not as fine as that of a image sensor with an enormous number of pixels where each is an images or light sensor.

Another concept with telescopes, microscopes, and other lens systems is called "**depth of field**", and this is the range of distance to and behind the object that it will still be in focus to a high degree of clarity. Generally, reducing the opening to a center area of a lens will increase the depth of field. This lens area is also a flatter (ie., less of a curve) portion of that lens. Microscope lenses are for very high magnification, and are more spherical in shape, and these small lens shapes, like a bead of clear glass with a relatively high curvature, have a very small depth of field distance. The lens must be very close the subject or object being viewed, and a cover slip (glass cover) is often placed over some objects being viewed so as the lens does not get debris on it. These objects to be viewed are initially placed onto a larger clear glass slide or

APPROXIMATING ANGLES AND TRIGONOMETRIC VALUES GRAPHICALLY

If a line segment, acting as a hypotenuse side of a right triangle, is given a length of one (1) unit (in "relative terms" or portions, this whole =1=100% value can represent any other whole or entire length value) and is rotated about the origin (O) of a rectangular coordinate measuring and location system, it will create an angle between that line and the X or horizontal axis. The Y coordinate of the endpoint (P) on the line will be equal to the sine of the angle, and the corresponding X coordinate of this endpoint will be equal to the cosine value of that angle. The point's location is therefore: P(x,y) = P(cos ¢, sin ¢). This is easily seen when lines are extended or "dropped" to both axis from that endpoint, and therefore creating a right-triangle construction that can be mathematically analyzed as a right- triangle. You may need a device that can measure (or create) angles such as a protractor. To "protract" is to extend outward as opposed to "retract" which means extend inward. In general, better or more accurate results are often easier to achieve by drawing a larger graph. [FIG 107]
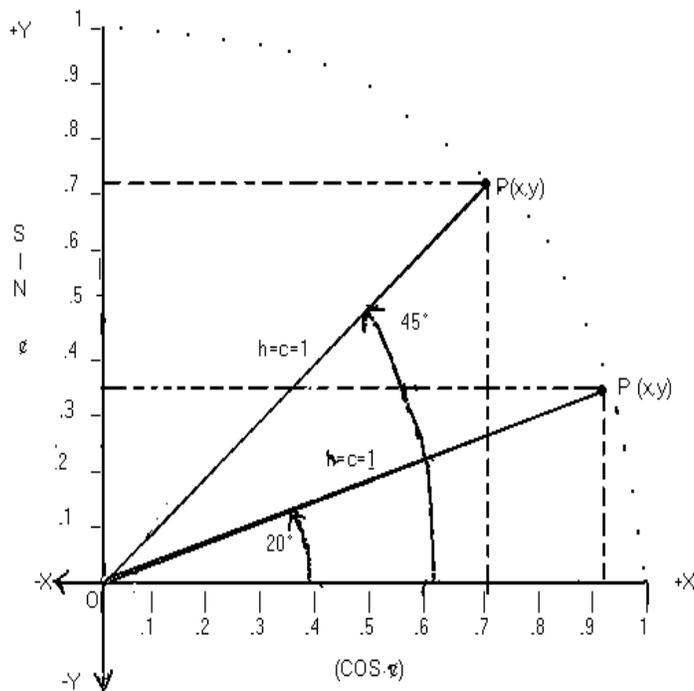
Using the graph above, and the definition of SIN ¢:

SIN ¢ = __opposite__   =   _____y_____
               hypotenuse   hypotenuse

Since the hypotenuse here is equal to 1, or is considered as a relative value of 100% = 1.0 portion of some other length:

SIN ¢ = y   : If the hypotenuse was a relative or percentage value of some length, then (y) and (x)
                               will also be a relative values.

Likewise, COS ¢ = __adjacent__   =   _____x_____   Since the hypotenuse here is equal to 1:
                          hypotenuse   hypotenuse

COS ¢ = x   : clearly in the graph, as the angle increases, the cosine of that angle decreases, and its sine value
                 increases, but **not** linearly, such as for example that sin¢ + cos ¢ = would always some constant value

trigonometric functions, then in a reverse type of manner as the angles get larger than 90°, the corresponding SIN ¢ values are decreasing from 1 to a value 0 at the angle of 180°.   Therefore, an angle in the "second quadrant" having a value between 90° and 180° will have a corresponding angle and trigonometric value of an angle less than or equal to a 90° angle in the first quadrant.

By applying the Pythagorean Theorem to the triangle constructions that can be created in the graph, alternate expressions for SIN ¢, COS ¢, and TAN ¢ can now be derived:

$h^2 = x^2 + y^2$ or $r^2 = x^2 + y^2$,   when h, the hypotenuse, is considered as the radius (r) of a circle.

Since x = COS ¢ and   y = SIN ¢, h = 1,   and substituting this into the above expression :

1 = (COS ¢)^2 + (SIN ¢)^2   **: You can remember this as the (trigonometric) "Pythagorean identity".**
                                      An alternate syntax or notation form often used is:

1 = COS^2 ¢ + SIN^2 ¢   : For this indicated notation, only the trigonometric value is squared. Also in this
                                special expression, (COS^2) and (SIN^2) are not being multiplied to the angle (¢).

Solving for SIN ¢ and COS ¢ from this expression:

SIN ¢ = $\sqrt{1 - COS^2 ¢}$ and   COS ¢ = $\sqrt{1- SIN^2 ¢}$: some TRIGONOMETRIC CONVERSIONS
                                                                              (Pythagorean-like equations)

The TAN ¢ in relation to the corresponding SIN ¢ and COS ¢ values can be found using:

from: TAN ¢ = _opposite_ = _y_ ,   and SIN ¢ = y,   and   COS ¢ = x
                adjacent   x

TAN ¢ = _SIN ¢_
          COS ¢

        Another method for approximating trigonometric values is by drawing a graph of all the angles from 0° to 360° on the x-axis, and their corresponding trigonometric function values on the y-axis. This can be easily accomplished by selecting several of the angles between and including 0° and 360° and their corresponding trigonometric values, and then connecting the points with a smooth curve to represent and approximate all the other possible point values. The resulting curve is often called a periodic (repeating, repetitive, "oscillating" or cyclic) waveform.   Below is the sine waveform. Besides trigonometry where its' concepts originated, it is used to (graphically, visually) represent and analyze many other periodic events such as alternating (increasing in value, and direction) current (ie., ac electricity) and sound waves. Since a triangles trigonometric values are usually for angles less than or equal to 90°, and the values are cyclical or repeating for the larger angles, you may wish to only draw a graph with great detail or precision for only the angles between and including 0° and 90°.

Plotting a few select "key" points (basically at the 45 degree increment points) and drawing a smooth curve to connect these points:

already 0.50 which is already at half of the maximum SIN value of 90° which is 1.0, and that 30° is only at one-third of the way to 90°. In short, the relationship between the angles and their corresponding trigonometric values is non-linear (ie., non-proportional, or some other direct mathematical relationship), and non-algebraic (non-exact, no simple algebraic formula, irrational in value). Many algebraic-like methods have been developed that simulate or express their relationship so as to come as close as desired (usually in terms of practicality) to the true value or solution. Some of these will be presented further ahead in this book. Also notice that throughout the full 360° angle range, that the angles trigonometric (absolute) values (except at 0°, 90°, 180° and 360°) are repeated a total of four times during one cycle from 0° to 360°. Starting from the first quadrant (0° to 90°), each same trigonometric value instance corresponds to another quadrant of the rectangular coordinate system. The cosine curve is very similar to the sine curve above except that it effectively begins at the 90° mark of the sine curve and has a starting value of 1 instead of 0. This can also be thought of as like a sine curve being ("phase") shifted (in time and-or values) by 90°, and is often spoken as that the cosine curve (physically, in time) "leads" (ie., begins first) those of the sine curve. The sine curve is then often spoken as "lagging" the cosine curve by 90° in time and-or values. [FIG 109]
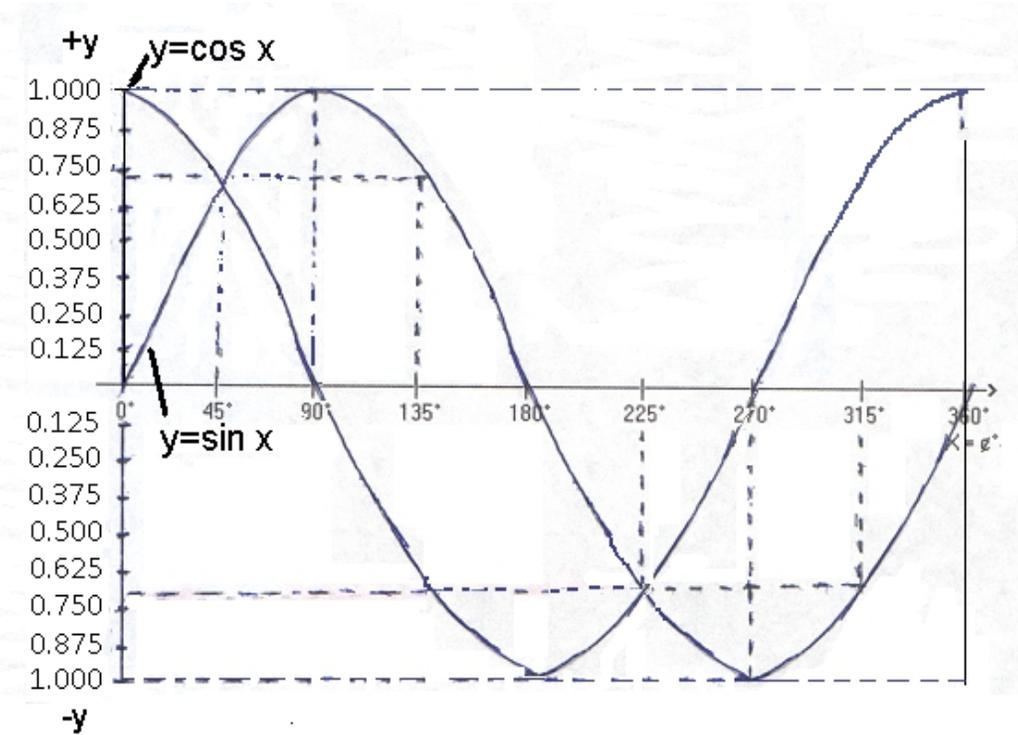
The SIN and-or the COS curves are known as periodic curves since the values will essentially keep repeating periodically as the angles grow larger over time. If the angle is greater than 360°, you can keep subtracting 360° from it to find its equivalent angle that is less than or equal to 360°. Since simple harmonic motion (ie., periodic motion or vibrations, that repeat after a certain amount of time, and are therefore said to have a frequency (ie. how often, or "hertz") or amount of cycles per amount of time, such as a second of time) such as the swing of a pendulum, sound and electro-mechanical generated electricity, and radio waves are periodic in nature, the SIN and-or COS curves are used to represent and mathematically analyze the physical creation, nature and motion or movements of this type. Here, the (x) axis becomes time, and the (y) axis becomes the voltage, current, or the height of a pendulum swinging which is its distance from an imaginary horizontal line directly equal to the height of the pendulum when it is not in motion.

As our planet Earth revolves around the Sun during a full year of time (indicated on the x-axis), the (apparent or effective) tilt of Earth (with respect to the solar (Sun) plane or "ecliptic", the direct line or orbital plane between the Earth and Sun) is (nearly) sinusoidal in nature. On about March 21 and September 21, the tilt (noted on the y-axis) of the Earth with respect to the Sun or solar plane is 0°. On about June 21 (toward the Sun) and December 21 (back or away from the Sun), the

linear in nature and is constantly changing at a different rate. Nonetheless, it can be averaged like that of a circular orbit approximation as shown in this book further ahead, and so as to express the basic concept of Earths tilt angle with respect to the Sun. As mentioned, the Earth does not tilt with respect to the star Polaris, and is rather an apparent tilt due to the orbit of Earth about the Sun.

COMPLEMENTARY ANGLES OF A RIGHT TRIANGLE

Since the right angle (C) of a right triangle is 90°, the other two angles of a right-triangle will always total to 90° since a triangles interior sum of angles is always a total of 180°. These other two angles that sum to 90° are called complementary angles, because they essentially complement each other so as their combined sum is equal to a (complete) 90° angle.   Given two complementary angles, if one increases by N degrees, the other will decrease by N degrees so as to maintain their 90 degree sum.

A + B + C = 180°   Here, C=90° and is the right angle of a right triangle:
A + B + 90° = 180°   after transposing 90°:
A + B = 90°   **A and B, each less than 90°, are called the complementary angles of a right triangle and-or a right angle which is a 90° angle.**

[FIG 111]
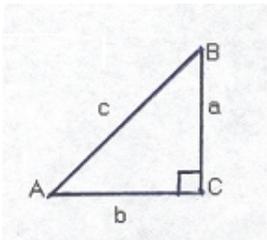
For the figure above, Angles A and B, the non-right angles of a right triangle, are said to be complementary angles since the value of either angle always complements the other angle in order to create the required sum of 90° so as the internal angle sum of a triangle is 180°.

The trigonometric co-functions of complementary angles are equal. This is described below in mathematical terms:

SIN A = COS B   and   SIN B = COS A

$$\frac{a}{c} = \frac{a}{c} \qquad\qquad \frac{b}{c} = \frac{b}{c}$$

If you had only a sine table, you would then also know the cosine of its complementary angle and vice-versa.

Ex. In a right triangle, if A = 50° , then the complementary angle, angle B = (90°- A) = (90°- 50°) = 40°

   SIN A = COS B
 SIN 50° = COS 40°
   0.766 = 0.766

   SIN B = COS A
 SIN 40° = COS (90° - 40°) = COS 50°
   0.64$\underline{3}$ = 0.64$\underline{3}$

   TAN A = COTAN B   and   TAN B = COTAN A
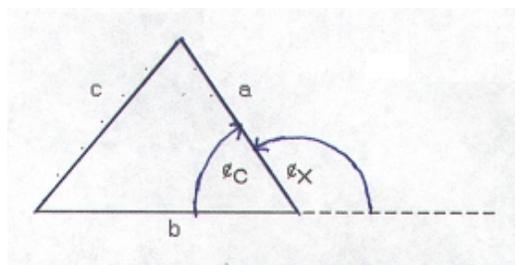
$180° = ¢C + ¢X$: by observation, ¢C and ¢X are supplementary angles that sum to 180°.

$180° = 60° + ¢X$ Solving for ¢X by subtracting (or transposing) ¢C = 60° (the supplementary angle of ¢X) from each side of the equation, and then combining like terms:

$¢X = 120°$: also of note is that this value is also equal to the sum of the other two interior angles, here ¢A and ¢B, since all three interior angles of a triangle always sum to 180°.

The sum of all the exterior angles of a triangle is always 900° which is 5 times more than 180°.

VERTICAL ANGLES

Though not specifically related to triangles, the concept of vertical angles can play an important role in the further study of trigonometry, constructions and other fields of study. Vertical angles were previously briefly mentioned in this section. Whenever two lines cross each other, 4 angles are produced, all of which will have their vertex at the point of intersection of those two lines. 2 pairs of vertical angles are produced, and the angle for each corresponding pair will be equivalent. The equivalent pairs are directly "opposite", "across" or on the other side of the vertex point. In the drawing below, A and B are corresponding vertical angles, and C and D are corresponding vertical angles. [FIG 114]
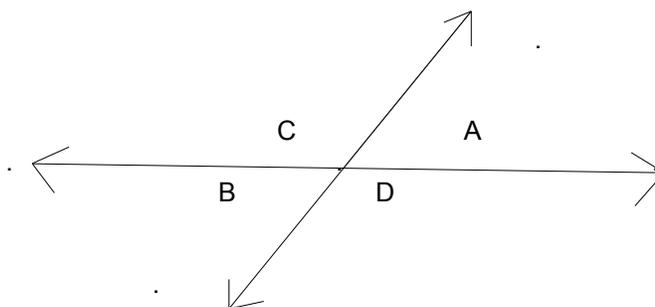
What is the proof that vertical angles are the same (besides the somewhat obvious visual graphical proof)? In a worded form, for example, any and all supplementary angles to an angle in question are equivalent. If angle C is the supplementary angle of angle A, and angle D is also a supplementary angle (though along a different line that effectively creates another 180° straight-angle) to angle A, it is only logical to equate angles C and D. Here's an algebraic proof:

Due to supplementary angles:

$A + C = 180°$ and:

$A + D = 180°$ therefore, equating the two equations since they are both equal to 180° :

these two lines are practically parallel, but the will intersect at a point near infinity as shown here

this ¢ is nearly 0°

great distance, near infinity
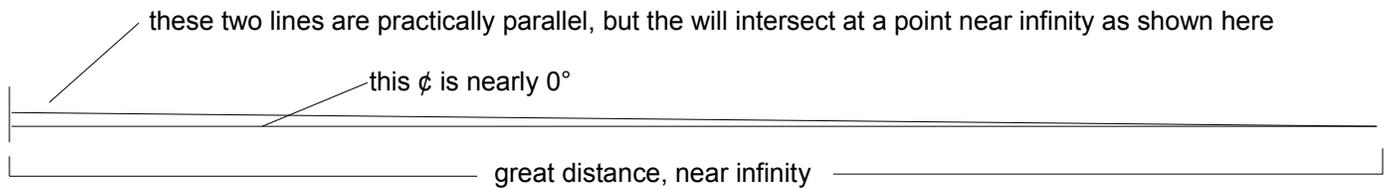
The theoretical angle (¢) measured at this (imaginary) point of intersection is very close to a value to 0°. If we consider this (almost imperceptible) angle as the smallest of all angles possible, and angles B and C as the other two angles within this triangle construction, we get:

$180° = B + C + ¢$:180° = interior sum of angles of a triangle

Since the angle (¢), as mentioned above is nearly 0°, for all practical purposes, it can then be considered as 0°.   Hence we get:

$180° = B + C + 0°$
$180° = B + C$ equating this to the supplementary angles of A and B, and C and D:

$180° = A + B = B + C = C + D$ therefore:

$A + B = B + C$ solving for A:
$A + B - B = B + C - B$
$A = C$

$B + C = C + D$ solving for B:
$B + C - C = C + D - C$
$B = D$

Also, though not quite obvious, angles B and C are supplementary angles due to that their sum is 180° as indicated above, and here is another verification that B and C sum to 180°:

Since: (A + B) = 180°   and   (C + D) = 180°   adding both equations, or all the angles:
          (A + B) + (C + D) = 180° + 180°
          A + B + C + D = 360°

Since lines 1 and 2 are parallel, no matter how close (even if "on top" [coinciding] of each other) or far apart they are, the angles created by a transversal line at the intersection of any one parallel line will always be the same as those created at the intersection of another parallel line by that transversal line or another transversal line parallel to it. Since lines 1 and 2 are parallel or correspond to each other, the angles at each intersection of the transversal line will correspond and be the same value.

Since C corresponds to A: C = A
Since B corresponds to D: B = D

          A + B + C + D = 360°   using algebraic substitution:
          C + B + C + B = 360°   combining like terms:
          2B + 2C = 360°   factoring 2 from each term on the left side:
          2(B + C) = 360°   after dividing each side by 2:
          B + C = 180°   : verification

Perhaps a simpler way of expressing the above method and result is:

A = B = D = C   : all the angles shown in the figure above have the same value.

Now that transversal lines and the corresponding angles created have been discussed, a verification that the sum of all three interior angles of any triangle is always 180° can be described. First, observe the angles created by the intersection of two lines: [FIG 118]
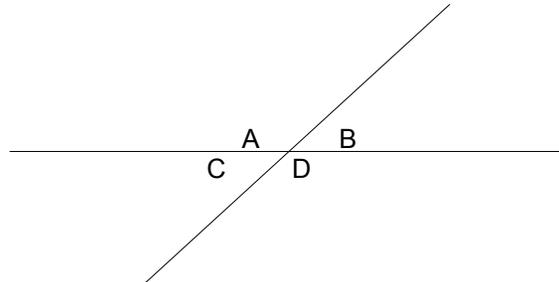
Since A and B are supplementary angles, and C and D are supplementary angles, the sum of angles around any point (as with the center of a circle) is:

$$180° \ + \ 180° \ = \ 360°$$

It will now be shown that the sum of the interior angles of four sided (of any lengths) figures called quadrilaterals (any four sided portion of a plane; quad meaning four, and lateral meaning side) is always 360°. Squares and rectangles are common examples of quadrilaterals. A drawing of an quadrilateral called a parallelogram is shown below with its sides extended for the analysis.   Note that a rectangle is a special instance of all possible parallelograms, and a square is a special instance of all possible rectangles. [FIG 119]
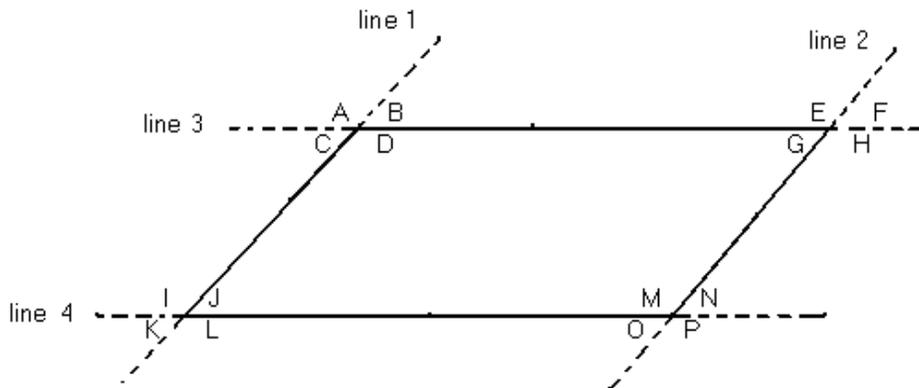
We see that a parallelogram can be drawn by intercepting two parallel lines by two parallel transversal lines.

Since there are 4 vertices or "corners" of the quadrilateral, and we know that the sum of the angles surrounding each corner or point is 360°, the total sum of all the angles indicated is:

$$(360°)(4) \ = \ 1,440°   \text{ : internal and external sum of angles of a quadrilateral}$$

We are to verify that the sum of interior angles of D, G, J, and M is (always) equal to 360°. By construction, line 1 and line 2 are parallel. Line 3 and line 4 are both parallel transversal lines to line 1 and line 2. From the concepts of vertical angles and corresponding angles, we can show some relationships of all the angles above (that total to 1,440°):

A = D = E = H   : vertical angles, and since they all equal the same value of D, they sum to: 4D
B = C = F = G   : vertical angles, and since they all equal the same value of G, they sum to: 4G

(J + M) = 180°

(D + G) + (J + M)　　　removing grouping symbols:
　D + G + J + M　　　:sum of the interior angles of the parallelogram
　　180°　+　180°
　　　　360°　　　　　　: checks

Here's another simple verification that uses all the angles on the "outside" (external) of the parallelogram to find those that are "inside" (internal) the parallelogram:

(A+B) = 180° ,　(E+F)=180,　(K+L)=180° ,　(O+P)=180°,　 (I+C)=180° ,　(N+H)=180°

The sum of the above pairs is:　6(180°) = 1080°

Subtracting this specific external angle sum from the total possible vertice or "corner" angle sum of (360° x 4) = 1440° we have the sum of just the parallelograms internal angles:

1440° - 1080°　=　360°　　　: checks

Perhaps the simplest verification that the interior angles of a parallelogram sum to 360°:

Here is a simple verification that the sum of the interior angles of a parallelogram is 360°.　[FIG 120]
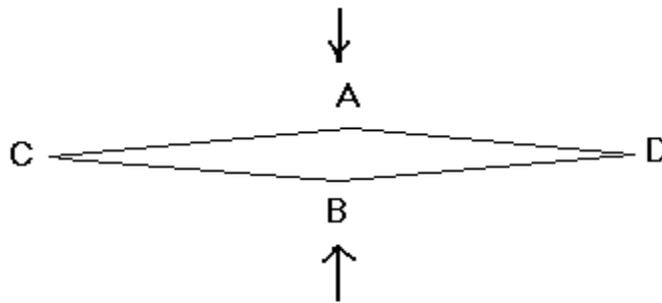
Consider A and B being pushed closer together as illustrated by the directional arrows. Those angles, A and B, will get larger and each will approach a value of a 180° "straight (line)" angle. At the same time, this will cause both angles of C and D to approach a value of 0 °.   The net sum of internal angles of this parallelogram will be practically:

　　　　A　 +　B　　+ C　+　D
　　　180°　+　180°　+　0°　+　0°　= 360°　　　,　 or something as: 359.99999....　 hence, approaching　----> 360°

　　　When a (diagonal) line is drawn from two diagonal (ie. not on the same line) or "opposite" corners of a parallelogram, it will divide it into two congruent (identical) triangles. If the diagonal is extended for the analysis, and with the concept of supplementary angles, it is easy to see that the total vertice (ie. peak, corner, where two sides meet) sum of 1440° will be divided into two 720° (half of 1440°) sums. It is also easy to see that this line divides the interior sum of 360° into two 180° sums. Each 180° sum corresponds to the interior angles of a general or representative triangle. [FIG 121]

THE HEIGHT WITHIN A RIGHT TRIANGLE

Given any side of a triangle and considering it a base side, the perpendicular distance to the corresponding internal angle which is directly opposite from it is called the height or "altitude" at that base side or vertex. Sometimes the base side of a triangle needs to be considered as extended so as to create the perpendicular height line.

The two sides, other than the hypotenuse side, of a right triangle are equal to the height (or "altitude") from a vertex to the corresponding base side it intercepts perpendicularly (at 90° to that side). The third height (shown as (h) in the figure below) within a right triangle is the distance from the vertex of the 90° angle to the hypotenuse side it intercepts perpendicularly. [FIG 122]
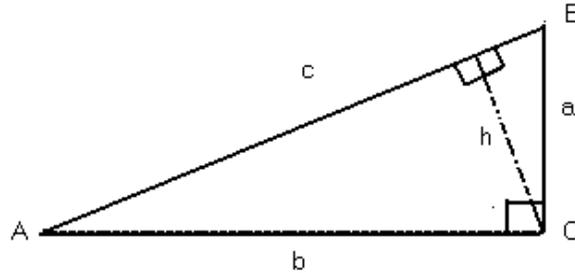
As seen in the drawing of a right-triangle above, the height at (base) side (b) to the vertex B is equal to side (a). The height at side (a) is equal to side (b). A height line always intersects its corresponding base (of the height) line perpendicularly = (90°). In the right triangle above: height at base side b = hb = a , and height at base side a = ha = b.

This height line to the hypotenuse (as a base) side will divide the right triangle into two smaller and similar triangles to the original outer triangle. This can easily be verified from the fact that each contains a right angle and that the complementary sum of the other two angles within each similar triangle is always 90°. In the drawing above, the pair of right triangles (the larger "outer" triangle, and the smallest interior triangle) have the same angle B. The other pair (the larger "outer" triangle, and the larger interior triangle) have the same angle A. Still, each of these similar triangles will and must have (so as to be similar) angles equivalent to A, B and C. Since we know that we have similar triangles, the simplest method to solve for the length of this height line (at base side c) is by using the concept of proportions (equivalent portions) of the sides of similar triangles sides, for example:

$$\frac{hypotenuse}{adjacent} = \frac{c}{b} = \frac{a}{h} \qquad \text{solving for h:}$$

$$h = \frac{ab}{c} \qquad \text{: INTERIOR HEIGHT FORMULA FOR A RIGHT TRIANGLE}$$
$$\text{: c = hypotenuse side}$$

Ex. In a right triangle, a=3, b=4, c=5, find the altitude (height) from the right angle to the hypotenuse side.

$$h = \frac{(3)(4)}{5} = \frac{12}{5} = 2.4 \qquad \text{:height or altitude at side c}$$

From the equation, also notice the possible methods to solve for a side if it is not known. For example, here is a simple formula for the hypotenuse (side c) if the height at side (c) is known:

$$c = \frac{ab}{h} \qquad \text{: side (c) is the "base side" to altitude (line) of (h)}$$

LAW OF SINES

By using algebraic manipulation on the trigonometric ratio formulas for the angles, we can derive some other useful formulas. [FIG 124]
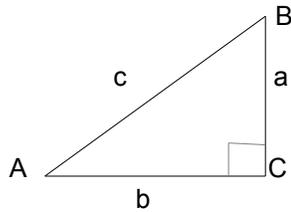
From: SIN A = $\dfrac{a}{c}$ and   SIN B = $\dfrac{b}{c}$ therefore, mathematically:

$$c = \dfrac{a}{SIN\ A} \quad \text{and} \quad c = \dfrac{b}{SIN\ B}$$

, therefore, this is another method of calculating the hypotenuse without using the Pythagorean Theorem.   The hypotenuse of a right triangle is equal to the ratio of any other side divided by the sine of its corresponding angle. For further practical use, the hypotenuse is also the diagonal line within a rectangle since a rectangle can be divided into two right triangles. In fact, the word hypotenuse seems to have originated first with rectangles since the root meaning of the word "hypotenuse" is to stretch or extend as we see with a diagonal line going (stretching or extending) from corner to opposite corner within a rectangle. Also notice that:

$$c = c \qquad \text{substituting the equivalent values for c:}$$

$$\dfrac{a}{SIN\ A} = \dfrac{b}{SIN\ B} \qquad \text{: alternate Hypotenuse Formulas}$$

Since each fraction represents c, and SIN C = SIN 90° = 1, we can include side (c) into the equivalence above.   This equivalence is known as the "law of sines".

$$\dfrac{a}{SIN\ A} = \dfrac{b}{SIN\ B} = \dfrac{c}{SIN\ C} \qquad \text{: LAW OF SINES (here, verified for right triangles only)}$$

From this we also have these equivalent fractions:

$$\dfrac{a}{b} = \dfrac{SIN\ A}{SIN\ B} = TAN\ A \qquad \text{: Ratio Of Sines (derived for right triangles)}$$

$$\dfrac{b}{a} = \dfrac{SIN\ B}{SIN\ A} = TAN\ B$$

Using the law of sines, if you have enough information to place into any two of the three equivalences shown, you can algebraically solve for a side or a trigonometric ratio value.

Ex. a = $\dfrac{b\ SIN\ A}{SIN\ B}$ also, = a = $\dfrac{b\ (SIN\ A)}{(COS\ A)}$ = b TAN A   :as expected since TAN A = a/b

From: SIN C = $\dfrac{hb}{a}$, we get: hb = a SIN C   Equating the two expressions for hb:

    c SIN A = a SIN C   algebraically:

    $\dfrac{a}{SIN\ A}$ = $\dfrac{c}{SIN\ C}$: for all triangles

Now observe the following drawing of the same triangle: [FIG 126]

B

c   a

ha

A   b   C
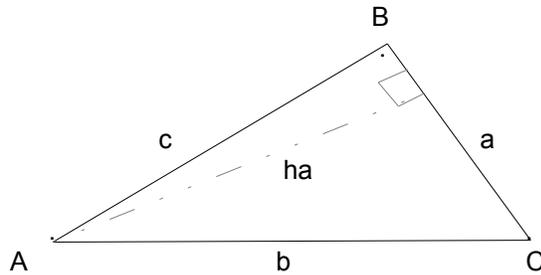
From: SIN B = $\dfrac{ha}{c}$, we get: ha = c SIN B

From: SIN C = $\dfrac{ha}{b}$, we get: ha = b SIN C

Equating the two expressions for ha:

    c SIN B = b SIN C   algebraically:

    $\dfrac{c}{SIN\ C}$ = $\dfrac{b}{SIN\ B}$: for all triangles

Equating this to the results of the previous drawing and equations (a use of what is known as
the Transitive Law where if a=b, and b=c, then a=c. In short, a = b = c):

    $\dfrac{a}{SIN\ A}$ = $\dfrac{b}{SIN\ B}$ = $\dfrac{c}{SIN\ C}$ **: LAW OF SINES (for all triangles)**

Some extra derivations from the above:

  $\dfrac{side1}{side2}$ = $\dfrac{SIN\ (corresponding\ angle\ to\ side1)}{SIN\ (corresponding\ angle\ to\ side2)}$, for example: $\dfrac{a}{b}$ = $\dfrac{SIN\ A}{SIN\ B}$

Since SIN 90° = 1, for a right triangle:

    $\dfrac{a}{SIN\ A}$ = $\dfrac{c}{SIN\ 90°}$ = $\dfrac{c}{1}$ = c   : where C is 90°, essentially, this is equivalent to SIN A = a/c
                        : for a right triangle only

Likewise:

By observing the equation of the area of an isosceles triangle, it then becomes clear that its area is only a certain fraction of that of a square having the same side length. The area of a square is s^2. Also note that the height of the isosceles triangle does not equal its side length (s) or that of a square with side (s). Looking at the above equation for the height of an isosceles triangle, we see that it is less than (s). Also, by the Pythagorean Theorem, if (s) is considered a hypotenuse and the longest side of a triangle, the height must be less.

An isosceles triangle has only two of its sides equal in length. The third side of different length is often called the "base" (side), and the two equivalent angles created at the intersection of sides that base are called the base-angles of the isosceles triangle: [FIG 128]

B

c   a   : ISOSCELES TRIANGLE

A   C
b
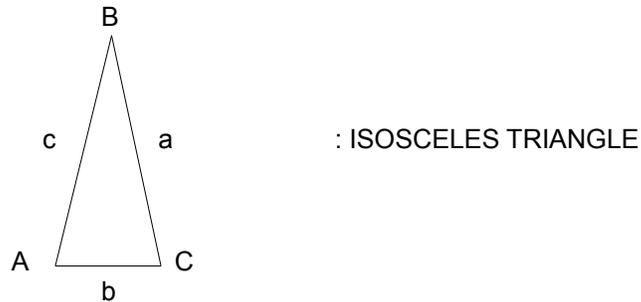
In the above drawing: a = c
                         A = C   : "base angles"   at   b = "base side"

You can verify that A=C by drawing the height at side b and observing the two equivalent right triangles created. The height (h) of an isosceles triangle can be found using the Pythagorean theorem with: $c^2 = h^2 + (b/2)^2 = h^2 + b^2/4$

A scalene triangle has all three of its sides different in length: [FIG 129]

B

c   a   : SCALENE TRIANGLE

A   C
b
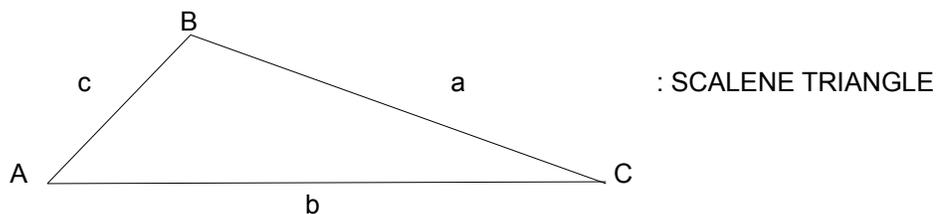
Congruent triangles are identical triangles. If all of the 6 parts (3 angles, and 3 sides) of two or more triangles have the same value,   then those triangles are congruent or identical triangles.   Even with a minimum of three specific corresponding parts having the same value is enough to determine that the triangles are congruent or identical, and all of the other corresponding parts will also be equivalent in value. Here is the minimum (3) number, and specific parts needed to determine if two triangles are congruent or identical:

    Three Sides   : commonly known as the "SSS" test ,   S = side

    An Angle And Its Two Adjacent Sides   : commonly known as the "SAS" test,   A = angle
    (or "two sides and the included angle")

    Two Angles And The Common Side Between Them   : commonly known as the "ASA" test
    (or "two angles and the included side")

T1

T2
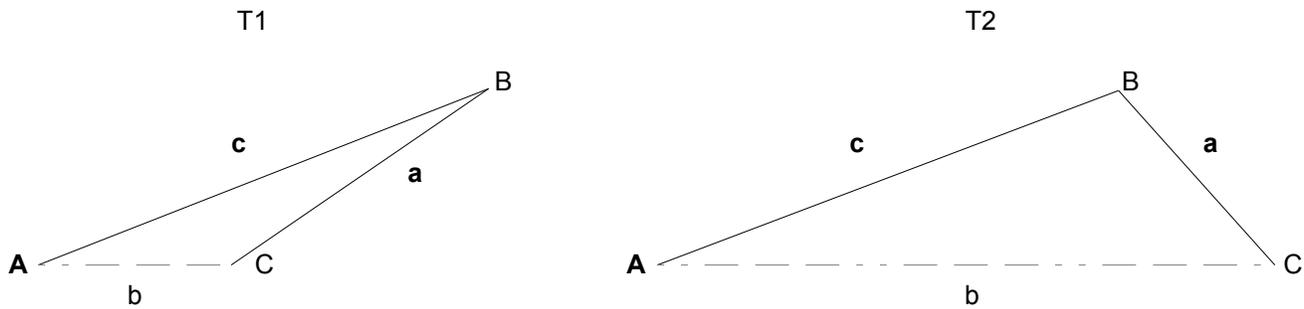
Given the triangle information, first consider if it contains a minimum of 3 and specific parts needed, such as one part being a side, so as to solve that triangle, and then consider:

When given just two sides of a triangle and one angle, if the angle is not at the intersection of those two given sides, then a condition of an ambiguity exists, and even though the law of sines could be used (on each different ambiguous triangle), as it does with the limited information given in this example. We are given sides (a) and (c), and the angle that we need is at the intersection of these two sides, and that angle is B which was not given. If B, and not A, was given, we would of had enough information to use the law of cosines formulas to solve for some other parts of this triangle, such as:

sideb^2 = sidea^2 + sidec^2 - 2(sidea)(sidec)(cos ¢B)   : a form of the law of cosines

Using the law of sines using: (a/Sin A) = (b/Sin B) = (c/Sin C), we can find C as about 23.90° OR (180° - 23.90°) = 156.01° due to the ambiguous condition being met. For both angles, their sine value is the same, here about: 0.405

Even though side (a) has the same length for both triangles, clearly, side (b) in T2 is longer than side (b) in T1. As mentioned previously, to continue solving either triangle, more information about the triangle must be known. This information can be found by measurement and-or by calculation using further known information about the triangle. Using the triangles above, here is a combined view of the two possible (ambiguous triangles): [FIG 132]
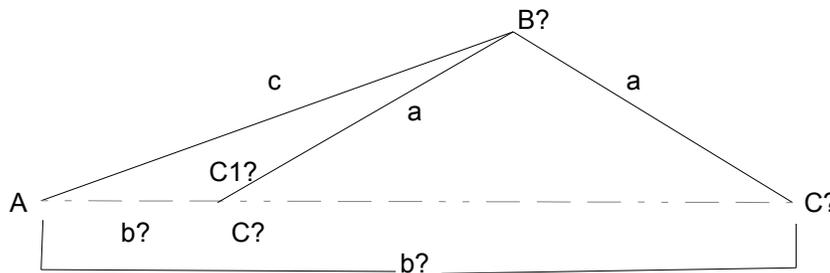
Given: a , A , c
Ambiguous: b , B , C
C + C1 = 180°
C1 = 180° - C

Notice the isosceles triangle indicated and having two sides equal to (a), and both base angles of it are equal to C. This fact will be of aid when solving for one of the two possible ambiguous triangle solutions. For one solution when ambiguity exists, C (which is the corresponding angle to the other given side, here side (c)) is greater than 90°, but if the given angle, here ¢A, is greater than or equal to 90°, there is no ambiguity since the other two angles will complete the 180° internal angle sum, and their sum will be equal to or less than 90°, and it is not then possible for another angle to be greater than 90°.   If side (a), and which corresponds to the given angle A, was longer than side (c), then the two equal sides of the isosceles triangle indicated cannot exist, and therefore, the isosceles construction will not exist, and there will be one unique solution only since the ambiguous condition does not then exist.

When given 4 or more correct parts of a triangle, the triangle can be completely solved, that is, there is no chance for an ambiguous part(s). When given 3 parts, the minimum number of parts to solve a triangle, one part must be a side (to distinguish it from all other possible similar triangles that have the same angles), however, the triangle can still be ambiguous (have 2 possible solutions). A triangle is ambiguous when given only two sides and an angle (<90°) that corresponds to one of those sides (ie., the given angle was not the angle between and at the intersection of those two given sides). We were not given angle (B) at the intersection of sides (a) and (c). All other (the three remaining) parts of

base
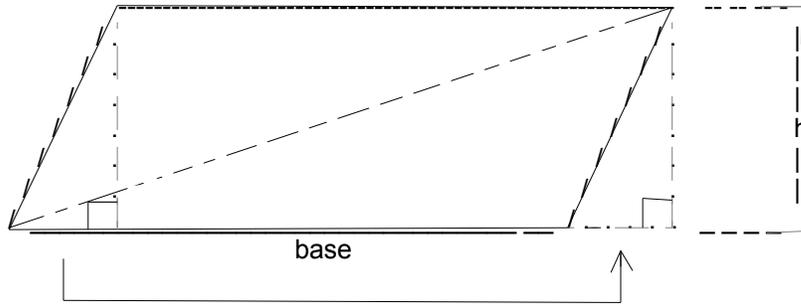
Using the Pythagorean Theorem on the two internal right triangles that compose the general triangle indicated in Fig 133:

$a^2 = b2^2 + h^2$ and $c^2 = b1^2 + h^2$ solving for h^2 in both equations:

$h^2 = a^2 - b2^2 = c^2 - b1^2$

$COS\ C = \dfrac{b2}{a} = \dfrac{b - b1}{a}$, hence b1 = (b - a COS C),   and   b2 = a COS C
therefore, b2^2 = (a COS C)^2 = a^2 COS^2 C

$a^2 - b^2 = c^2 - b1^2 = c^2 - (b - a\ COS\ C)^2$ expanding the binomial:

$a^2 - b^2 = c^2 - (b^2 - 2ab\ COS\ C + a^2\ COS^2\ C)$ using substitution, and distribution:

$a^2 - b^2 = c^2 - (b^2 - 2ab\ COS\ C + b^2) = c^2 - b^2 + 2ab\ COS\ C - b^2$ combining terms:

$a^2 - b^2 = c^2 - 2b^2 + 2ab\ COS\ C$ after solving for c^2 :

$c^2 = a^2 + b^2 - 2ab\ COS\ C$ **: LAW OF COSINES (for all triangles, side c)**
The formula is very similar to the Pythagorean Theorem,
but has another term added to it. Below is an example.

$c = \sqrt{a^2 + b^2 - 2ab\ COS\ C}$: side c, also:

$COS\ C = \dfrac{a^2 + b^2 - c^2}{2ab}$ **: useful when only three sides are given**
and you need to find some angles

¢C = ARCCOS (COS C)   : ARC COS essentially means to find the corresponding angle given
its corresponding COSine value. As cosine ¢ is a trigonometric
function (can be thought of as an equation or process) that depends
on the sides of that angle. arccos ¢ is a function that is essentially
the inverse of the cos ¢ function.

Similar formulas for sides (a) and (b) can be derived as:

$a^2 = b^2 + c^2 - 2bc\ COS\ A$: LAW OF COSINES (side a)

$b^2 = a^2 + c^2 - 2ac\ COS\ B$: LAW OF COSINES (side b)

The basic format (which has a Pythagorean Theorem-like form) of each of the above equations for the law of cosines is therefore:

CALCULATING ANY HEIGHT WITHIN A TRIANGLE

Here is a formula to calculate the (perpendicular) height (ie., "altitude") from any vertex of a triangle to its corresponding base side. The height between a side and the corresponding angle is also the shortest distance from that vertex to the opposite side. Though there are other methods, such as the law of cosines to solve for the height if enough information (triangle parts) about the triangle is known, the formula presented below can be helpful when a triangle construction is based upon a base side value and the (base) angles adjoining to that base side. [FIG 136]
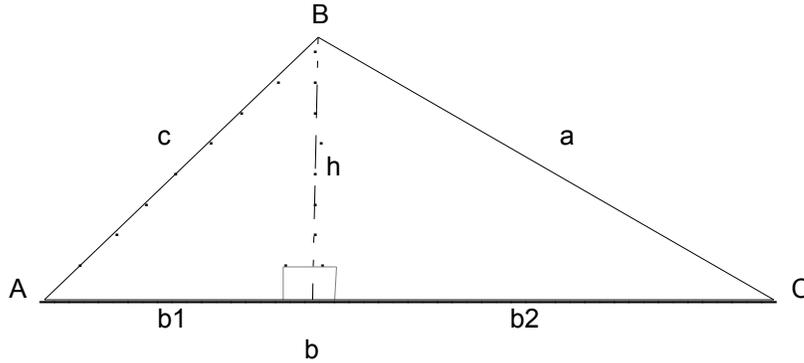
TAN A = $\dfrac{h}{b1}$　　　　TAN C = $\dfrac{h}{b2}$　　　　therefore:

b1 = $\dfrac{h}{\text{TAN A}}$　　　　b2 = $\dfrac{h}{\text{TAN C}}$

b = b1 + b2

b = $\dfrac{h}{\text{TAN A}}$ + $\dfrac{h}{\text{TAN C}}$　　　　: An example of solving for a base side using the base angles and the height at that base side. After combining fractions:

b = $\dfrac{h\ (\text{TAN A} + \text{TAN C})}{\text{TAN A TAN C}}$　　　　: an optional formula. Solving for h = hb:

hb = $\dfrac{b\ \text{TAN A TAN C}}{\text{TAN A} + \text{TAN C}}$　　　　: height at base side b given the adjacent angles. Likewise:

ha = $\dfrac{a\ \text{TAN B TAN C}}{\text{TAN B} + \text{TAN C}}$

hc = $\dfrac{c\ \text{TAN A TAN B}}{\text{TAN A} + \text{TAN B}}$

Observing the above formulas, here is a generalized or non specific HEIGHT FORMULA given a base side and the two adjoining angles to that base side:

height = $\dfrac{\text{(base perpendicular to height) (product of the TAN values of both base angles)}}{\text{(sum of the TAN values of both base angles)}}$

There is a way to calculate the height of any triangle without using any trigonometric functions. This method utilizes a unique formula for the area of any triangle which was derived long ago in about the year 60AD by the Greek mathematician and inventor named **Heron** (aka Hero of Alexandria, Egypt). An easy to understand, modern version of the derivation of this formula is given in a book such as: **A Journey Through Genius, The Great Theorems Of Mathematics, by William Dunham**, and which could be described as an interesting, helpful, "good read", and which gives many mathematical topics a practical analysis and understanding.

Given: $S = \dfrac{a + b + c}{2}$: half the triangles perimeter (variable S = Semi-perimeter distance),
which is half the length of the sum of a triangle's' sides

TRIANGLE AREA = At = $\sqrt{S (S - a) (S - b) (S - c)}$: **HERON'S FORMULA** , for the area of a triangle.
S = half the perimeter = semi-perimeter

As an interesting note for this unique formula and its structure, consider that half a square area with side (s) is equal to a triangle with half the area of that square, and if you square both sides of Heron's Formula , you will have this similarity:

$(At^{\wedge})2 = S (S-a) (S-b) (S-c)$, and for a square: $As = s^{\wedge}2$ and therefore: $(As)^{\wedge}2 = (s^{\wedge}2)^{\wedge}2 = s^{\wedge}4 = (s)(s)(s)(s)$

Since: At = $\dfrac{(base)(height)}{2}$: base and height are perpendicular to each other. For example, if c is the base side, the corresponding height (h) is hc and its length is the perpendicular line from side c to the vertex of the two other sides (a and b) that define and are adjacent to angle C.
(base)(height at that base) or= (side)(height at that side). Mathematically:

$h = \dfrac{2At}{base}$: another Height Formula (no TAN values needed)
: the height and base correspond and are perpendicular (right angle) to each other

$h = \dfrac{2 \sqrt{S (S - a)(S - b)(S - c)}}{base}$: a height formula based on Heron's formula for triangle area

From: SIN A = $\dfrac{hb}{c}$ and side b is perpendicular to hb = the height at side b, we can get:

$hb = \dfrac{2At}{b}$ using substitution:

SIN A = $\dfrac{hb}{c} = \dfrac{\frac{2At}{b}}{c} = \dfrac{2At}{bc} = \dfrac{2 \sqrt{S (S - a) (S - b) (S - c)}}{bc}$

¢A = ARCSIN (SIN A)   : formal mathematical notation for finding the corresponding angle of the corresponding sine value of that angle

SIN A = $\dfrac{2 At}{bc}$ SIN B = $\dfrac{2 At}{ac}$ SIN C = $\dfrac{2 At}{ab}$ and the general format is:

SIN ¢ = $\dfrac{2\ Area}{product\ of\ both\ sides\ adjoining\ that\ angle}$: adjoining=composing=creating=defining
From this, we can also derive another formula for the area of the triangle (At):

At = $\dfrac{(product\ of\ both\ sides\ adjoining\ the\ angle)\ Sin\ ¢}{2}$

Taking just the last three right side equivalences and "dividing through (all)" by (0.5) so as to "clear", "rid" or set all the numerical coefficients to 1, we have this equality or mathematical relationship among the sides and heights:

a ha = b hb = c hc : **EQUAL PRODUCTS OF BASE SIDES AND CORRESPONDING HEIGHTS**
All products of a side and the corresponding height at that side are equal.
Note also that twice the area = 2 At = a ha = b hb = c hc
Note the "reverse ratios", for ex.: (a / b) = (hb / ha)

Many of the above values can also be derived and solved for if the other parts of the triangle are known. For example:

$a = \dfrac{b\ hb}{ha} = \dfrac{c\ hc}{ha}$ These have the basic or general format of:

side1 = $\dfrac{(side2)\ (height\ of\ side2)}{(height\ of\ side1)}$

$ha = \dfrac{b\ hb}{a} = \dfrac{c\ hc}{a}$: in a right triangle, (ha) = b, and (hb) = a.   *   Basic format:

(height side1) = $\dfrac{(side2)(height\ of\ side2)}{(side1)}$: **this can also be derived from the previous derivation above**

Considering the above, and the law of sines, for example: (a / b) = (sin A / sin B):

$a = \dfrac{b\ hb}{ha} = \dfrac{b\ \sin A}{\sin B}$ dividing each fraction by b:

$\dfrac{a}{b} = \dfrac{hb}{ha} = \dfrac{\sin A}{\sin B}$ The basic format here is:

$\dfrac{side1}{side2} = \dfrac{\sin\ angle1}{\sin\ angle2} = \dfrac{height\ of\ side2}{height\ of\ side1}$: note that the ratio of the heights is in reverse ratio to
both the sides and sines of the corresponding angles

* From the previously given, we also can derive: hc = hb b/c   and

  If the triangle is a right triangle: hc = hb cos A , therefore: cos A = hc / hb = b / c

  hb = hc / cos A = hc c / b

  hc = ha a / c = ha sin A , therefore: sin A = hc / ha = a / c

  ha = hc / sin A = hc c / a

  tan A = a / b = sin A / cos A = (hc / ha) / (hc / hb) = hb / ha

7. If given just 3 angles, the triangle cannot be solved since it is indistinguishable from all the other similar triangles with the same exact angles, except that the triangle(s) may be magnified in size by magnifying all its sides by the same factor.

8. Sometimes you may be given some of the heights of a triangle and can solve for some of the 6 fundamental triangle parts from that. Here is a basic method for solving for a side or its corresponding height (ie., "altitude"), and this equation shows all the equal products of each base sides and its corresponding height:

$a\ h_a = b\ h_b = c\ h_c$, from this, we can find some other mathematical relationships:
: h_a = height at side (a)

$\dfrac{a}{c} = \dfrac{h_c}{h_a} = \dfrac{Sin\ A}{Sin\ C}$, from this we have:

$h_a\ Sin\ A = h_c\ Sin\ C$: $= h_b\ Sin\ B$

9. You can make a check on all the given and-or found 6 parts by using the law of sines, and all the ratio values should be near in value to each other. If they are not near in value, then check the values used and the results.   It always helps to draw the triangle during and for the analysis, and this may also prevent an ambiguous triangle.

$$\dfrac{a}{Sin\ A} = \dfrac{b}{Sin\ B} = \dfrac{c}{Sin\ C}$$

c = $\dfrac{h}{\text{SIN A}}$ since h = a SIN ¢1:

c = $\dfrac{\text{a SIN ¢1}}{\text{SIN A}}$ algebraically:

a = $\dfrac{\text{c SIN A}}{\text{SIN ¢1}}$

SIN ¢1 = $\dfrac{\text{c SIN A}}{a}$ and   SIN A = $\dfrac{\text{a SIN ¢1}}{c}$

¢1 = ARC (SIN ¢1) = ARCSIN $\left(\dfrac{\text{c SIN A}}{a}\right)$, and   C = 180° - ¢1,   B = 180° - (A + C)

From the above equation for SIN ¢1, we can derive a "sort of" law of sines which can easily be memorized:

$\dfrac{a}{\text{SIN A}}$ = $\dfrac{c}{\text{SIN ¢1}}$ from this we can also get: $\dfrac{a}{c}$ = $\dfrac{\text{SIN A}}{\text{SIN ¢1}}$

Since ¢1 is the supplementary angle of: (180° - ¢1) = C, their trigonometric functions are equal: SIN ¢1 = SIN C, and the law of sines similarity above can now be expressed as the more familiar looking:

$\dfrac{a}{\text{SIN A}}$ = $\dfrac{c}{\text{SIN C}}$ and: $\dfrac{a}{c}$ = $\dfrac{\text{SIN A}}{\text{SIN C}}$ = $\dfrac{hc}{ha}$: hc = height at side (c)
                                   ha = height at side (a)

Can the relationship of side (b) and SIN B also be included into this law of sines for oblique triangles, therefore resulting into the same law of sines as shown for right triangles? Yes, and this will be shown ahead.

If your also looking for a Pythagorean-like relationship among this triangles sides, here is one which includes a "fourth" or extended side:

From: c^2 = (b+d)^2 + h^2   , and   a^2 = d^2 + h^2
          (b+d)^2 = c^2 - h^2   , and   d^2 = a^2 - h^2   extending the left hand side:
          b^2 + 2bd + d^2 = c^2 - h^2   after algebraic substitution for d^2:
          b^2 + 2bd + a^2 - h^2 = c^2 - h^2   T. (-h^2), and switching sides:

          c^2 = a^2 + b^2 + 2bd   : PYTHAGOREAN-LIKE RELATIONSHIP
                                          FOR AN OBTUSE TRIANGLE
                                          (For when C > 90°, or B < the complementary
                                          angle of A, or when b < c COS A since
                                          COS A = adj./hyp., or when a > c SIN A since
                                          SIN A = opp./hyp.)

          d = $\dfrac{\text{c^2 - a^2 - b^2}}{2b}$

When side (a) is extended to show the height at side (a) (or base (a)), this will effectively create another right triangle construction: [FIG 139]

Equating the two expressions for ha:

b SIN C = c SIN B   algebraically:

$$\frac{b}{SIN\ B} = \frac{c}{SIN\ C}$$ expressing this equivalence to what was derived previously:

$$\frac{a}{SIN\ A} = \frac{b}{SIN\ B} = \frac{c}{SIN\ C}$$: LAW OF SINES (general, for all triangle)
Note also, mathematically, for ex.:

a / b = SIN A / SIN B = hb / ha

a = b (SIN A / SIN B) = b (hb / ha) = c (SIN A / SIN C) = c (hc / ha)

SIN A = (a/b) SIN B = (a/c) SIN C

A + B = 90°

B = 90° - A = 90° - 36.87° = 53.13°

Ex. Here is another triangle to analyze: [FIG 141]
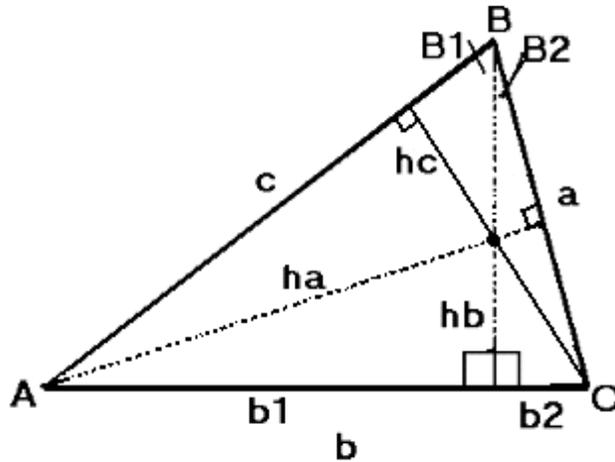
For this non right triangle, hb=3, b1=4, c = 5. and b2=1. By the values given, the left side portion of this triangle is similar in some ways to the ("3, 4, 5") right triangle given above.

By the Pythagorean Theorem: $a = \sqrt{b2^2 + hb^2} = \sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10} = 3.16228$

To find angle B2: $\tan B2 = \dfrac{\text{opposite B2}}{\text{adjacent B2}} = \dfrac{b2}{hb} = \dfrac{1}{3} = 0.3333$

B2 = arctan(tan B2) = arctan(0.3333) = 18.435°

$\tan A = \dfrac{\text{opposite A}}{\text{adjacent A}} = \dfrac{hb}{b1} = \dfrac{3}{4} = 0.75$

A = ArcTan(Tan A) = ArcTan(0.75) = 36.87°

By observing the internal right triangle construction, B1 is a complementary angle to angle A:

A + B1 = 90°

B1 = 90° - A = 90° - 36.87° = 53.13°

B = B1 + B2 = 53.13° + 18.435° = 71.565°

Since C is a supplementary angle to (A + B) when considered as one angle value:

180° = (A + B) + C

B1 + C1 = 90°

C1 = 90° - B1 = 90° - 18.435° = 71.565°

Since C is a supplementary angle to C1:

180° = C + C1

C = 180° - C1 = 180° - 71.565 = 108.435°

Since the sum of a triangles interior angles is always 180°:

180° = A + B + C

B = 180° - A - C   or = 180 - (A + C)

B = 180° - 36.87° - 108.435°
B = 34.695°

B + B1 = 34.695° + 18.435° = 53.13°

Notice that the extended sides will create two similar right triangle constructions. Each of these triangles contains angles of 90°, C1, and the complementary angle to c1 which is (90° - c1)° = B1

TO EASILY CONSTRUCT A 45°, 90°, 45° TRIANGLE:

First, before this discussion, a protractor is an angle gauge. It can be used to draw and-or measure angles. This book has an article on making your own protractor.

First draw a square, and then draw a diagonal line within it. This will create two triangles each being a 45°, 90°, 45° right triangle. Each side of this square, and the "leg" sides (a and b) of this triangle, will have the same measurement. As a check for "rightness" or correctness of the right angle, the hypotenuse side (c) of the triangle will be equal the square root of two (about 1.4142) times longer than the leg side; according to the Pythagorean Theorem.

TO EASILY CONSTRUCT A 30°, 90°, 60° TRIANGLE: (and possibly create smaller angles)

Draw an isosceles triangle. Each angle of an isosceles triangle is 60° Draw an altitude or height line from a base side to the corresponding opposite vertex. This line will divide that base side in half and create two 30°, 90°, 60° triangles. As a check to determine if the isosceles triangle was drawn or constructed right, the height should be about 0.866 times the hypotenuse of that right triangle, or any equivalent side of that isosceles triangle. This can be derived and checked with the Pythagorean Theorem.   If smaller angles are needed: If a compass for drawing circles, or arcs of circles, is available, take the 30° angle created and create a 30° circular arc. Draw a ("chord") line at the endpoints of the arc. Divide this line into two, and this will be used to create an angle bisector line. This angle is 15°.

Before we move onto other topics of trigonometry, the concept of triangulation is of interest. **Triangulation** is very useful for when a location and-or direction of some object or event can be determined, such as on a map, by using two or more sensed (ex., eye witnesses) directions, angles, time, etc. Where each of these direction lines intersect in reality and-or on a map, is a very good estimate of a location, direction, angle and-or event.

same distance from the vertex at (0,0).   The shortest distance between any two points, such as A and B, is always the straight-line distance between those points. This line is indicated and it crosses the angle bisector line perpendicularly creating two congruent (identical) right triangles having the same sides and angles. At half the distance on this perpendicular line is a point of angle bisection, that is, it is a point on the angle bisector line. The larger triangle construction is that of an isosceles triangle where the angle bisector line is the height of that triangle, and therefore, it intercepts the base side (here, between points A and B, which are of equal length) perpendicularly. If the triangle was not an isosceles triangle which has two equivalent sides, the angle bisector line would not intercept its opposite side perpendicularly.

In actual practice, if you are trying to find or draw an bisector line and a point on it away from the vertex point of the angle, and where an angle is to be bisected, the longer that the two equidistant adjoining sides of that angle are from the vertex, a greater accuracy can be achieved. Another point(s) of angle bisection can be used as a check of the angle bisector line.

Now that we have found a point of angle bisection, it might be assumed, due to the above discussion, that the (perpendicular, from angle bisector line) indicated distance to point A or point B from the equidistant (from, or to the angles sides) angle bisector point is the shortest distance to the side of that angle. This is an incorrect assumption. In the drawing below, a circle is drawn between the two sides of the angle, and the center of this circle is at a point of angle bisection. The radius line of the circle will intersect the sides of the angle perpendicularly (at 90°), or vice-versa, since the angles sides will be "tangent" (meaning: slightly intercept at one point, but not go through) to the circle at the point of interception. The radius, being a straight line, is the shortest distance from the center of the circle to it's outer perimeter (ie. circumference), and therefore, it is also the shortest distance to the side of the angle.   By observation, this perpendicular distance (ie. the radius) is the shortest distance since the other distance indicated is the hypotenuse (ie. the longest side) of the right triangle indicated. [FIG 145]
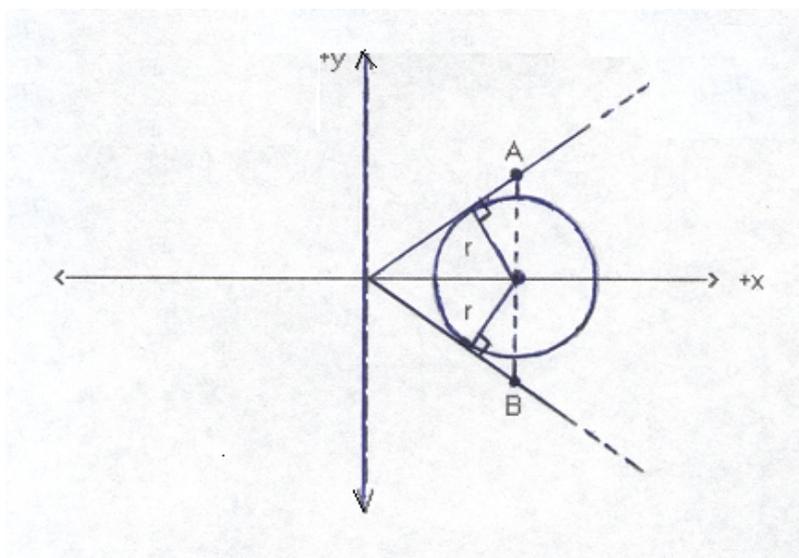
Any point along the angle bisector line can be the center of a circle whose perimeter (circumference) intersects the sides of the angle. Obviously, the farther that the center of the circle is from the vertex of the angle, the larger the circle, and therefore the larger the radius of that circle. At some point along the angle bisector line of an angle within a triangle, the corresponding circle's perimeter (circumference) will intercept the side opposite that angle. This is the largest possible circle that can be inscribed (scribed [write, draw, scribble, indicate]) within both a triangles angle and entire triangle with three angles. The center point of the largest inscribed circle within a triangle is called the **incenter** of the triangle. All three angle bisector lines will intersect at the (same) incenter point. The incenter is also (perpendicularly) equidistant (having the same distance) from all three sides of the triangle. The drawing below shows the incenter of a triangle:
[FIG 146]

SOLVING A TRIANGLE USING AN ANGLE BISECTOR LINE

This topic shows that there are often some new, insightful and beneficial analysis methods to explore. At the point where an angle bisector line intersects (and not generally perpendicularly at 90°, and-or at half of) the third side of a triangle, it divides that side into two line segments whose ratio is equal to the ratio of the other two sides of the triangle. Observe the drawing below where (side a) = (a1+a2) is to be found: [FIG 148]
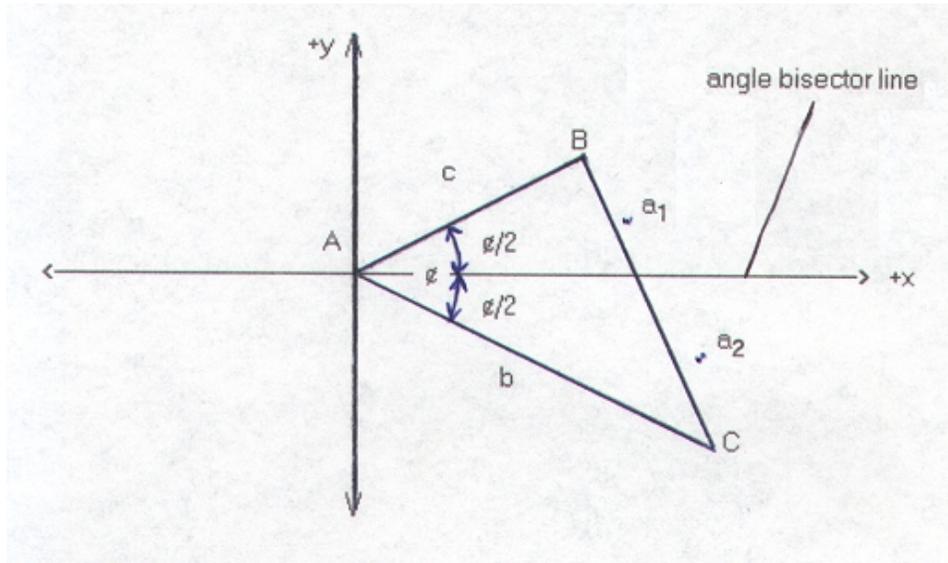
$$\frac{a1}{a2} = \frac{c}{b} \quad \text{or} \quad \frac{a1}{c} = \frac{a2}{b}$$

: note that (a1) and (c) are on the same side of the angle bisector line, and (a1) and (b) are on the same side also.

They are equivalent since these are ratios of corresponding sides of similar (right) triangles. This is verified below: [FIG 149]
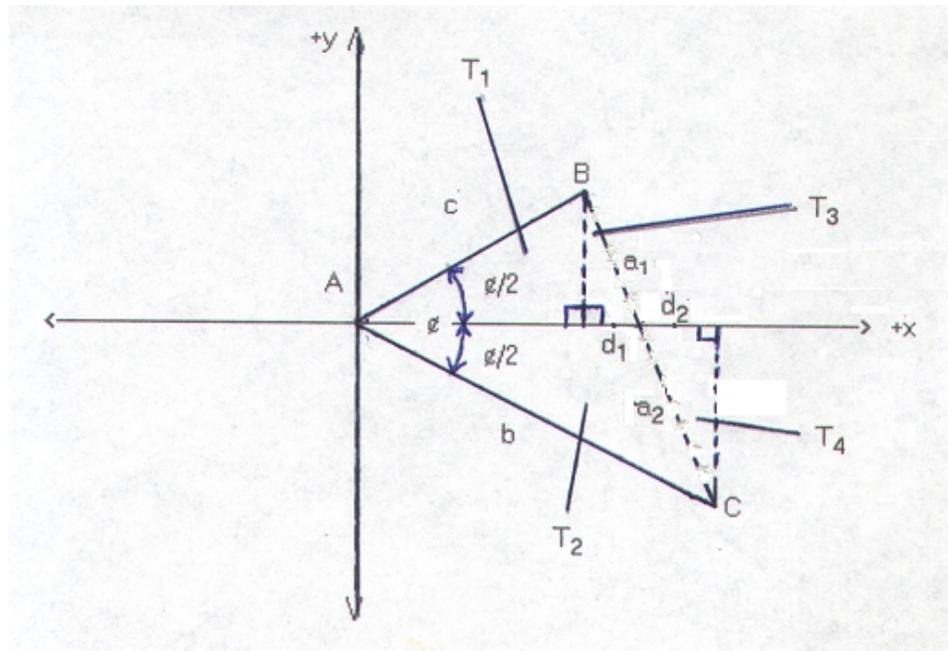
For point 1:   From: COS ¢ = adj. / hyp. = x / h = x / r   and   SIN ¢ = opp. / hyp. = y / h = y / r   :

(x1 , y1) = (r1 COS ¢ , r1 SIN ¢)   : point 1, substituting the values for (r) which is the hypotenuse side
                                             (here, using side (b) or (c)) of the two right triangles:

Be sure to use only the half-angle values shown. (r), or radius, is equivalent to the hypotenuse of the right triangle, hence:

(x1 , y1) = (b COS ¢ , b SIN ¢)   : point 1,   b=length of side b

For point 2:

(x2 , y2) = (r2 COS ¢ , - (r2 SIN ¢))   : point 2,   therefore:
(x2 , y2) = (c COS ¢ , - (c SIN ¢))   : point 2,   c=length of side c

The negative sign is used since the angle is arbitrarily considered positive in sign or signless for this analysis (often, clockwise angles are considered negative in sign (ie. signed angles)), and (y) coordinates of points are negative in the fourth quadrant (or anywhere "below" the x-axis) Alternatively, you could remove the sign and consider the angle as a fourth quadrant angle (since the ending side of that angle is in the fourth quadrant) where the SIN ¢ values are known to be negative in value due to the signs of the co-ordinates of points in that quadrant. For example: p(5 , -2).

distance = side a = $\sqrt{(x2 - x1)^{\wedge}2 + (y2 - y1)^{\wedge}2}$

        Below is an example of solving for the lengths of the line segments of a side (whose length is already known) intersected (and not necessarily perpendicularly) by an angle bisector line. This method shown does not directly utilize any angles, and it is essentially a simultaneous equation method.

In the drawing below, find the lengths of line segments a1 and a2 created at the intersection of side (a) due to the angle bisector line.   [FIG 150]
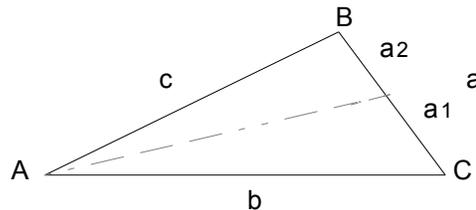
        a = 3.606 ,   b = 6 ,   c = 5

        3.606 = a = a1 + a2   and from the facts about an angle bisector line and the bisected side:

        $\dfrac{a1}{a2}$ = $\dfrac{b}{c}$: side divided by an angle bisector line rule

        $\dfrac{a1}{a2}$ = $\dfrac{6}{5}$ = 1.2   solving for a1:

        a1 = 1.2 a2   using substitution for a1

        3.606 = 1.2 a2 + a2   : a solvable equation with one variable, combining like terms:

CENTRAL AND INSCRIBED ANGLES

Central and inscribed angles are formal names given to certain angles because of their position within a circle. A central angle has its vertex located at the center point within a circle: [FIG 151]

: central angle
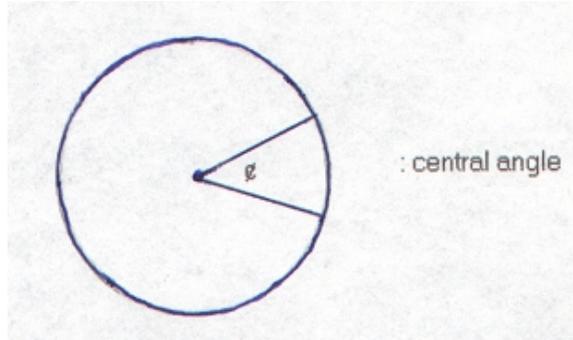
An inscribed angle has its vertex located at a point on the circumference of a circle: [FIG 152]

: inscribed angles
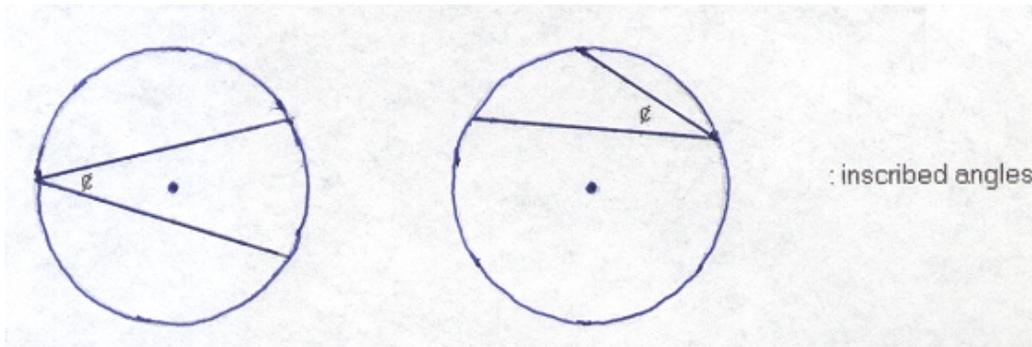
The portion of the circles circumference that subtends (being directly across or opposite to, or beneath) the angles vertex is called an arc (segment of the circles circumference). The angle and its corresponding arc are directly and proportionally (same fraction or portion of the whole or entirety) mathematically related. The greater the angle, the greater the corresponding arc length and vice-versa. This is very similar to that of any angle or triangle; the greater the angle, the greater the side length opposite that angle. Arc length can be calculated easily by setting up an equivalent portions (proportions) or equivalent fractions relationship:

From:

$$\frac{C}{360°} = \frac{S}{¢°}$$

: C = circumference of circle = an arc length of a 360° angle
: S = **ARC LENGTH** (of a central angle, or "centrally inscribed"). Letter S is used since it is technically a semi-perimeter or just a part of the entire circumference.
: ¢ is the central angle that subtends the arc S

OR:

$$\frac{S}{C} = \frac{¢°}{360°}$$

An arc of a circle can be described in various ways such as:
A specific arc length, or the arc due to an certain angle. Ex. "an arc of 10°"

$$S = \frac{¢° \ C}{360°}$$

: **Arc Length**, this also shows that S is essentially a certain fraction of the circumference C.

Also shown above is an angle bisector line that divides both angles in half, and a chord (a line that passes through two points on a circle's circumference; for example, the diameter is the longest chord possible for a circle) line that subtends (here, immediately below or across from) the arc of both angles. To broaden the analysis for use with triangles also, a more general representation using only the half angles is shown below where E = (¢i/2) and A = (¢c/2) : [FIG 155]
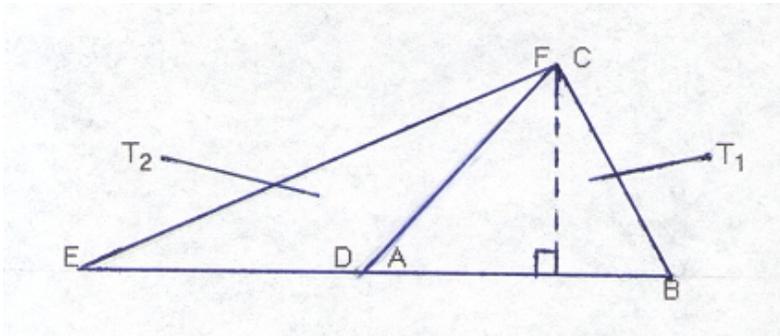
The vertex point of angles D and A is the center of the circle as shown in the previous figure, therefore lines ED = AB since they are radius lines of that same circle. It will be shown that angle E is half of the inscribed angle, angle A. By construction, line A to B, or line AB, equals line AC of T1 (triangle 1) since they are equivalent to the radius line of the same circle. Likewise, line DF and line DE of T2 (triangle 2) are also equal. Line EB is equivalent to the diameter of the circumscribing circle shown in the previous figure. Points B, C (and F, which coincides with point C) and E are points on the circumference of the same circle construction as shown in the previous figure above. Lines such as BC which pass through the circle and intersects it at two points are sometimes called a **chord line**. The entire construction shown will create the two (equivalent in length) sides of angle D which adjoin or intersect to the base side EF (E to F) of an isosceles triangle (T2) construction. Note that angles B and C, and the line that joins their vertex point play no specific role in this analysis other than the fact that this line is a side opposite to both angles E and A whose numerical relationship is being found in this discussion.

Since   E = F   : base angles of T2 (an isosceles triangle)
         E + F = E + E = 2E

         2E + D   =   180°   : T2 interior angular sum of 180°

         D + A = 180°   : supplementary angle sum,   therefore:
         D = 180° - A   by algebraic substitution into T2 interior angular sum:

         2E + (180° - A) = 180°   distribute (+1) to clear grouping symbols:
         2E + 180° - A = 180°   after transposing +180° :
         2E - A = 0   solving for E, transposing (-A): so add -(-A) = +A to each side :
         2E = A   isolating E :

         E = $\dfrac{A}{2}$: verification that an inscribed angle is half a central angle when
                                                            the arc lengths have the same value

Since half-angles (of both the inscribed and central angles) were considered for this analysis, multiplying each angle by 2 to consider the full angle values, we find:

         (2E) = $\dfrac{(2A)}{2}$ = A   or: ¢i = $\dfrac{¢c}{2}$: the central angle (¢c) is twice that of the inscribed angle (¢i)
                                                      of a circle, and they both have the same value of arc length

Also, if not obvious in the above derivation, D is supplementary (sum to 180°) to A. The sum of (E + F) = (2E) is supplementary to D. Since 2E is supplementary to D, and A is supplementary to D, then 2E is equal to A (as expressed

$$R \; = \; \frac{c}{2} \; = \; \frac{\sqrt{a^2 + b^2}}{2}$$: radius of circumscribing circle for right triangles,
(equals half of the hypotenuse side, here, side (c))
The illustration is also one way to create and draw a right angle.

This formula below is be used for non-right triangles that are circumscribed by a circle. For acute triangles, the excenter will be within the triangle, and for obtuse triangles (have an ¢ > 90°), the excenter will be outside the triangle. Here is a derivation of a formula for the radius of the circumscribing circle of any triangle: [FIG 157]
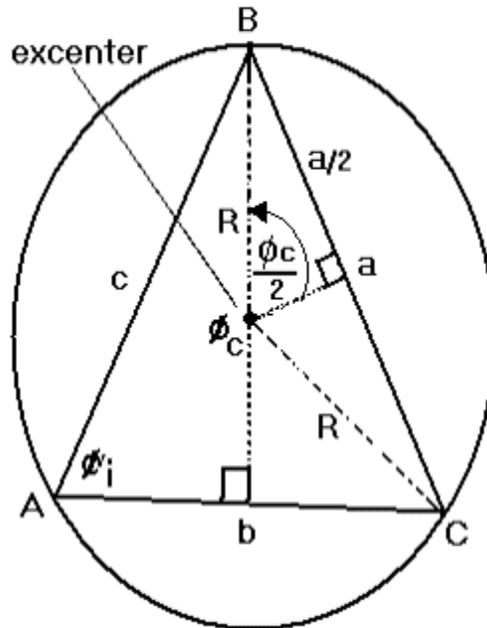
$$A \; = \; \frac{¢c}{2}$$: angle A is an inscribed angle and is equal to half of the corresponding central angle (¢c) which has the same arc length. These angles will have the same trigonometric function values.

$$SIN\,A \; = \; SIN\left(\frac{¢c}{2}\right) \; = \; \frac{side\ opposite}{hypotenuse} \; = \; \frac{\frac{a}{2}}{\frac{R}{1}} \; = \; \frac{a}{2R}$$ algebraically:

$$R \; = \; \frac{a}{2\,SIN\,A}$$ likewise:

$$R \; = \; \frac{a}{2\,SIN\,A} \; = \; \frac{b}{2\,SIN\,B} \; = \; \frac{c}{2\,SIN\,C}$$: Radius of a circumscribing circle of a triangle.
(when C=90°, SIN 90° = 1, and we see that R = c/2)

Note also, for example, that $SIN\,A = \dfrac{a}{2R} = \dfrac{a}{D}$: D = diameter of the circumscribing circle.
Also, for example: D = a / SIN A = b / SIN B = c / SIN C

A generalized formula is: $SIN\,¢ \; = \; \dfrac{corresponding\ side\ (or\ circles\ chord)\ to\ that\ angle}{diameter\ of\ the\ circumscribing\ circle}$
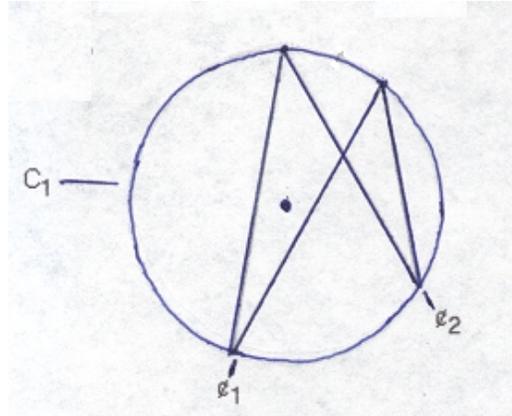
In the drawing below, C2 is identical to C1 (as above) and contains the same exact angle as ¢1 of C1.   This circle (and angle) is rotated slightly and is offset from C1 in such a way that it's angle (¢3) coincides exactly with ¢2 of C1. That is, ¢3 = ¢2:   [FIG 161]
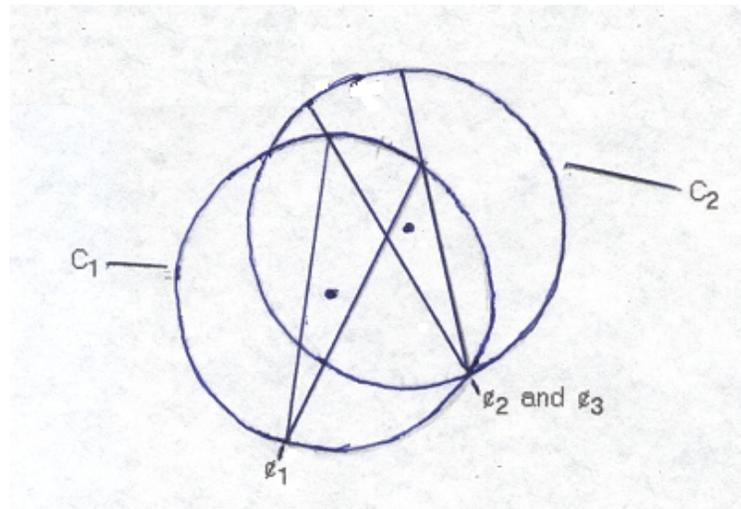
Since: ¢1 = ¢3
 And: ¢2 = ¢3   we can conclude:

        ¢1 = ¢2   : verification   : When the arc lengths are the same, the angles are the same.
                                                The angles being compared must be both central angles or both
                                                inscribed angles.

It must be noted that this verification is more of a "verification by visualization" rather than an "exact" algebraic method.

One method to tell if two angles are the same, without using a protractor, is that the distance between a selected point on one side of an angle, and a selected point on the other side of that angle, will be the same for both angles. These selected points along both angles are usually located to be at the same distance from the vertex of each angle. Though the angles might be found to be identical in size, we don't know their specific value without further calculation. As seen with inscribed and central angles having different angle values and yet having the same chord or arc length of a circle, you therefore cannot tell the actual size of any angles that have equivalent chord lengths, opposite [corresponding] side lengths or arc lengths. Other data ,such as side lengths corresponding to that angle must then be considered.
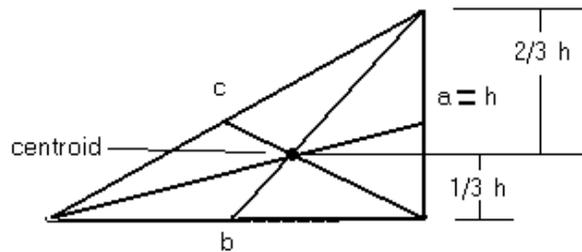
Notice the area of the small triangle created which has (b) as its' base side. Since the base is the same as the base of the total area of the given triangle, to equal (1/3) of the overall area, the height side must then only be 1/3 of the total height. The same can also be said about the other two smaller triangles. Half the area of each of the three small internal triangle constructions is: (total area/3)/2 = (1A/6), or one-sixth of the total area of the original triangle construction, therefore, any two of these smaller areas sum to: (1/6)+(1/6) = (2/6) = (1/3) of the total area, and any three (in particular, those adjoining each other) smaller areas sum to (1/6 + 1/6 + 1/6) = (3/6) = (1/2) of the total area of the triangle.

An ancient mathematician known as Euclid is credited with mathematical understanding of the length of a median line by using the concepts of basic geometry with triangles, and here is a general format for the length of a median line:

median length to side1 = $0.5 \sqrt{2\ \text{side2}^2\ +\ 2\ \text{side3}^2\ -\ \text{side1}^2}$: **length of a media line in a triangle**
Formula credited to Euclid
of Alexandria city in Egypt.
Note: Can also use 1/2 for 0.5

The formula for the median line is based on the Pythagorean Theorem, Apollonius Theorem, and Parallelogram Law. The derivation is not overly difficult and it is available elsewhere.

As a relatively easy verification to the above formula, you can use an equilateral triangle with sides say of 1 unit, and which also has there internal angles of 60°. Here s1 = s2 = s3 = s, and the above formula will simplify to:

median length to any side s = $0.5 \sqrt{3\ \text{s}^2}$ = $0.5 \sqrt{3}\ \sqrt{3^2}$ = 0.5 (1.732051) s = 0.866025 (1) = 0.866025

The **Apollonius Theorem** is usually something mentioned in more advanced geometric studies, and it states that given the three sides of a triangle, the sum of the squares of any two sides will equal the sum of the squares of half the third side, plus the square of the median line to that third side.

The **Parallelogram Law** states that the sum of the squares of all the sides of a (four sided) parallelogram will equal the sum of the squares of its two internal diagonal lengths. A parallelogram, being somewhat like a rectangle, will also have two pair of equal sides, and the two diagonals will also be an equal valued pair. If the parallelogram is a rectangle, and if we divide each side of this equation by 2, we could have the familiar Pythagorean Theorem.

With the concept of median lines, the lengths of both (1/3) and (2/3) of it can be found geometrically.

Even though many of these these formally defined (interior and-or exterior) triangle lines such as medians, altitudes, and angle bisectors have been known about and studied for hundreds of years previously, any and all lines from a vertex of a triangle to the opposite side (even if extended) of that angle are more recently and formally called a **cevian** ("chevian") line.   In 1678, **Giovanni Ceva** published and popularized a theorem and formula for the ratio of the lengths of a triangles side segments intercepted by cevians. Ceva's Theorem states that given a triangle with 3 interior and-or exterior lines that intersect at a concurrent (common), interior or exterior, (cevian) point, the product of all the 3 ratios of each two line

CONSTRUCTING A PROTRACTOR

Below are several methods that you can use as guides for constructing a homemade protractor, however, if you already have access to a protractor, you can use it as a basic template (pattern). The protractors below have a scale from 0° to 90°, hence a "single quadrant protractor". If you want, you can make one with a scale from 0° to 180° (a semi/half-circle protractor), or from 0° to 360° for a full circle protractor. Common protractors have indicated divisions or "graduations" on its' scale as small as 1°. A "rough or close estimate" protractor might only have divisions as small as 5°. You can draw your protractor on paper, cut it out, and then glue it onto a stiff paper or wood backing. Mentioned previously, if you draw an equilateral triangle and divide one side in half, you can easily construct a triangle with angles of 30°, 60°, and 90°. You can then, for example, bisect the 30° degree angle to 15°, and then the 15° angle into three 5° angles.

Method 1:

Draw a quadrant of a circle (typically) by using a device called a compass. A compass is basically anything that can assist you drawing various sized arcs or circles by adjusting the distance (essentially a radius distance) between its two indication or scribing points. To encompass something is to surround or encircle it. If you do not have a compass, you can use anything circular, such as a bowl or cup as a template to trace around. In general, the larger the radius of this circle the more accurate your protractor can possibly be, and therefore, the more accurate the drawn or measured angles The larger you make the protractor, it can also be more precise, perhaps allowing measurements as small as 1°. A radius of about 4 inches is good to start with. Measure the radius of this circle for it will be used in the calculations below. With a compass, the center of the circumscribed circle is clearly marked, otherwise, use something like the following method to find the circles center so as to then find the circles diameter and radius:

A line that intercepts a circle, or other curve, at only one point is called a tangent line, and the line is said to be tangent to the circle. The diameter line or radius line extended (perhaps with the aid of some right-angle, perpendicular guide, or just a plain ruler) to this (tangent) point will always be perpendicular (90°) to this tangent line. Likewise, tangent lines are always perpendicular to the radius line extended to the tangent lines point of interception with the circle. To find the center using a diameter line, draw a (or several) tangent line and then draw a perpendicular (diameter) line to it. Measure and divide the diameter line in half to find the value of the radius and the location of the center. To find the center using the radius line method (no measurement needed), draw another tangent and radius line. The point of interception of both radius lines is the center of the circle: [FIG 165]
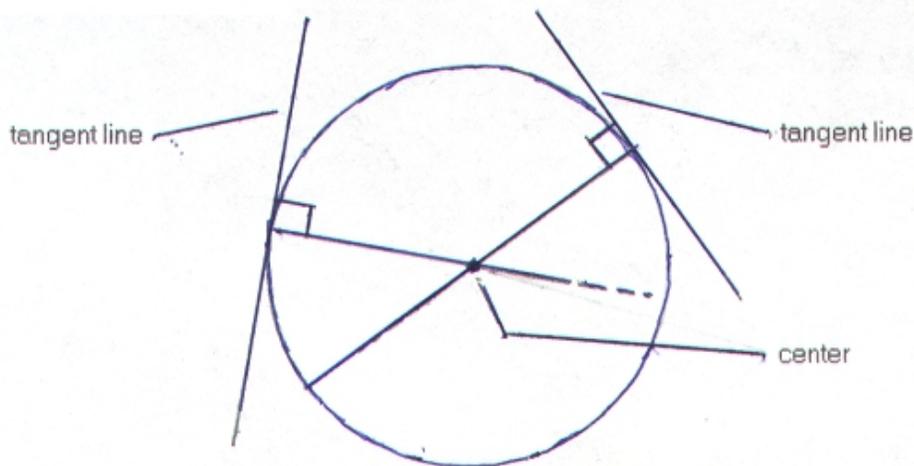
Optionally, to find the center of a circle, you can find the excenter of a triangle that has all three of its vertices on the circumference of this circle. The excenter of the (circumscribed) triangle will coincide (be the same point) with the center
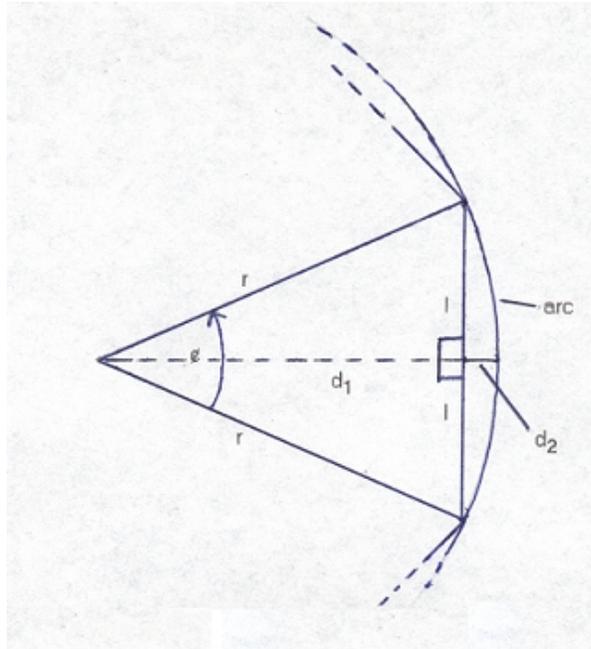
To make the analysis simple, the angle is bisected into two half-angles, creating two equivalent right-triangles. In the formulas below, use half the angle in question.

From :   SIN ¢ = $\dfrac{opp.}{hyp.}$ = $\dfrac{l}{r}$: here, l is half the length of the chord

           l = r SIN ¢   : be sure to use the half-angle value only

Twice the length of (l) is the (chord) length (L) of the line-segment between the endpoints on the arc:

  L = 2l   : Length of the entire line segment "beneath" the arc.
                               Such a line segment is called a "chord" of a circle.
                               Note that the (radius) line drawn from the center of the
                               circle to the midpoint of any chord will be perpendicular
                               to that chord line. If this line is extended to the circumference
                               of the circle, it divides the arc in half, and a line tangent to
                               the circle at this point will be parallel to the chord.

Additionally, if you want to know the length of d1 and d2:

From :   COS ¢ = $\dfrac{adj.}{hyp.}$ = $\dfrac{d1}{r}$

      d1 = r COS ¢   : be sure to use the half-angle value only

Since: r = d1 + d2   therefore:

      d2 = r - d1

an angle and the adjacent or base side. The only exception to this is that say if you double the base side of a triangle, to make a similar triangle the height is likewise doubled or increased by the same factor. For similar triangles the ratio of all corresponding sides of the two triangles is the same value, and this is due to that each side is a factor increase or magnification of the other.   Similar triangles are said to be proportional to each other.

If for example you find that 5° would be a height of 1inch, you cannot use a height of 2 inches for 10° and so on.

Converting the fractional part of this value to a number of 32nds or 64ths of an inch:

Since: $\dfrac{1"}{32}$ = 0.03125"   dividing the fractional part of 1.4559" by this fraction (1/32) of an inch:

$\dfrac{0.4559}{0.03125}$ = 14.6 , (one) thirty-two seconds of an inch

If you need some more clarification on the result, consider this:

$$\dfrac{\dfrac{0.4559\ in.}{1}}{\dfrac{1in.}{32}} \quad = \quad \dfrac{0.4559\ in\ (32)}{1\ in.} \quad = \quad 0.4559\,(32) \quad = \quad 14.6$$

Or by using proportions:

$$\dfrac{1\ inch}{32,\ 32\text{-seconds}} \quad = \quad \dfrac{0.4599\ inch}{x\ \ 32\ parts} \qquad :\ 1,\ 32\ seconds\ =\ 1\ of\ 32\ parts\ =\ 1/32$$

$$x\ \ 32\text{-seconds}\ =\ \dfrac{0.4599"\ (32\ 32\text{-seconds})}{1"}\ =\ 0.4599\,(32\ 32\text{-seconds})\ =\ 14.6\ \ thirty\text{-}two\ seconds$$

To convert a number of inches to its equivalent number of thirty-two seconds, we see that you can simply multiply by 32. Using the same reasoning, to convert inches to sixty-fourths, simply multiply by 64.

Using the results above, indicate the 20° angle at 1 inch and 14.6 thirty-two seconds high on the right side.

For angles along the top side:

From :   TAN ¢ = $\dfrac{opp.}{adj.}$ = $\dfrac{4}{adj.}$

adj. = $\dfrac{4"}{TAN\ ¢}$

Ex.   For an angle of 60°:

adj. = $\dfrac{4"}{TAN\ 60°}$ = 2.3094"

After converting the fractional portion to its equivalent value with units of thirty-two seconds of an inch, the angle should be indicated at 2 inches and 9.9 thirty-two-seconds rightward on the top side. This is possible on the top side since the top and bottom lines are parallel to each other and both are perpendicular to the left side line.

For a triangular shaped protractor, you can first create the square shaped protractor and draw a diagonal line from the

CONSTRUCTING A DEVICE TO MEASURE ANGLES FROM THE HORIZONTAL

A quadrant (a device to measure up to a 90 degree angle), astrolabe, sextant (often used for celestial (stars, space,"heavens") navigation which is often used for ship navigation by stars), theodolite, torquetum ["torketum", a device specifically made for celestial star coordinates mapping], or even a protractor are ancient devices which can measure either vertical and-or horizontal angles such as the angle to the top of a building, a mountain, or the angle a star or other celestial object that is "above" or with respect to your (local) horizon or level (technically, a tangent line to the surface of the Earth at a location, and which is perpendicular to a radius line to the center of the Earth). As shown ahead, it basically consists of an angled scale, such as a protractor, and a pointer, aiming or sighting indicator. The sighting indicator device can be a straight and narrow flat surface. For the sextant to work properly it must always be level (parallel) to the ground since the angles will be measured with respect to the ground or any other line or surface near to, and essentially parallel to, the ground or horizon. For additional aid you can place the device on a level surface at the top of a sturdy tripod stand. Once the protractor scale is level (set, calibrated, or indicated to 0° when the sighting board or plane is parallel to the ground or horizon (horizontal line)), the actual object is "sighted" (aimed or pointed to) by the measuring instrument and the angle will be indicated on the protractor that is located alongside the edge of the sighting device. A figure of such a device will be shown below in [FIG 171].

With the aid of a plumb-bob device, see FIG 169, the proper positioning, calibrating or leveling of the protractor is not a concern, that is, it will effectively take place automatically after the calibration (proper initial setup for use) of the device has taken place. The sighting device will remain "fixed" (does not rotate about the vertex to measure the angle), and the protractor effectively rotates vertically. The word "plumb" (or "plum") in the context of a **plumb-bob** device means that which is vertical (ie. perpendicular) to the surface of the earth. If something is actually vertical it is said to be "plum" (meaning "true" to being vertical). A plumb-bob is hung from a thin cord called a plumb-line so as to indicate a vertical line that is perpendicular (here, vertical) to the horizon line or surface of the earth. The word "plumb" in plumb-bob means a weight, and which was often a lead weight. An old world for lead is actually the word "plumb" (ie., for plumbers, a "true" vertical or horizontal lead pipe) of which now has a periodic table of elements entry identifying it as such: Pb. The word "bob" in plumb-bob means to be movie in an up and down manner as would a (fishing) "bobber" (with a "bobbing motion") used as a float and indicator when fishing. The plumb-line can be used to indicate and-or measure an angle with respect to the local horizontal 0° or vertical reference line. The plumb-bob weight usually has a point at the farthest end (near the ground) for maximum use of the device, however, for the final construction discussed, practically anything can be used as the weight at long as it does not interfere with the free movement of the plumb-line which will indicate the angle. The plumb-line can be made from common household sewing thread or fishing line. The common word "plumbers" who today, describes people who install and fix pipes.

Why is the plumb-line automatically "true" vertically, or always perpendicular (90°) to the ground?

The force called gravity, originating from the matter (ie., materials) of the Earth and concentrated in the direction of its center, will pull the plumb-bob weight towards the center of Earth. If the plumb-bob is suspended by a plumb-line, the line will be made vertical due to gravity constantly pulling upon the plumb-bob and causing a tension force to be applied to the line. This line, if considered greatly extended, will always pass through the center of Earth just like a radius line, and radius lines are known to be are perpendicular to tangent lines. The (true, at the ground level) horizon is actually a tangent line at any point where a vertical line crosses above and below the surface of Earth. A horizontal line is perpendicular to this vertical line through the center of the Earth. As you go farther above the surface of Earth (ie. above sea level) the viewable portion of the surface of Earth will increase since the greatest possible "line of sight" distance between you and the (seen) edge of Earth is now larger. The max. viewable distance is where your line of sight is tangent to the surface of Earth. Since the circumference of the Earth is so large, it usually does not appear to be curved but appears flat when standing on it at a low height such as at sea-level (0ft altitude), and for most applications not involving high altitudes and-or distances, the curvature of the Earth does not need to be considered in any calculations. Due to these facts, any other line (such as the side of a wall or other construction such as a building) that is parallel to the plumb-line will also be vertical. Here is a magnified image of this discussion: [FIG 169]

base — surface — horizontal indicator
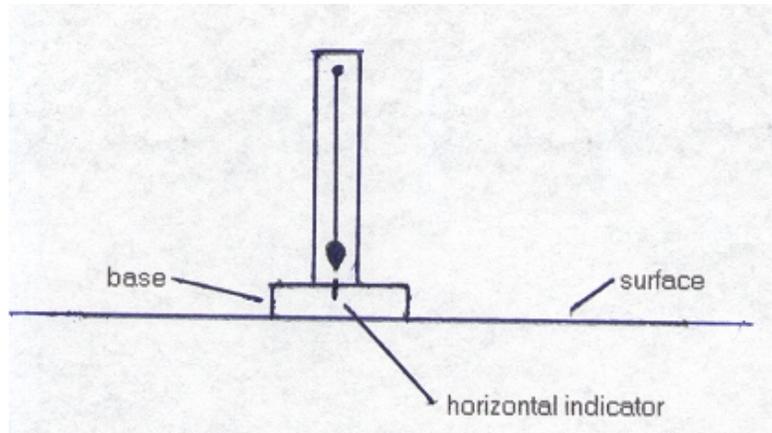
Now that we know the parts and construction of some basic angle measuring devices, the description of its operation is discussed and verified below.

To calibrate the sighting device, set the sighting device in parallel to the ground or horizontal level. The angle indicated should be set to 0° degrees. Note that a floating object of equal dimensions and having a flat surface on top can indicate a line or "level line" (no tilt or angle) that is parallel to the horizontal line at your location.

To use the angle measuring device, view ("line up", "aim") a distance object the "sighting (viewing) edge" of the device. The angle indicated on the angular scale or protractor is the angle between the horizontal (0°, "level" line of reference) and the line of sight to the distant object: [FIG 171]

line of sight — distant object
ℓ_e
level/horizontal line
eye
90°
45° ℓ_P 0°
angular scale
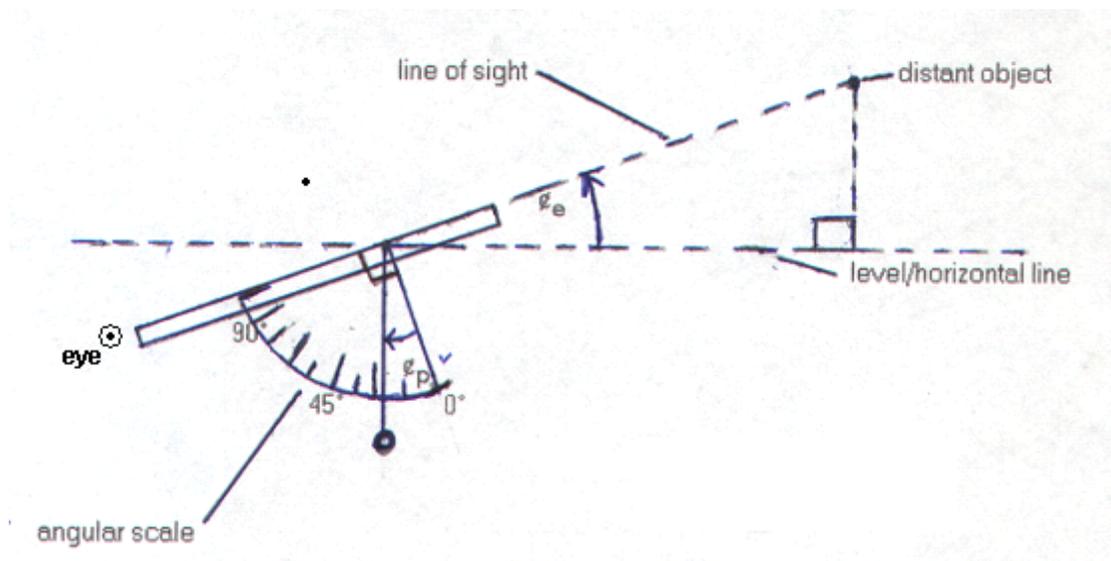
Below is a verification that the angle indicated on the protractor (¢p) is equivalent to the actual angle (¢e) between the line of sight and the horizontal "level" line of the Earth or any other line parallel to this line. For this example, the angle is an (upward) angle of elevation since the object is above the horizon. To measure a downward angle (where the object is below the ("level") horizon or 0° reference line), you can simply turn the device around, that is, sight the object from the other end of the device, or construct the device to measure all possible (vertical, up and down) angles. [FIG 172]

SOME CAUTION WHEN CALCULATING AN OBSERVED HEIGHT

When you are close to an object and are solving for the height of it with the aid of an angle measuring device and (right-angled) trigonometry, you are actually measuring the height that the object is above the your observation point, (eye) level line (parallel to the horizon line) as shown in the drawing below. This topic was also briefly indicated previously in this book: [FIG 173]
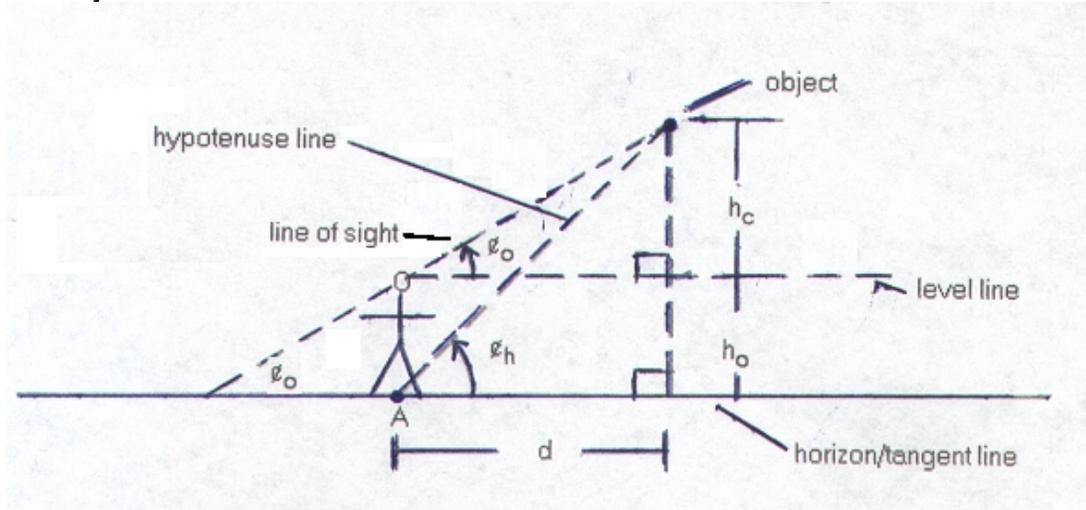
The angle (¢o) measured by the observer (O) standing above point A will be the angle between the (eye) level line and the line of sight to the object. The actual angle (¢h), for the actual height of the object above the horizon line as seen at point A, is the angle between the horizontal line and the hypotenuse line intercepting point A as shown. Be sure to notice that ¢o and ¢h do not have the same value.

The calculated height (hc) by the observer will be smaller than the actual height of the object. To correct this, simply add the height (ho) that the observers eye is above the horizontal, base or ground line to the calculated height.

actual height = hc + ho

If needed, the actual angle (¢h) at point A can now be calculated using trigonometry:

TAN ¢h = $\dfrac{\text{actual height}}{d}$ = $\dfrac{(hc + ho)}{d}$

¢h = ARC TAN (TAN ¢h)

If the object is very high (as compared to the observers relatively small height above the horizontal base or ground line) and distant, the observers height is practically irrelevant to the objects height and is therefore often omitted from the calculations. In this situation, the hypotenuse line and the line of sight become nearly parallel and along the same line, therefore, the angles (at ground level, and observed) will practically have the same value: [FIG 174]

Let: a = vertical change,   b = horizontal change,   and   c = line in question = hypotenuse of the right triangle analysis

slope =   vertical change   =   rise   =   height   =   a   : slope of line c
              horizontal change   run   base   b

Clearly, this is equal to the TANgent of the angle that the line (c) makes with respect to the horizontal line (b). In this example, it is angle A and it is indicated at or as vertex point A of the triangle.

slope = TAN A = TAN ¢A =   a   and: ¢A = ARCTAN (TAN A) = ARCTAN (a / b)
                                        b

The angle and slope values are mathematically directly related. As one gets larger, the other does also. Slopes greater than 0 "slope" or rise upward at an angle with respect to the horizontal ("level-line", 0 slope) or base line.   Slopes of 1 correspond to an ¢ of 45°, and the rate of change in the variables is 1 because their changes were the same value. Slopes less than 1, correspond to an ¢ less than 45°.   Slopes of 0 indicate a horizontal line. Slopes less than 0 are of lines that slope downward below the horizontal or base line .

Ex.   What is the slope of the line between points A and B, and what angle does it correspond to with respect to the level ground or horizon line? [FIG 177]
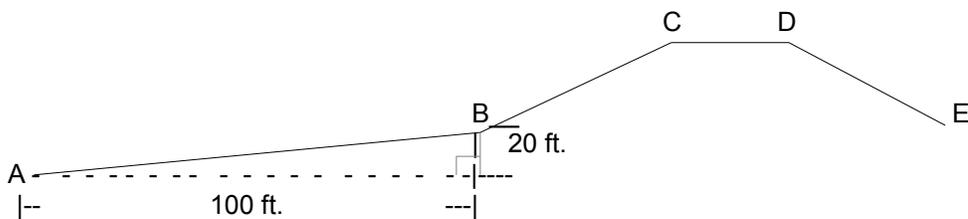
slope =   vert. change   =   20 ft.   =   0.2   = 0.2 = tan ¢   : the slope between points B and C is clearly more
              horz. change   100 ft.   1

¢ = arctan (tan ¢) = arctan 0.2 = 11.3°

Ex.   What is the slope and the angle of the mountain between points A and B with respect to a horizontal level line? [FIG 178]
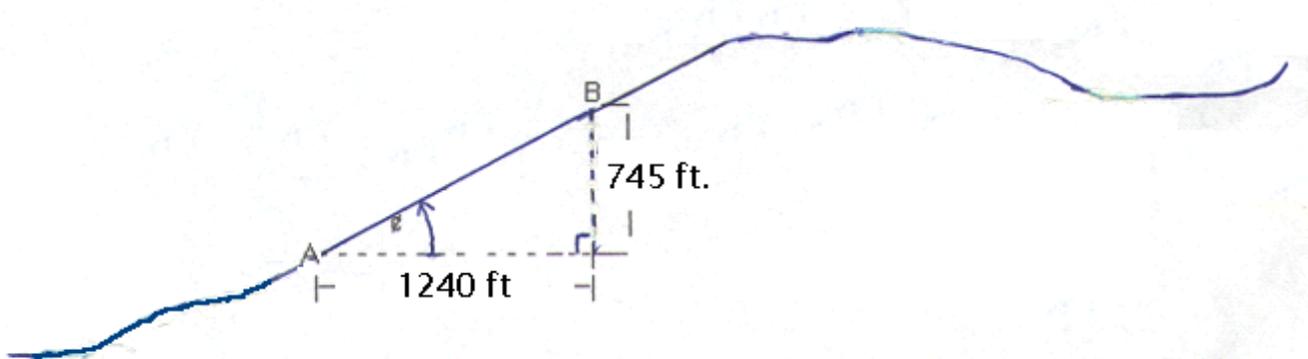
Ex.   Below, you are given a drawing of a construction with its' corresponding measurements. This construction happens to be a portion of a triangular shaped construction with an entire "corner" or vertex portion of the triangle removed. Solve for the angle at this removed or unseen vertex. [FIG 179A]
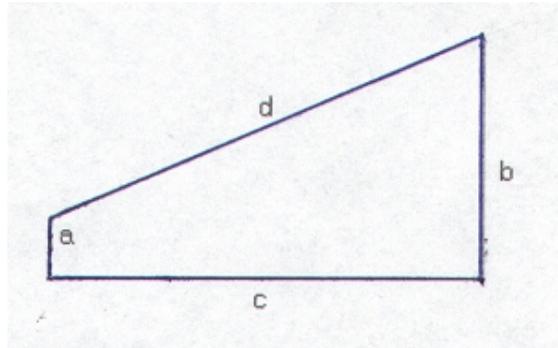
Drawing a (dotted) line parallel to line (c), we create a small internal triangle (with sides equivalent to: c, x, and d) that is similar to the entire triangle construction that is being found. [FIG 180]
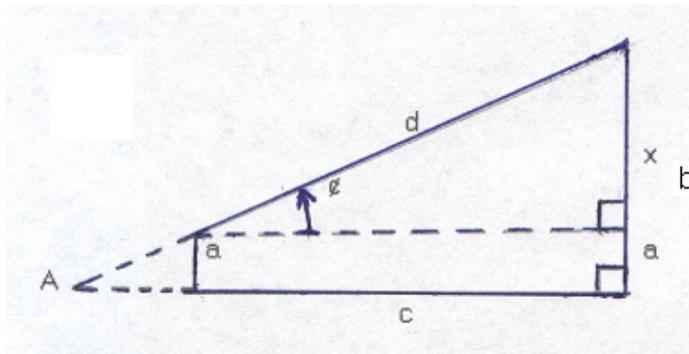
We also note that line (b) includes the same length of line (a) plus an unknown length that we will call (x) that we can solve for:

b = a + x   therefore:
x = b - a   also:

This angle can be found from:

From: COS ¢ = adj. / hypotenuse = c / d   : when considering ¢ A   ,   and:

¢ = ARC COS (COS ¢)

We could solve for (x) using the Pythagorean Theorem: $x = \sqrt{d^2 - c^2}$

We could also solve for (x) using a trigonometric function:

From: TAN ¢ = $\frac{opp.}{adj.}$ = $\frac{x}{c}$

        x = c TAN ¢

The slope (m) of line (d) is equivalent to TAN ¢ which is also equivalent to TAN A due to corresponding angles of similar

SLOPES OF PERPENDICULAR LINES, DERIVATION

Before two perpendicular lines are shown, observe the following drawing of a single line that is also indicated as two line segments of that same line. [FIG 181]
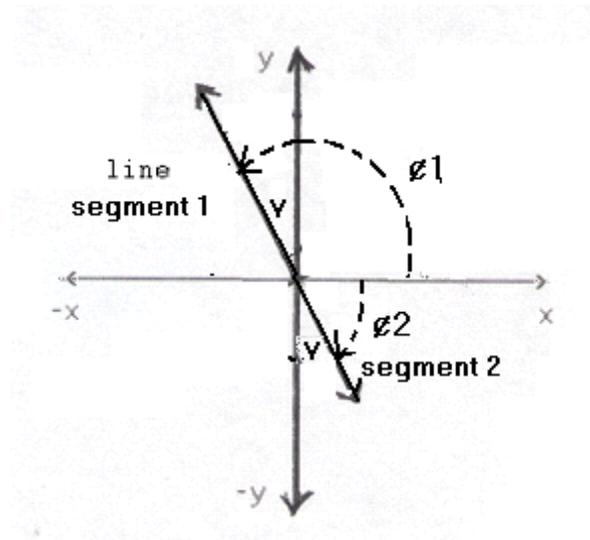
v = equal vertical angles

First note that ¢1 and ¢2 are in reference to the exact same line with respect to the x reference axis of angle measurement.   The right or positive side of this horizontal or x-axis is considered as the ¢° reference line for measuring angles.

The two angles indicated as (v) are vertical angles, and they are equal in value, and therefore have the same corresponding trigonometric values. ¢1 and ¢2 are supplementary angles, and therefore sum to 180°:

¢1 + ¢2 = 180°   , therefore: ¢1 = 180° - ¢2   and   ¢2 = 180° - ¢1

Any two angles that sum to 90° are called complementary angles.   The trigonometric cofunction values of each complementary angle are equal. For example:

      60° and 30° are complementary angles.

$$\tan 60° \;=\; \cot 30° \;=\; \frac{1}{\tan 30°} \;=\; 1.732$$

Surely since ¢1 and ¢2 refer to the same line, the slopes of the indicated line segments are somehow similar, and it is that the line can be thought of as going in the leftward direction for line segment 1, and also as going in the rightward direction as for line segment 2. It is still the same line so these slopes must be the same value as either would have in reference to only one of those directions.   Their slopes are negative values of each others slope value:

    slope of any line segment of a line   =   - (slope of an other line segment considered in the opposite direction).

Lines that slope upward in one direction or angle (¢), will slope downward in the other direction or angle (180° - ¢).   From this we can conclude that lines (or segments) that are separated by a 180°, and-or supplementary angles that sum to

Trigonometric functions of supplementary angles are equal in value.   One way to verify this is:

If ¢1=¢1   and   ¢2 = (¢1 + 90°)   and   ¢3 = (¢2 + 90)   ,   therefore: ¢3 = (¢1 + 90°) + 90° = ¢1 + 180°

Therefore, ¢1 and ¢3 are supplementary angles
¢2 is essentially perpendicular to ¢1, and ¢3 is essentially perpendicular to ¢2, and a (180°) straight angle to ¢1.

  trig. function ¢1 = trig. cofunction ¢2 = trig. cofunction ¢3

Ex.   sin 30° = cos 120° = sin 210°

    sin ¢ = sin (¢ + 180°)   or   sin ¢ = sin(¢ - 180°)   : same absolute value, but the signs may not the same

Trigonometric cofunctions of complementary angles are equal in absolute value.

    sin ¢ = cos (90° - ¢)   or   cos ¢ = sin (90° - ¢)   : Ex.   sin 60° = cos 30°   and   cos 60° = sin 30°

Now we will consider two lines which cross each other perpendicularly (at 90°) with respect to, or in reference to each other. The left and right images below show two of the many possible ways to analyze this type of situation.
[FIG 183]
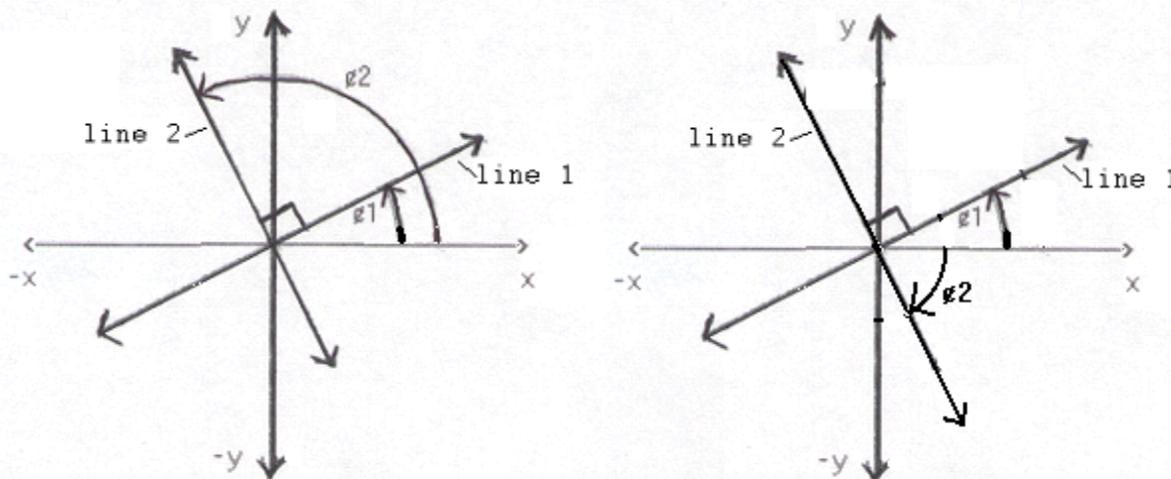
By observation, any two lines that cross perpendicularly (at 90°, with respect or in reference to each other), one line will slope upward and have a positive slope value, and one line will slope downward and have a negative slope value. Don't assume that lines always cross at the axis of the coordinate system. The drawing above does not indicate the 0° reference value of the entire coordinate system, and the point of intersection of the two given lines could be any point in the coordinate system.   Is there a certain mathematical relationship of the slope values of perpendicular lines?   The following analysis will answer this question.

If the two angles (¢1 and ¢2) differ by 90° the corresponding lines are perpendicular. ¢1 and ¢2 are complementary angles that sum to 90°, and their trigonometric cofunctions are equivalent in value.

SLOPE AS A MATHEMATICAL APPROXIMATION TOOL

Below are several basic methods to approximate a value of any equation where the value to be found is between two other close values that were previously calculated or from a table. An obvious choice of finding the approximate value would be to take the average of the two given values. This will work, but for better results, use something like the more advanced methods described below.

Let's say that you calculated, or found listed in a table, the value of two sine ¢ values or two square root values where the arguments (ie. the independent value or variable of the function) used are close in value. For example:

    SIN 36°
    SIN 37°

Let's assign (y) as the dependent variable equal to the value of the expression, and we then will have an equation which is often called a function since a variable(s) and its value(s) are essentially processed by the equation to produce a result. The equation is like a machine that applies a certain (preprogrammed, set, determined) process or function to the input values(s) so as to produce an output or result that was partially or significantly determined by that input values(s). Here, with an equation that is like a mathematical machine for an input value, the output is another mathematical value. Here are some ways to express what was just mentioned so as be helpful to its understanding:

    input_value -----> process ---------> output value   : the input value can be called the independent value
    input_value -----> expression -----> output value   : the output value can be called the dependent value
    input_value -----> function ---------> output value   : a function contains expressions that will use the input value
    function(input_value) ---------------> output value = "(function) return value"   : computer, pseudo or generic code
    function(independent value) ------> dependent or return value   : computer, pseudo or generic code
                                                      The independent value(s) is also called the
                                       "function argument(s)" or "argument(s) sent to the function"

Ex.   If the input value, or the value to be put into: (5 + x) is 3, we know that (5+x) is an expression, process or function, and the value of (x) will be set or assigned the value of 3. We can set (y) as equal to the output or resulting value of that equation or function:

    input_value -----> expression ----->output value

            3 ------> (5+x) ------------> y   or expressed more commonly as, in a reverse direction:
            y<------- (5+x) <------------ 3   and:
            y = (5+x)   : if, or letting, x=3, inputting, entering, setting, assigning or substituting 3 for x:
            y = (5+3)
            y = 8

Ex.   SIN(x) = y   or by switching sides:

    y = SIN x

Formally, (y) is said to be the result of and equal to this particular function of (x). The actual and-or specific value of (y) depends on the specific value of (x) and how it is processed by the entire expression that (x) is in.   The mathematical relationship between (y) and (x) depends on the specific expression that (x) is in.

    (x) is the independent variable (sometimes known as a declared, formal argument or parameter).
    (y) is the dependent variable whose value depends upon the specific value of the independent
        variable (here, x) and then how it is processed by the equation to produce an output value
        based on both the equation and the input value (here, x).

$y_{n+1} = y_n + m$: when the (change in x) = 1

If (x) were to increment by 1 again (for a total increment or change of 2) the next corresponding value of (y) would be:

$y_3 = y_2 + m$ algebraically substituting the equivalent expression for y2:
$y_3 = (y_1 + m) + m$ clearing grouping symbols:
$y_3 = y_1 + m + m$ combining like terms:
$y_3 = y_1 + 2m$

This same substitution for y1 can be used to find the next value, y4, and so on. Clearly we see a pattern developing and it is that the coefficient of (m) is equivalent to the (change in x). In general, if (change in x) = c, the corresponding (change in y) equals (cm) and:

$y_{n+c} = y_n + cm$ or by using the commutative law:
$y_{n+c} = mc + y_n$: finding (y) given a previous value of (y). This still has the basic line form of: y = mx + b

This equation is a linear (line) equation and therefore it should hold valid for all values of (c), and not just for 1 and-or just for integers. This equation will be used ahead when calculating the square root of 10.7 when given the square roots of both 10 and 11.

For example, if the Y value of point A represents the square-root of 10, and the Y value of point B represents the square-root of 11, what is the square root of 10.7?

Given: $\sqrt{11}$ = 3.31662479   : B = point B = PB = Pb = Pb(Xb , Yb) = (Xb, Yb) = ($\sqrt{11}$, 3.31662479)

Find $\sqrt{10.7}$ =   ?   : C = point C = PC = Pc = Pc(Xc , Yc) = (Xc, Yc) = ($\sqrt{10.7}$, ?)

Given: $\sqrt{10}$ = 3.16227766   : A = point A = PA = Pa = Pa(Xa , Ya) = (Xa, Ya) = ($\sqrt{10}$, 3.16227766)

The above data of corresponding values of the square root of 10 and 11 may be from a calculation, a graph, or even from a table of square root values.

From: Y = function (X)   : "Y is a function of X"   or: "The value of Y is a function of X"   or:
                                "Y is determined by this function of X".
                                Here, the function of X is the square root function: $y = f(x) = \sqrt{x}$

      P = P(X, Y)   : a general point notation for any function, and location on a graph.
                        point = position = location = address = corresponding pair of values
                        point = p(location) = p(independent value, dependent value) =
                             = p(input, output) = p(x, f(x)) = p(x,y)

      PA = (XA, YA) = (10 , 3.16227766)   : previously calculated, or given from a table
      PC = (XC, YC) = (10.7,   YC)   : YC is to be found by calculation
      PB = (XB, YB) = (11 , 3.31662479)   : previously calculated, or given from a table

Solving for YC:

YC = YA + $\dfrac{(YB - YA)(XC - XA)}{(XB - XA)}$: Interpolating (here, for a corresponding value of point C) using
                                          proportions (equivalent fractions) or rates (slopes) of the changes
                                          of two points).

YC = $\sqrt{10.7}$ = 3.16227766 + $\dfrac{(3.31662479 - 3.16227766) (10.7 - 10.0)}{(11.0 - 10.0)}$

YC = $\sqrt{10.7}$ = 3.270320651   : accurate to 2 decimal places beyond the decimal point.
                                         You might think the accuracy should be higher here, but this method
                                         assumes there is a linear or constant relationship between the points,
                                         and that is not so, since the equation was not a linear equation, and was
                                         only assumed to be one for that small line segment approximation for a
                                         portion or segment of that curve which actually has a varying slope (m),
                                          and therefore, it is not so line-like or linear as when the slope (m) is constant.

    .

        The concepts of similar triangles can also be applied. However, an adjustment method is used since the triangle, like any line, is not and should not be assumed to be (originate or have a point) at the origin of the system. The adjustment allows the triangles to be anywhere within the coordinate system. The resulting formula is very similar to that shown above. Here, the height (h) of each triangle is the difference of the (y) coordinates of the points, and the base value of each triangle is the difference of the (x) coordinates of the points: [FIG 185]

$$\% = \frac{\text{length 1}}{\text{length 2}} = \frac{(\text{max. 1} - \text{min. 1})}{(\text{max. 2} - \text{min. 2})}$$

$$\% = \frac{b2}{b1} = \frac{(x3 - x1)}{(x2 - x1)}$$

Using the last examples values:

$$\% = \frac{(10.7 - 10)}{(11 - 10)} = \frac{0.7}{1} = 0.7 = 70\% \qquad : 10.7 \text{ is } 70\% \text{ of the way from 10 to 11}$$

From the concepts of similar triangles, where to construct a similar triangle, all the parts are magnified by the same value, resulting in that the ratio of corresponding parts of the two constructions is constant and is equivalent to that magnifying or factor value:

If b2 is 70% of b1, (b2/b1) = 0.70, h2 should also be 70% of h1. The resulting equations are similar to the last method.

$$\frac{h1}{b1} = \frac{nh1}{nb1} = \frac{h2}{b2} \qquad : \text{here, } n=0.7 = 70\%$$

$$h2 = \frac{b2h1}{b1} = \frac{(x3 - x1)(y2 - y1)}{(x2 - x1)} \qquad : \text{Interpolating by equivalent fractions}$$

$$y3 = y1 + h2$$

By having a segment of a line, we can find the equation of that entire line given just a small piece or segment of it. We can find the equation of this line since we are essentially given two points on this line, which will satisfy the minimum number of points needed to define that line and make it corresponding equation. Once we have this equation, any other or "third point" on the line can then be found, and specifically for this discussion, the point must be between the two points which bound (ie. are endpoints of) that line segment approximation of any curve.

The equation we are to write is a simple linear (line) equation of the form:

$$y = mx + b \qquad : \text{"slope-intercept" form of a line, algebraically:}$$

$$m = \frac{y - b}{x} \qquad : (m) \text{ is a constant for a given line, algebraically:}$$

$$b = y - mx \qquad : \text{as mentioned previously, (b) is defined as a (arbitrary) constant for a given line.}$$
$$\text{(It could be thought of as the "(initial) height" or starting value of the line when } x=0.)$$

We don't not know yet the value of (b) (technically the "y axis intercept" where x=0, this is point(0,b)) in order to calculate (m). Since (m) is equivalent to the slope of the line, and that we can calculate the slope of any line given two points on it:

$$m = \frac{(y2 - y1)}{(x2 - x1)} = \frac{\text{corresponding change in y values}}{\text{corresponding change in x values}} \qquad : \text{slope of the line segment between } p(x1,y1) \text{ and } p(x2,y2)$$

Using the above values:

$$m = \frac{\sqrt{11} - \sqrt{10}}{11 - 10} = 0.15434713$$

other two sets of that data. In general, it is assumed that each set of data or points lays upon the same line, and specifically for this type of analysis, a line segment when the points are close so as any other or "third point" between them will also be close in value.

For example, you are given a chart or table of data and are to find a (corresponding) set of data that is between two other sets of data, specifically, you are to find the corresponding (y) value for a given (x) value. You can identify or assign variables to the given sets of data or points, such as: (x1, y1) and (x3, y3), and you are to find the "third, middle or unknown" set of data of: (x2, y2): [FIG 186]

TABLE OF DATA ,OR "LOOK-UP" CHART

```
   ___x___|___y___        : corresponding values, or coordinates (locations) of points, of successive data of a curve
      .    |    .
      .    |    .
    (x1    |   y1)         : point 1  =  P1 , a pair or corresponding set of data
      .    |    .          : the dots represent any possible intermediate values such as between (x1,y1) and (x2,y2)
      .    |    .
    (x2    |   y2)         : point 2  =  P2,  which is to be found.   x2 is given, and its corresponding value of y2
      .    |    .                          is to be found
      .    |    .
    (x3    |   y3)         : point 3  =  P3
      .    |    .
      .    |    .
```

To help simplify the description and analysis of this method, we will identify the differences between corresponding members of two sets, and we will assign them (the differences) identifiers or variable names. For assistance, look at the drawing below of a portion of a "look-up" or reference table of values:

The differences ((d), "lengths", or "changes") are:

a = x3 - x1   : could also use something like: d1 for a,   d2 for b,   d3 for c,   d4 for d
b = y3 - y1
c = x2 - x1
d = y2 - y1   : y2 is unknown and being solved for. Note that: y2 = y1 + d   , [FIG 187]

```
              ( x , y )  : A LIST OR TABLE OF CORRESPONDING DATA VALUES OR POINTS ON A LINE GRAPH
          .   ‖   .
          .   ‖   .               : pairs of corresponding values, or coordinates (locations) of points, of successive
          .   ‖   .                 data of a curve, equation, mathematical relationship, or table of "lookup" values
     ┌------(x1   |   y1) ------┐
     |      |   ‖   |           |
     |      c   ‖   d           |
   a |      |   ‖   |           | b
     |     (x2   ‖   y2)        |   : A  typical reference table-list of (increasing or decreasing) data values.
     |      .       .           |     We are given two points or sets of corresponding data:  (x1,y1) and (x3,y3).
     |      .       .           |     Find (x2,y2), the unlisted value being found that is "in-between" the given data.
     |      .       .           |
     └----- (x3   |   y3) ------┘
            .   |   .
            .   |   . . .
            .   |   .
```

m = $\dfrac{\text{change in y}}{\text{change in x}}$ = $\dfrac{b}{a}$ = $\dfrac{y3 - y1}{x3 - x1}$: slope of all line segments of, or on a line will all have the same value

m = $\dfrac{\text{change in y}}{\text{change in x}}$ = $\dfrac{d}{c}$ = $\dfrac{y2 - y1}{x2 - x1}$: slope of all line segments of, or on a line will all have the same value

Therefore, if you can remember this general formula for slope (m), you can also remember this simple formula for (d), which you can think of as: "the difference":

d = (change in y) = m (change in x)   and then:

y2 = y1 + (change in y)

y2 = y1 + d

Or showing this in a more algebraic sense:

$\dfrac{y2 - y1}{x2 - x1}$ = m   therefore:

y2 - y1 = m (x2 - x1)   : has the basic linear form of: y = mx + b, Transposing y1:

y2 = y1 + m (x2 - x1)   : or   y2 = y1 + m(change in x) = y1 + d
                              This has the basic linear form of: y = mx + b , y2 = m(x2-x1) + y1

where the slope constantly varies (changes). The faster the object goes (ie., its speed), the greater the distance it will travel per unit of time, mathematically giving a greater slope or rate of change of distance with respect to time, and the line will then be steeper on the graph. When an object has a change in speed, this change is called acceleration. It could be either positive, or negative in value such as when it slowed down due to some force acting upon or being applied to it.

The concept of slope is used extensively in understanding the fundamental concepts of a branch of mathematics called calculus, where essentially a whole entire value can be solved for by accounting for all the (infinite number of infinitesimally small) smaller values or parts that make up or define that whole value. Here, the slope value is generally not between any two points on a curve (or a line which can be considered a special curve and-or graph), but between successive (adjacent, infinitely close, "instantaneous", in an instant in time) points. Therefore, these very close (as possible) points essentially (or practically) define very small line segments which compose the entire curve. This change (of the independent values or variables, and then the corresponding change in dependent values or variables) in variables under this condition is very small, which usually occur in a very small change or an "instant" of time. Therefore, the slope value of an equation in calculus is considered as the "instantaneous slope" or "instantaneous rate of change" of one variable with respect to, or in reference to, another variable, rather than an overall average slope value.   This slope value is only the slope of the curve at only, or essentially at one particular point on that curve, and is generally not the (variable, changing) slope value for the entire curve.   Except for lines with a constant slope, since the slope of a curve is constantly changing, it is variable in value and it could be said that the slope is variable. A constant can not then be used to define a (changing) slope of a curve, but a special and constant (slope) equation can be derived and used for the slope value.

For this discussion, observe the drawing below which indicates the slope of a curve at one point, here point A: [FIG 190]
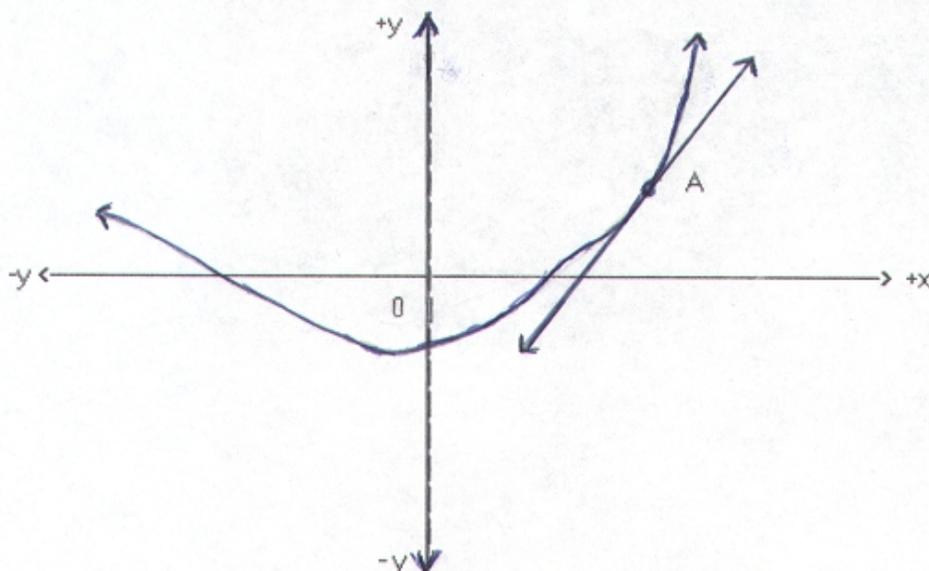
You are to find the slope of the curve at point A. As an aid to the analysis, the line indicated, if properly drawn, is simply an extension of the very small line (segment) created by the two adjacent points located at point A.   You could say this is a "tangent line" to the curve at that point, location or section used for this analysis. To find the slope at this point on the curve, any two points on this line, or extended line, can then be utilized to calculate it since the slope of a line is constant at all points along it.   In this example, since (x) is the horizontal or independent variable, and (y) is the vertical or dependent variable, the slope at any point on this curve is equal to the corresponding change in (y) divided by its corresponding change in (x):

slope = _change in (y) of the (small) line_   : The slope at a particular point on the curve is the
                 change in (x) of the (small) line   instantaneous rate of change of (y) with respect to (x)

DERIVATIVE

Given a mathematical relationship (expressed as an equation) or function of variables, a single and special equation called a derivative can be mathematically created by deriving it out from that initial equation. The derivative equation will mathematically represent the value of the instantaneous rate of change, or "slope", of variables between any two consecutive (adjacent, near identical) points in question of the function or curve.

For an example of a derivative, velocity is the name often used for the derivative of the speed equation:

speed = distance , Velocity is the (instantaneous) rate of the change of distance with respect to time:
time   In short, velocity is the speed at only a certain instant of time.

velocity = (instantaneous change in distance)   : clearly , this equation is much like that for the slope of a line, and
(instantaneous change in time)   velocity can be thought of as an instantaneous slope, speed or rate.

Since velocity (or think of speed) can also change with respect to time, another derivative formally called acceleration can be derived which is the rate of change of velocity (or speed) with respect to time. A derivative equation of another derivative equation is mathematically called a "second derivative" since it will be a derivative of the first derivative. The derivative of velocity is called acceleration:

acceleration = (instantaneous change in velocity)   : acceleration could be thought of as the slope of velocity
(instantaneous change in time)

Whenever an object slows down or speeds up due to some force applied to that object, its velocity will therefore change, and hence there is an acceleration or change in motion and speed. If the object increases speed, the change in speed is called an acceleration, or a deceleration when an object "slows down" or reduces is speed or velocity. When speed or velocity is not changing (ie., not increasing or decreasing) and is staying the same constant value, there is therefore no (0) acceleration happening.

Considering just lines where the slope (ie., "speed" rate) is a constant value, it is not too difficult to realize that the derivative of any line (linear) equation is a constant value equivalent to that slope.

To give you a brief insight on derivatives and inverse derivatives (often called anti-derivatives which are associated with the topic of calculus known as integration or integral calculus which sums all the infinitely small parts of a value, so as to find the entire value, a short, but very powerful discussion related to basic calculus is given below. The discussion will help you understand and give verification to some of the topics discussed further ahead in this book. Be sure to also review the brief discussion about derivatives in the Squares Example previously shown in the Basic Algebra section of this book.

$$\frac{dy}{dx} = \frac{y2 - y1}{\text{very small change in (x)}} = \frac{f(x + \text{change in x}) - f(x)}{\text{very small change in (x)}}$$

: This is usually called the "delta process". You can think of delta as meaning a very small change or differential (difference = d).

Above, the words "very small" also mean instantaneous, or infinitely small. For even the smallest change in a variable to occur it will take at least a very small amount of time, and hence the concept of an "instant in time" or something being "instantaneous". An instant of time is a small or "short" or "quick" amount of time. An instantaneous change in (x) will produce a corresponding instantaneous change in (y), however as mentioned, do not assume that this corresponding change in (y) is always equal to that of the change in (x). If they were always equal, their ratio (ie. slope) would always be the constant value of 1 which is not very useful to describe all the various possible curves or mathematical relationships with varying slopes and-or rates of changes between the variables.

If $y = 5x^1$, the derivative of this function is: $5x^0 = 5(1) = 5$, and this result will be clarified below.

Notation(s) for this derivative: $y' = f'(x) = dy/dx = d(5x)/dx = 5.$, Also note mathematically: $dy = f'(x)\, dx$

Notice that y=5x is a linear equation, and therefore its slope and derivative must be a constant numerical value, and in this example, it's 5. If the equation had an arbitrary constant (such as b) added in to it, the slope and derivative would still be the same. The constant simply shifts (moves) an existing curve "up or down" vertically, and all the corresponding slope values still remain the same. In simple words, the mathematical relationship of the variables and shape of the curve is still the same regardless if it's shifted (moved) up or down vertically.

Below is an example of differentiating the formal (strictly algebraic, ie. with variables only) linear equation of:

$$y = m x + b \qquad \text{: basic linear equation}$$

For showing corresponding pairs of the variables, such as the corresponding coordinates of a point, this equation can be expressed with subscript notation as:

$$y_n = m x_n + b$$

: Basic linear equation, here, n is a subscript value and used to distinguish each instance of pair of corresponding (x,y) variables. Ex. (xn, yn) , (x5,y5), and so on.

The process below will produce the same results if values were used instead of variables, for example:

$$y = 2x + 3$$

, however, we are looking for a formal or general expression for the derivative of any linear equation, rather than a specific derivative expression or value of one specific linear equation.

Considering any two pairs or sets of corresponding values of (x) and (y), which defines two points on any curve:

$y1 = m x1 + b$: for one point, ex.:   p1(x1, y1)
$y2 = m x2 + b$: for another point, ex.: p2(x2, y2) on the same curve from the same equation

Since: $(x2 - x1) = (\text{change in x})$ we have:
$x2 = x1 + (\text{change in x})$

And: $(y2 - y1) = (\text{change in y})$ we have:
$y2 = y1 + (\text{change in y})$

With: $y2 = m x2 + b$ using algebraic substitution of the above expressions:
$(y1 + \text{change in y}) = m (x1 + \text{change in x}) + b$

If y = 3x^2   , its derivative is 6x^1 = 6x.   Checking using the "change" or "delta" process:

$$\frac{dy}{dx} = \frac{(change\ in\ y)}{(change\ in\ x)} = \frac{y2 - y1}{(change\ in\ x)} = \frac{f(x + change\ in\ x)\ -\ f(x)}{(change\ in\ x)}$$

(Where for the delta process, and for finding derivatives (a processes called differentiation), (change in x) and hence the corresponding (change in y) is as minute or infinitesimally small as possible. This essentially will give the slope value at a particular point or instance on the curve which visually or graphically shows the mathematical relationship between two variables at that specific location, point or instance).

The right hand expression is all in "terms (expressed as, or with) of the variable", here its (x). Substituting the given values:

$$\frac{dy}{dx} = \frac{3\ (x + (change\ in\ x))^2\ -\ 3x^2}{(change\ in\ x)} \qquad : extending\ the\ binomial\ in\ parenthesis:$$

$$\frac{dy}{dx} = \frac{3\ (x^2\ +2x(change\ in\ x) + (change\ in\ x)^2)\ -\ 3x^2}{(change\ in\ x)} \qquad distributing\ 3 :$$

$$\frac{dy}{dx} = \frac{3x^2\ + 6x(change\ in\ x) + 3(change\ in\ x)^2\ -\ 3x^2}{(change\ in\ x)}$$

After canceling like terms, and the fact that (change in x)^2 is a very, very small insignificant value compared to (change in x)^1 which itself is already as small as can be, (change in x)^2 can be considered as practically meaningless and a value of 0:

$$\frac{dy}{dx} = \frac{6x(change\ in\ x) + 3(0)}{(change\ in\ x)} = \frac{6x(change\ in\ x)}{(change\ in\ x)}$$

$$\frac{dy}{dx} = 6x \qquad : the\ derivative\ of\ the\ function\ 3x^2\ =\ dy/dx\ =\ 6x \qquad : The\ (instantaneous,\ at\ a\ specific\ point)\ rate\ of$$
change of (y) with respect to (x) is 6x.

Note that the value of the derivative above is not a constant value like it would be for a linear (line) equation where the slope (= an expressed numerical or mathematical relationship between the variables) is the constant value of (m), but with the above derivative expression of 6x, the slope can vary and now depends upon the specific value of (x).   The result on a graph of: y=3x^2 is that the curve does not have a constant slope like a line does, and therefore, the curve will not look straight like a line.   The slope of the curve when x=2 is: 6(2) = 12. The slope of the curve when x=3 is: 6(3)=18. Clearly this is not a constant value.

First, another example description of a derivative for further understanding of these things:

Let: y = mx + b = 2x + 1   : this is a linear equation and-or function, and that y is a function of x.

When there is a change in the value of x, there will be a change in the value of y. To find out how (y) changes when there is a change in (x), we can divide the two, and which is essentially a rate since changes take some amount of time, even if infinitely quick, a short or instantaneous amount of time.

$$\frac{(change\ in\ y)}{(change\ in\ x)} = \frac{(y2 - y1)}{(x2 - x1)} \ ,\quad if\ (x)\ changes\ by\ 1:\ =\ \frac{(y2 - y1)}{1} = \frac{(2\ x2 +1) - (2\ x1 + 1)}{1} =$$

$$\frac{2\ x2 + 1 - 2\ x1 -1}{1}\ =\ \frac{2\ x2 - 2\ x1}{1} = 2\ (x2 - x1) = 2(change\ in\ x)\ ,\ and\ here\ = 2(1)\ = 2\ ,\ a\ constant\ ,\ and$$
a constant rate

If y = x = x^1 , its derivative is 1x^0 = 1(1) = 1 , a constant. y = x = x + 0, a linear equation, with a=m=1, and b=0.

If y = 3x = 3x^1 , its derivative is 3x^0 = 3(1) = 3 , a constant. Since the derivative is constant, the slope of the
         curve (here, actually a line) is constant. Since the mathematical relationship between the variables is
         a constant mathematical relationship, the equation is called a linear equation which also graphs as a
         line. Note that here, the anti-derivative (ie. finding the source function (F) of which the derivative
         expression was derived from) of 3 is: 3(1) = 3x^(0+1) = 3x^1

If y = f(x) = $\frac{x^3}{3}$, its derivative is: $\frac{d\,f(x)}{d\,x}$ = $\frac{dy}{dx}$ = y ' = f '(x) = $\frac{3x^2}{3}$ = x^2   : optionally showing much notation.

Observing the pattern seen in the derivatives of the above type expressions, we find: Multiply the exponent of (x) to the
numerical co-efficient of (x), and then reduce the exponent of (x) by one. Knowing such patterns is very practical and
commonly used in calculus so as to find quick and practical solutions. This specific pattern just shown is commonly
known as the (derivative) "power rule".

In algebraic notation, the derivative of a function of this form or type: ax^n

                                             is: nax^(n-1)   : **DERIVATIVE POWER RULE**

This can be expressed as: Given a function that is a power of a variable: (coefficient) variable^exponent

                                 It's derivative is: (coefficient)(exponent) variable^(exponent-1)

Taking the derivative of this first derivative will give the second derivative of that same function of: ax^n:

                                             n(n-1)ax^((n-1)-1)
                                     is: n(n-1)ax^(n-2)

Taking the derivative of the second derivative will give the third derivative of that same function of: ax^n:

                                             n(n-1)(n-2)ax^(n-3)

Ex. The derivative of the square root, or even the cube root, of (x) can be found using the derivative power rule:

   $\frac{d\,\sqrt{x}}{d\,x}$ = $\frac{d\,x^{0.5}}{d\,x}$ = 0.5 x^(0.5 - 1) = 0.5 x^- 0.5 = $\frac{0.5}{(1)\,x^{0.5}}$ = $\frac{1}{2\,x^{0.5}}$ =

   $\frac{d\,\sqrt{x}}{d\,x}$ = $\frac{1}{2\,\sqrt{x}}$: this is also the slope at any point (x,y) on the curve of: y = $\sqrt{x}$

For this derivative of a square root equation, you can see that as (x) gets larger, the slope will decrease, hence, here the
slope is inversely related to (x). As (x) gets very large, the slope approaches a value of 0 indicating that there is not much
of a change in the corresponding (y) values when (x) changes.

You can use then this derivative in the interpolation formula derived previously so as to find a close approximation for the
square root of values near to (x). This method of interpolation using a derivative is also demonstrated in the tables shown
in this book so as to find an unlisted value that is between two listed calculated values.

  Let d = (change in y) = m (change in x)   replacing (m), the slope value, with the above derivative value:
                                                      d = a difference in the (y) values of two points.

  d = $\frac{1}{2\,\sqrt{x}}$ (change in x)   = $\frac{(change\ in\ x)}{2\,\sqrt{x}}$

dy = 14x^1 dx   Expressing the sum of all these bits, or very small changes or parts of the "whole thing", we can find the anti-derivative and the "whole thing" from which all the bits came from. This process is also called integration. To **integrate** is to combine things, and for math, that then means to add things so as to produce a new thing or sum. An integral part of a whole thing is a part that composes that whole thing. We can say that the part has been integrated into that whole thing. In math, the whole thing is called the sum or **integral** (as in calculus) since it is composed of all the smaller parts than have been integrate into it. In calculus, all the parts are generally similar and infinitely small in size and-or value, and this then cause an infinite-like sum since there are so many. A line segment has an infinite number of points, but we know it still has a definite or specific length, unlike a general line of infinite length which can be called an indefinite integral and can only be represented as an equation. The line segment is then analogous to a definite integral, and which can be thought of an isolated or bound part of a line. A line segment is a specific amount of line.

$\displaystyle\int$ dy = $\displaystyle\int$ 14x^1 dx   :   : is a summation symbol, it looks like the letter S and is the first letter of the the word Summation and is the act to, and-or the result of adding and-or combining things and-or numeric values.

    Y = 7x^2

You might now ask what happened to dx in this process. Just like the sum of all the dy bits is equal to (y), the sum of all the dx or bits of (x) is equal to the whole (x), and this (multiplier) is what increases the power of (x) when it goes through the integration process. The appendix of this book includes more discussion about integration called: Basic Concepts Of Integration. The small bits have been summed and have produced a non-specific sum and-or another equation, as can bee seen due to the variable (x). This is the result of what is called an **indefinite integral** that is unbounded or non-specific and is rather a general representation of the resulting (infinite) sum of terms.

Ex. What is the anti-derivative of y = f(a) = a^2 ?

    $F(a) = \dfrac{a^{(2+1)}}{(2+1)} = \dfrac{a^3}{3}$ Taking the derivative of this as a check: $\dfrac{3\,a^{(3-1)}}{3} = a^2 = f(a)$: checks

The derivative of a sum of functions of (x) is equal to the sum of the derivatives of each function of (x). This is known as the **derivative sum rule** and can easily be verified with an example:

If function 1 = 2x + 3 = f1(x) = y   and:
If function 2 = 5x + 7 = f2(x) = y

  $\dfrac{dy}{dx} = \dfrac{d\,(f1)}{d\,x} = \dfrac{d\,(2x+3)}{d\,x} = 2$ and:

  $\dfrac{dy}{dx} = \dfrac{d\,(f2)}{d\,x} = \dfrac{d\,(5x+7)}{d\,x} = 5$

f3 = f1 + f2 = (2x + 3) + (5x + 7)
f3 = f1 + f2 = 2x + 3 + 5x + 7   combining terms:
f3 = f1 + f2 = 7x + 10   : the sum of the two given functions

  $\dfrac{dy}{dx} = \dfrac{d\,(f3)}{d\,x} = \dfrac{d\,(f1+f2)}{d\,x} = \dfrac{d\,(7x+10)}{d\,x} = 7 = \dfrac{d\,(f1)}{dx} + \dfrac{d\,(f2)}{d\,x} = 2 + 5 = 7$: checks

By the first equation, the derivative of (v) is the derivative of the entire equation given with respect to (u) since (v) is a function of (u). By the second equation (which is equivalent to the first equation), it explicitly implies or indicates that the derivative of (v) with respect to (x) is the same as the derivative of (v) with respect to (u), but will also include a factor that is the derivative of (u) with respect to (x) since (u) itself is a function of (x). If we let variable name (y) = (v) for notation conformity to what we have been using up to this point:

1. $y = f(u)$: y is a function of another function (here, function u), and since u = f(x), therefore

   $y = f(f(x))$: y is a function of u which is a function of x,   y is therefore also a function of x:

   $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$: **Derivative Chain Rule, or Derivative Of A Function Of Another Function Rule**

As a simple verification to this formula consider:

   Let y = v
   v is a function of u, and
   u is a function of x

   $y = v = v(u) = v(f(x))$

The derivative of y with respect to x is therefore:

First let:

   $\dfrac{dy}{dx} = \dfrac{dv}{dx}$ continuing:

Since (du/du)=1 (anything divided by itself equals 1), and multiplying the numerator and denominator on the right hand side by du (creating an equivalent fraction), we have:

   $\dfrac{dy}{dx} = \dfrac{dv}{dx}\dfrac{du}{du}$ which can be mathematically expressed as:

   $\dfrac{dy}{dx} = \dfrac{dv}{du}\dfrac{du}{dx}$ Since y = v, this can be expressed as:

   $\dfrac{dv}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$: derivative chain rule

Ex. If $u = x^2$ and $y = (x^2)^3$: both (u) and (y) are functions of x. u=f(x) and y=f(x)

Clearly, we see that: $y = f(u) = u^3$: y=f(x) = f(u)   or y=f(f(x))

   $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

   $\dfrac{dy}{dx} = \dfrac{d(u^3)}{du}\dfrac{d(x^2)}{dx}$

   $\dfrac{dy}{dx} = \dfrac{d((x^2)^3)}{du}\dfrac{d(x^2)}{dx}$

As an aid to understand this, consider these two functions which are both functions of (x), and that u = f(x) = x^1:

1. y = x^2 , y = f(x) = f(u)
2. z = x^3   z = f(x) = f(u)

Both functions contain a factor of x^1, hence both contain a factor of (u)=f(x) = x^1

Since u=x^1 is the same for both functions, du/dx is therefore the same for both functions, and we also know that the derivatives of both functions can't possibly be the same since those two given functions are different. The derivative of each function with respect to (x) will actually contain (as a factor) the derivative of that function with respect to (u) that is a function of (x).

Consider that since (u) contains a factor of (x), and that (u) is a factor of (y), a derivative of (y) with respect to (x), will contain a derivative of that factor (u) with respect to (x), and a derivative of (y) with respect to (u) will obviously contain a derivative of (y) with respect to (u).

Consider if function2 = (f2) is itself a function of another function, say function1 = (f1). In other words, f2 contains a part of f1, and is dependent upon f1. Since the value of f1 depends on the variables and equation used in that specific function, the values of f2 depends on the variables and equation in both itself, f2, and f1.

A small change in (x) will first cause a small change in (u). The derivative of (u) with respect to (x) will be a certain value: du/dx = slope1 = value1   This change in (u) will cause a small change in (y). The derivative of (y) with respect to (u) will be a certain value: dy/du = slope2 = value2. Therefore, the rate of change of (y) with respect to (x), is also determined by the rate of change of (u) with respect to (x). Therefore, the derivative of (y) with respect to (x) is also partly determined by the derivative of (u) with respect to (x). and value2 is partly determined by value1. One rate of change is party determined by the value of another rate of change. This almost seems like a derivative of another derivative concept here, but as mentioned, the derivative of (y) with respect to (x) is partly determined by both (u) with respect to (x), and (y) with respect to (u), and hence it is partly determined by each.

Considering that a derivative is a form of a rate. Here is a simple way to explain why one derivative is being multiplied by another derivative. and that each is a factor of the result.

value1 = rate1 value2   : note for example, mathematically: value1/value2 = rate1
value2 = rate2 value3

Since value2 is partly determined by value3, and if value1 is partly determined by value2, essentially then, value 1 is partly determined by value3. Because of the rates shown, of which can vary upon each given situation, the values are not completely determined by the other values alone, but also by the (varying or current) rates.

value1 = (rate1 value2) = (rate1) ((rate2) value3)   this can be expressed as:
value1 = (rate1) (rate2) value3

 value1 = rate1 rate2   : the rate of value1 with respect or in reference to value3 is the
 value3   product of rates.

Ex. value1 = 50% value2   or: value1 = 0.50 value2
      value2 = 30% value3   or: value2 = 0.30 value3

      value1 = 0.50 value2 = 0.50 (0.30 value3) = (0.50)(0.30) value3 = 0.15 value3

Ex. Find the derivative of the area (A) of a circle, with respect to its radius (r):

  $A = (pi)r^2$,   " dy/dx " = dA / dr = $2(pi)r$: = about 6.28r ,   also: A / r^2 = (pi) =~ 3.14..., a constant

  This derivative value is not a constant since it contains a variable, hence the derivative or rate of the increase or change of A with respect to a increase or change in (r) depends on the specific or current value of (r). This derivative is a "first order" or linear equation where the independent variable has a power of 1, and was derived from a "second order" equation where the independent variable has a power of 2.

  As (r) increases: When (r) = 1unit, A = 6.28square-units, and A is increasing or changing by 6.28(1) = 6.28sq-units
                When (r) = 2units, A = 12.56square-units, and A is increasing or changing by 6.28(2) = 12.56sq-un.
                for each increase in (r). The area, like a distance, is now increasing at a much faster rate (rate) or speed which has been accelerated or increased over 1 time unit which is similar to increasing (r) by 1. In a pseudo-distance equivalence representation: $A = (pi)r^2$ = distance = (speed)(time^2).

Notice that the derivative for a linear or first-order equation was a number, a constant, and this corresponds to a dimensionless point. We just seen that the derivative of a length, such as a circumference is a constant (here, (2)(pi)). The derivative of a second-order equation was a first-order equation. We just seen that the derivative of an area with two dimensions was a one-dimensional linear equation. Likewise, the derivative of a volume which has three dimensions, is a second-order equation:

  V = three-dimensions = (1 dimension)^3 = (1 unit)^3 ,

  dV / d (1 dimension)   = d(1 unit)^3 / d (1 dimension)   = $\frac{2 (1 \text{ unit})^2}{3}$

The equation for the volume of a sphere is: Vs = $\frac{4 (pi)r^3}{3}$,   The derivative of Vs with respect to (r) is:

  dVs / dr = (3)(4)(pi)(r^(3-1)) / 3 = 12(pi)r^2 / 3 = 4(pi)r^2 , This value is also equal to the **surface area of a sphere**
                                          It is also equal to 4 times the area of a circle with same r.

The derivative of the surface area, (Ss), of a sphere with respect to its radius (r) is:

  dSs / dr = d 4(pi)r^2 / d r = 8(pi)r = 25.13274123 r

A fundamental part or lower dimension of a line is a point. A fundamental part or lower dimension of an area (two dimensions) is a line (one dimension). A fundamental part or lower dimension of a volume (three dimensions) is an area (two dimensions).

Extra: The ratio of a spheres volume to its surface area is not a constant value, but depends on its specific radius value:

    $\frac{Vs}{As}$ = (4(pi)r^3 / 3) / 4(pi)r^2 = r / 3   : the volume of a sphere is greater than its surface area by
                                                    r / 3 = 0.33333 r   ,   also, from this we find:

    Vs = $\frac{As\ r}{3}$ = 0.3333 (As r)   also:

    $\frac{Vs}{As\ r}$ = 0.3333...

SECTION 4: ADVANCED TOPICS

This section is about making some important calculations when a scientific calculator or reference ("look-up") table(s) is not available. Specifically, this section shows you how to calculate logarithms, antilogarithms (inverse logarithms), trigonometric functions values of an angle, and inverse trigonometric functions, that is, calculating the corresponding angle given its corresponding trigonometric function value. Actually, this section could be called "Required Topics" since how could you perform these crucial mathematical operations out of "thin air", without a "look-up" table, or some direction as to how to proceed in their calculation? Since these calculations (especially division) could become tedious when repeatedly performed with hand, pen and paper, something such as a modernly inexpensive 4-function, home-use type of electronic calculator can be utilized to its fullest potential as if it were practically a full-function scientific calculator. Many of these types of electronic calculators also have a generous square-root, "fifth-function", key which is of great aid, especially for calculating some trigonometric values. Much of this section can also be adapted for computer programming languages that may or may not have these preprogrammed functions. In the image below, on the right is a typical 5-function "home-use calculator". On the left is an example of a scientific or multifunction calculator. Today, calculators are often inexpensively available at thrift stores, and some have a solar power option. [Image magnified 25%] [FIG 192]
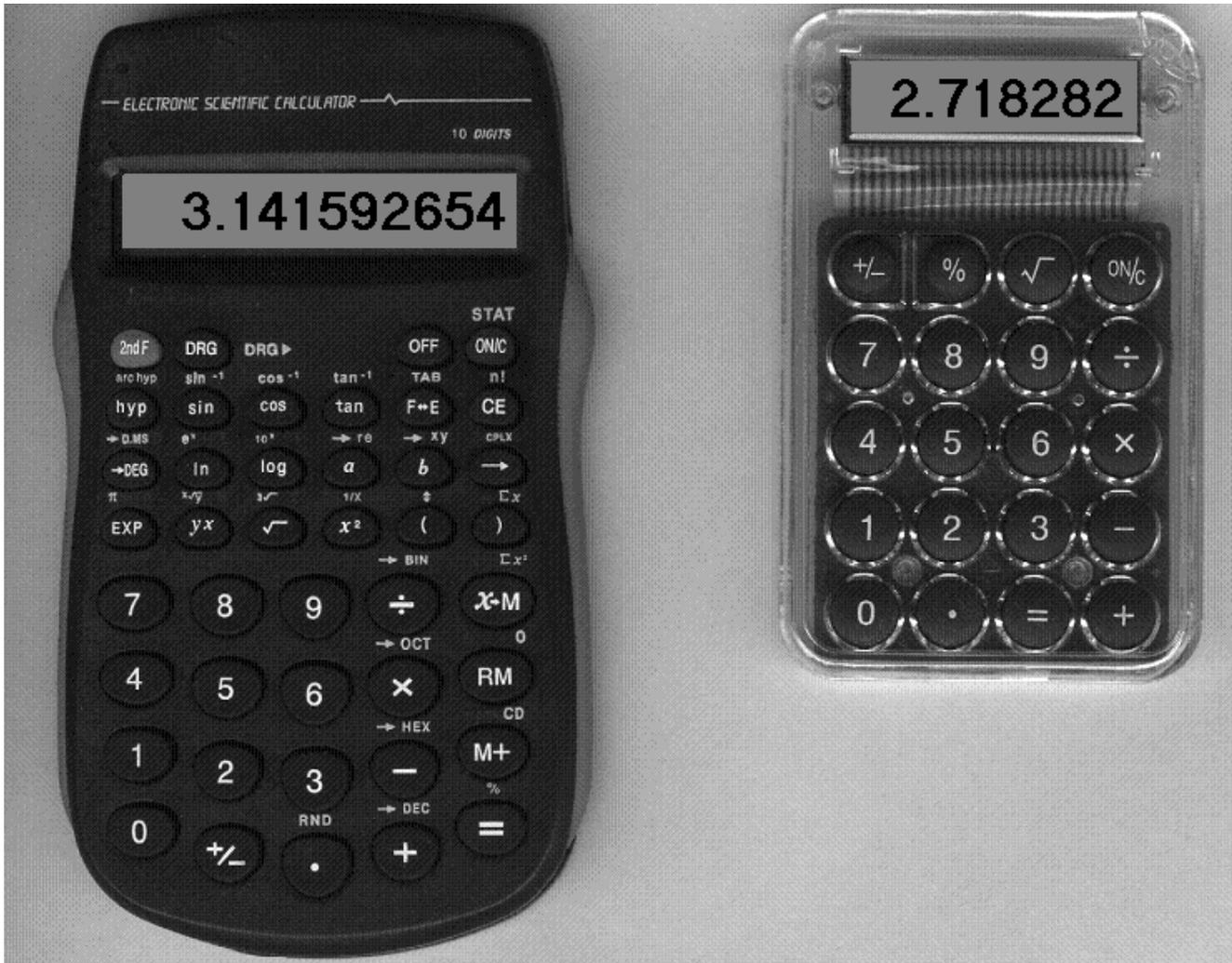
Note that a radian angle generally has nothing to do with the actual size of a circle or the specific distance (ie., radius) from the center point of rotation. All the possible and different arc lengths of concentric (have the same center) paths or orbits of points with each having their own corresponding distance (ie., radius) from the center point of rotation, can describe the exact same angle. For the same fixed or given angle, the farther a point is from the center of rotation, the greater the corresponding arc length corresponding to that angle, and this is similar to how the corresponding side opposite an angle in a triangle also depends on the size of that angle and the distance from it's vertex: [FIG 194]
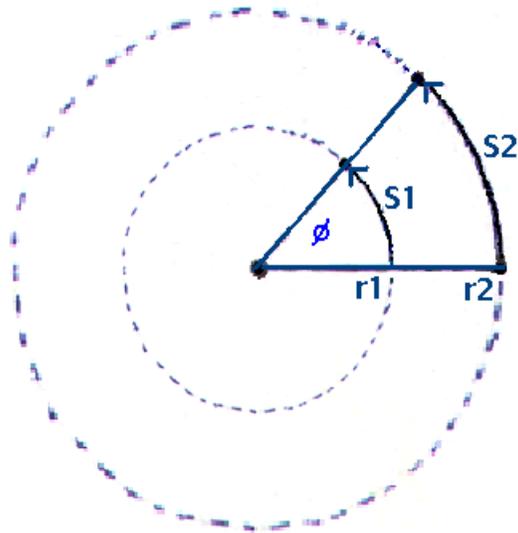
This figure above shows that different arc lengths and radius values can correspond to the same angle, and vice-versa. This is due to that all circle constructions are similar and the internal proportions of one is the same as that as another. Specifically, as an example for this discussion of concentric circles:

(arc length) / (radius) = a constant ratio value equal to the (radian) angle.

Since arc length and radius length have the same units, their units will be canceled during the division, hence radian angle measurement, in a strict sense, is a (strict) "numeric only" or unitless quantity, and is often indicated that way as just a (ratio) value without any indicated units. Generally, if an angle measurement does not have a degree symbol (°) or some other indication that the units are degrees, it could then be considered as a radian angle measurement. Many scientific calculators have a radian or degree angle, working mode setting (button, key, switch) for entering (as operands) angles, and displaying angles (the results). From the above equation we mathematically have this handy formula:

arc length = ¢ radius   : ¢ is a radian angle

We see that arc length is directly related to the angle and the radius.
If either the angle or radius changes by a factor (n), arc length also change
by that same factor. For example, if the angle doubles (multiplied by 2) the
corresponding arc length will also double. If the radius, orbit, or rotation distance
doubles, then the corresponding arc length will double:

(n) arc length = ¢ (n) radius   : for example, for a given angle, if the radius is halved by multiplying by 0.5, the
corresponding arc length will be halved. This equation can also be expressed as:

(n) arc length = (n) ¢ radius   : for example, for a fixed (in value, constant) specific radius, if the angle doubles,
the corresponding arc length doubles, or vice-versa

(180°/π) or = 57.29577951°.   For example, to convert 0.5 rads (radian units of angle measurement) to it's equivalent amount of degrees, multiply each side of the above equation by 0.5:

Ex.   0.5 (1) rads =   0.5 $\frac{(180°)}{(π)}$ =   about 28.648°   : For a radians to degrees conversion, use the "degrees
                                                                                                        on top" from the fraction (180°/π) for the multiplication.
                                                                                                        This is easier to remember than 57.29577951

Given 2π = 360°, solving for (1) degree:

   360° = 2π rads   dividing each side by 360:

   $\frac{360°}{360}$ = $\frac{2p \text{ rads}}{360°}$ after canceling:

    1° = $\frac{π}{180}$: about 0.017453292 radians.   Here, it is obviously easier to remember the numerator
                                                     and denominator of this fraction, rather than its quotient value.

To convert a degree angle measurement to its equivalent radian angle measurement, multiply the degree angle measurement by (π/180°) = 0.017453292. This is equivalent to multiplying both sides of the above equation by the same value (here, 10°).

Ex.   10° (1°) = $\frac{10° (π)}{1 (180°)}$ = $\frac{π}{18}$ = about 0.1745 radians   : For a degrees to radian conversion, use "pi on top"
                                                                                                        from the fraction (π/180°) for the multiplication.

Using a full circle as a starting reference, here's how you can get a perspective of corresponding radian and degree values:

From: 360° = 2(180°) = 2π   (about 6.28 radians)   : full circle, an angle of a "full turn" or rotation

Dividing by 2 to find out the amount of radians equivalent to 180°:

180° = 1 π =   π   (about 3.14 radians)   : half a circle   : "two quarters" (50 percent) of a circle
                                                                                                    π , "pi", as an (radian) angle, can be thought of as
                                                                                                    half a "full turn" or rotation.

Dividing by 2 to find the radian angle equivalent of 90°:

90° = $\frac{π}{2}$ =   0.5 π   (about 3.14/2 = 1.57 radians)   : "one quarter" (25 percent) of a circle

Since 270° = 90° x 3 , and   90° = π/2   we have:

270° = $\frac{3(π)}{(2)}$ = 1.5 π   (about (3)(1.57) = 4.71 radians)   : "three quarters" (75 percent) of a circle

Below is a table that summarizes these values:

SERIES

A mathematical, or numerical series or progression is a sequence (or "string") of numbers or terms that are mathematically related in some way, where either or both the next term or previous term in that specific series can be calculated by knowing the mathematical relationship (ie., a mathematical pattern) between, or of those ("adjacent") numbers or terms.

Every and all terms in a series can be formally represented or expressed algebraically by a single representative term that expresses the general format of each successive term in that series.

Formulas can also be developed to calculate the value of any specific term of a series, or for the sum of a certain number of terms in that series.   Later, some very important series will be developed to allow us to calculate things such as trigonometric, natural logarithm, and natural anti-logarithm values from the sum of the terms of these special series.

Here is a pseudo-algebraic description of a series (of mathematically related terms or values):

```
  term1
  term2  =  mathematical operation(s) applied to term1        : Though the mathematical operation(s) is constant, the
  term3  =  mathematical operation(s) applied to term2           result of that mathematical operation(s) that is to be
  term4  =  mathematical operation(s) applied to term3           applied to each term is not necessarily a constant value.
   (termN  or term(N) using a special notation (and which is not multiplication) and so on for other terms of the series.)
```

We see that the general format of each successive, or next, term is:

   term(n+1) = mathematical operation applied to term(n)

For example, if given a term value of 5, and we keep adding 2 to each term so as to find the value of the next term, we will have this series:

successive term numbers = n: 1 ,   2   ,   3   , 4   , . . .

successive term expressions: 5 ,   5+2 ,   (5+2)+2 , ((5+2)+2)+2 , . . .   : (term n+1) = (term n)+ 2   , or:
                                                                                           "next term" = "current term" + 2
                                                                                           Mathematically:
                                                                                           (term n) = (term n+1) - 2   , or:
                                                                                           "previous term" = term - 2

        Simplifying the expressions:

        successive term values: 5 ,   7   ,   9   ,   11 ,   , . . .   : a portion of an odd numbers series

        A certain mathematical operation was applied to a term so as to find the next term. If given that value of the next term, and so as to find the value of the previous term, the inverse, "reverse" or "undoing" operation of that given operation is to be applied to that given term value. The operand will be the same value as that first applied. This can be expressed as:

      (term)----> operation -----> (next term)   therefore: (next term) -----> inverse operation ------> term
                                                          or=:   term ------> inverse operation ------> previous term

   Ex.   5 -------> +2 -------> 7   therefore: 7 ---------> -2 --------> 5   : checking: 5 + 2 = 7, and   7-2 = 5.
                                                                                          Subtraction and addition are inverse
                                                                                          operations with respect to each other

If you were to add the same value (here, d) to each new term or sum created, this can be expressed as:

value   = value + d;   : n = the corresponding term number , or since the values are the values of the terms,
    (n+1)   (n)   (n+1) would correspond to the next term, and (n-1) would correspond to the previous
                                 term. (n) would correspond to the current term:

                                 . . ., "previous terms" ,   term   , "next terms" , . . .
                                 . . . , value(n-1) ,   value(n) ,   value(n+1) , . . .   or:
                                 . . . ,   term(n-1)   ,   term(n)   ,   term(n+1) ,   . . .

Solving for the added in constant value, or (d):

    value(n+1) - value(n) = d   Given a term in this series and solving for the previous term value:
or: value(n) - value(n-1) = d

    value(n+1) - d = value(n)   : since value represents the value of a term, the word "term" can be used instead
or: value(n) - d = value(n-1)   of the word value:

    term(n+1) = term(n) + d

    term(n)
    term(n+1) = term(n) + 1d
    term(n+2) = term(n+1) + d = (term(n) + d) + d = term(n) + 2d

Ex. The whole number series where d = +1:

    0, 1, 2, 3, 4, 5, . . .   : an ascending (increasing in value) series

Ex. The integer number series where d = +1:

    . . . , -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, . . .

Ex. In this series, d = +2 :   :the "even series"

    0, +2, +4, +6, 8, . . .

Ex. In this series, (d) is also = +2 :   :the "odd series"

    +1, +3 , + 5 , +7, . . .   : with d = +2, this very similar to the "even series", except that the first term here is 1

Ex. In this series, d = -2 :

    . . . , +8, +6, +4, +2, 0, -2, -4, -6, . . .   : a descending (decreasing in value) series

Ex. In the series below, d = +2.5 :

    . . . , 0, 2.5, 5.0, 7.5, 10.0, 12.5, . . .

For the series above, what is the next term after 12.5?

next term = current term + change

(number of terms from an initial term)(d) = (total difference in the terms values)   therefore:

d = _____total difference in terms values_____
            number of terms from an initial term

Ex. What is the common difference (d) of a series if the first term is (considered as) 6 and the fourth term is 10.5?

d = _10.5 - 6_ = _4.5_ = 1.5 , A partial (fragment, portion, section, or segment (of the)) series with this is shown below:
       4 - 1   3

. . . 6, 7.5, 9, 10.5 , . . .   : d = 1.5

You can also find (d) by writing an equation. If the first term is 6, then the second term is (6 + d), the third term is
((6 + d) + d) = 6 + d + d , or after combining like terms: third term = 6 + 2d. Since we know that the value of the
fourth term is 10.5, we get:

term in question - first term = total difference to term in question   algebraically:

first term + total difference to term in question = term in question   which can be expressed as:
first term + (number of terms from initial term)d = term in question

6 + 3d = 10.5   solving for (d), we get:

3d = 10.5 - 6
3d = 4.5
d = 4.5/3 = 1.5

Considering what has just been discussed, a simple formula to find the nth term of an arithmetic series is derived as
follows:

Let n = nth term in question
Let F = value of the first or initial term taken, given or considered as the first term of the series

   n   =   1   2   3   4   . . . n   : term number

Term =   F   (F+d)   (F+d)+d   ((F+d)+d)+d . . .   clearing grouping symbols :
Term =   F   F+1d   F+2d   F+3d   . . . F + (n-1)d   : after combining like terms

By observing the above sequence for any patterns, we can arrive at a general (algebraic) formula for the nth term. Notice
that the numerical coefficient of d is 1 less than the term in question, hence:

nth term = F + (n-1)d   : **FORMULA FOR THE nth TERM OF AN ARITHMETIC SERIES**
                                  Since a series can have an infinite number of terms, there may not be a first term
                                  or start of that series, so (n=1) is the number of the term that is taken or considered
                                  as the first term of this particular string or segment of terms under consideration.

Using the last example's series, what is the value of the 5th term?

4th term = 6 + (4-1)1.5
4th term = 6 + (3)(1.5)   distributing:

```
       Term n :       1    2    3    4     5     6     7     8     9    10  . . .
       Term value:   . . .2 ,  4 ,  6 ,  8 ,  10 ,  12 ,  14 ,  16 ,  18 ,  20 , . . .
```

Though less used, to solve for previous terms in an arithmetic series, the change (regardless if its positive or negative in value) is not added, but subtracted instead since:

From: next term = previous term + change   , and if the next term becomes the new current term of reference:

    current term = previous term + change   therefore:
    previous term   =   current term - change   or:

    current term   = previous term + d   therefore:
    previous term = current term   - d

With F still considered as the first term, this leads to a slightly rewritten formula for the nth (previous) term of an arithmetic series:

```
            first term  =   F                                          : n=0
     first previous term  =  (F  -  d)                               : n=1
  second previous term  =  (F  -  d) - d   =   F  -  2d          : n=2
```

nth previous term   = F - nd   : **FORMULA FOR THE nth PREVIOUS TERM OF AN
                                                         ARITHMETIC SERIES**

As a final note, considering any three consecutive terms of an arithmetic series, the second term, or "middle term", can be found without knowing the difference (d) between any terms.

```
Term #:     n          n+1          n+1          considering term n as the first term for this specific analysis:
Term:       F          F+d          F+2d
```

Adding or "summing" the first term and third term (essentially, the previous term and next term considering if they were in reference to the second or middle term):

```
   Sum = F +  (F + 2d)        clearing grouping symbols:
   Sum = F + F + 2d            combining like terms:
   Sum = 2F + 2d                factoring 2:
   Sum = 2(F + d)               dividing both sides by 2:
   Sum = (F + d)                : We arrive at the value of the second term.  This derivation indicates, that the middle
      2                              term can simply be found using the arithmetic mean of the two given terms.
```

Note also that (d) can be found by:

```
   d  =  Sum  - F          : due to the derivation, as noted above,  Sum =  previous term + next term, when
            2                      considering the (middle) term between those two terms in the series.  Essentially,
                                   three consecutive terms of the series will be found:  first,  second or middle,  third
                                   or terms:  n  ,  (n+1),  (n+2)
```

Subtracting the first term from the third term, (2d) and also (d) can be found:

    2d = (F + 2d) - (F)   Let 2d equal the total difference between the first and third terms. Distributing (-1):

GEOMETRIC SERIES

In a geometric mathematical or number series, successive or adjacent terms or numbers differ or vary by a constant factor value. This factor is simply called the ratio (r). To find the next term of a geometric series, multiply the current term by the common ratio value between the terms. In a reverse type of manner, to find the previous term of a geometric series, divide the current term by the ratio value. If the ratio for a geometric series is not known, it can be found by dividing the next term by the current term in the series, or dividing the current term by the previous term.

(term) x (ratio or multiplier) = (next term)   : letting (term) = (current term) in question or as reference to
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ both the (previous term) and (next term)

$$\text{ratio} = \frac{\text{next term}}{\text{term}} = \frac{\text{term}}{\text{previous term}}$$

If the next term is now considered as the current term, the "old" current term effectively becomes the previous term:

previous term x ratio = term   solving for previous term:

previous term = $\dfrac{\text{term}}{\text{ratio}}$ or =   term/ratio = (term) (1/r) = term or current term times the reciprocal of the ratio.
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ In one direction, to find the next term, we multiply by (r), and to find the next
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ term in the other direction, we multiply by (1/r), the reciprocal value, which is
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ the same as dividing by that ratio value. Still, in either way or direction, the
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ multiplier is the essentially the reciprocal of the value used for the other direction.

This can be expressed in a more general way as:

$\quad$ value(n+1) = (r) value(n)   : n = the corresponding term number , or since the values are the values of the terms:

$\quad$ term(n+1) = (r) term(n)   mathematically:

$\quad \dfrac{\text{value2}}{\text{value1}} = \dfrac{\text{term(n+1)}}{\text{term(n)}} = r$ mathematically:

$\quad \dfrac{\text{term(n+1)}}{r} = \text{term(n)}$

Ex. In a series, the first term is 3, and r = +2:

$\quad$ For one direction: multiplier = factor = r = 2

$\qquad$ term x multiplier = next term

$\qquad$ 3
$\qquad$ 3 x 2 = 6
$\qquad$ 6 x 2 = 12
$\qquad$ 12 x 2 = 24
$\qquad$ 24 x 2 = 48

$\quad$ **. . . ,** 3, 6, 12, 24, 48, **. . .**   : an ascending (in value) series, since: (r > 1)

To solve, first find the value of (r) for this particular series. We know:

F = 2   : first term, and where n=1: Fr^(n-1) = 2r^(1-1) = 2r^0 = 2(1) = 2

Fr^(3-1)   =   Fr^2   =   18   : third term

Choosing to solve these simultaneous equations for (r) by algebraic substitution:

Fr^2   =   18   substituting the known value of the first term:
(2)r^2   =   18   solving for (r)

r^2   =   18_
             2

r^2   = 9   taking the square root of both sides of the equation:

r = $\sqrt{9}$ = 3

The second term is therefore:

Fr^(2-1)   =   Fr^1   =   2(3^1)   = 6

And the (partial) series is shown below:

. . ., 2,   6,   18,   . . .   : r=3

When r<1 and positive (usually indicated as 0<r<1, ie. a fraction of 1, since it is less than 1), that the values of the next terms will get smaller and smaller till they are effectively meaningless or practically equal to zero. If the terms of this type of series are summed from any point (ie. a term) in that series, the sum of those terms will approach a definite or specific value. The sum is said to converge or "zero-in" (having only a small negligible difference, practically 0) to a certain value. This type of series is called a converging series. Even with this said, it is possible for the terms of some geometric series to get smaller and smaller and yet the sum does not converge to a specific value. These types of series are called diverging or divergent series. When making calculations where only one specific value is being sought, converging series will obviously be utilized.

There are many other useful formulas for working with series, such as the sum of the first (n) terms of a series that is shown in the appendix section of this book. This basic discussion on series is meant to prepare you for the discussions that follow in this book. It should also be mentioned that the series, such as the trigonometric series, to be discussed further ahead, do not have a constant difference or ratio between successive terms. Each term is still predictable due to a pattern and can be calculated (ie. via an algebraic formula), and that the sum of these series still converges to a specific value. This specific value of the sum of terms depends upon the specific value of the independent value or variable used in the terms that are to be summed. This calculation using a series of terms is still an approximation (ie., not "algebraic" or exact as with a simple formula) of the true result, and as close (ie., acceptable precision, fineness) as desired.

How do I solve for a previous term of a geometric series?

Though less used, the method to solve for the previous term of a series is based on the same method to find the next term of a series:

From: (term) (r) = (next term)   mathematically:

Dividing the previous term by (r) so as to find the term that preceded it (ie., the 2nd term in this series):

$$\frac{\dfrac{F}{r}}{\dfrac{r}{1}} \quad = \quad \frac{F}{r^2}$$

Continuing this process of dividing by (r) so as to find the next previous term, we note a pattern that the resulting or indicated power of (r) and the term in question are the same, and we arrive at a formula for the nth previous term:

nth previous term = $\dfrac{F}{r^n}$ = Fr^-n   : **FORMULA FOR THE nth PREVIOUS TERM**
                                                  OF A GEOMETRIC SERIES
                                                  (r is the normal ratio of the series, n corresponds to the
                                                   nth previous term)

Here is a simple verification to this formula: Consider the "formal" or algebraic expressions for the terms of a geometric series:

. . ., Fn, Fr^(n+1), Fr^(n+2), Fr^(n+3), . . .   for our purposes here, letting n=0:

. . ., F, Fr^1, Fr^2, Fr^3, . . .

Now, consider any term as the first term. It is quite obvious that the power of (r) of the next term is one higher, and that the power of (r) in the previous term is one less.   Letting **Fr^0 = F** be the first term or reference term:

previous terms   . . ., Fr^-3, Fr^-2, Fr^-1, **Fr^0**, Fr^1, Fr^2, Fr^3, . . .   next terms

. . ., $\dfrac{F}{r^3}$, $\dfrac{F}{r^2}$, $\dfrac{F}{r^1}$, **F** , Fr^1 , Fr^2 , Fr^3 , . . .

We see that the first previous term has the divisor of (r) to the first power, the second has (r) raised to the second power and so on.

From the last example given, the 3rd term is (7 - 3) = 4th previous term:

4th previous term = $\dfrac{F}{r^4}$ = $\dfrac{64}{2^4}$ = $\dfrac{64}{16}$ = 4   :checks

As a final note for this discussion about geometric series, if given any three consecutive terms of a geometric series, the second term, or "middle term", is the geometric (as opposed to arithmetic) mean of the first and third terms. Mathematically, a geometric mean of two values is the square root of the product of those two (factor) values:

Term #:   1   2   3
Term: F   Fr   Fr^2

Multiplying the first and third terms:

(F)(Fr^2) = F^2r^2

Taking the square root of this value, we arrive at the value of the "middle" term, and we verify the definition:

Ex. Here is a series where r=2:

   . . . , 2, 4, 8, 16, 32, . . .

Note here that you can find the value of (r) by several ways. You can divide any term by the previous term, or consider the values somewhat like the corresponding values in the equation to find the slope (m) of a line since the ratio (r) is the (constant) rate of change for a line and for a geometric series of terms. That is, r=m:

   First let: y1 = m x1   and   y2 = m x2   : linear equations, m = r = yn/xn

              y1 = r x1   and   y2 = r x2

     Ex.   4 = (2) 2   and   32 = (2)16   : these are values in the series above
                                                        p1(x1,y1) = p(2,4)   and   p2(x2,y2) = p(16,32)

              m = r = $\dfrac{y2 - y1}{x2 - x1}$

     Ex.   m = r = $\dfrac{16 - 4}{8 - 2}$ = $\dfrac{12}{6}$ = 2

Checking: $\dfrac{(n\,r\,r\,r) - nr}{(n\,r\,r) - n}$ = $\dfrac{nr^3 - nr}{nr^2 - n}$ = $\dfrac{r(\cancel{nr^2 - n})}{(\cancel{nr^2 - n})}$ = r   : For the specific example: F = n = 2

Extra: If the sum of 4 consecutive terms of this series is 30, what is the first term of those 4 consecutive terms if r=2?

       n1 + n2 + n3 + n4 = 30   considering n1 as the first term, we can express each term in terms of n1:
       n1 + n1r + n1r r + n1 r r r = 30
       n1 + n1r + n1r^2 + n1r^3   = 30   dividing each term by n1 :
       n1 (1 + r + r^2 + r^3) = 30   and if r = 2:
       n1 (1 + 2 + 2^2 + 2^3) = 30
       n1 (15) = 30   solving for n1:
       n1 = 30/15 = 2

The appendix contains a discussion about summing consecutive terms of a geometric series.

SIN ¢ SERIES

The SIN ¢ series is presented below. Of all the various trigonometric series, it is a good idea, at some point in your mathematical journey, to try memorize the basic format of the SIN ¢ series, and so as to always have a mathematical foundation to find the various trigonometric values of an angle. The COS ¢ and TAN ¢ values can also be calculated from the SIN ¢ value. This is so, since any angle has a specific set of corresponding (SIN, COS, and TAN) trigonometric values associated with it.   The appendix also contains a few more series such as the COS ¢ series, and a derivation of the SIN ¢ series.   Here is the common "formula" for the SIN ¢ series:

$$\text{SIN } X = \sum_{n=0}^{n=\infty} \frac{(-1)^n \ X^{(2n+1)}}{(2n+1)!}$$

: X is a radian angle. The ∞ symbol means "infinity". Though infinity does not have a specific value, it indicates an endless process, and here, an endless number of terms to be summed together, in theory, are needed for the true result, In practice, the number of terms used is relatively few due to that the sum is converging or becoming equal to the true value very quickly with only a few terms with just a small difference between them, and therefore, only a small difference between the sum and the true result. The expression on the right is the general format for each term that is to be summed together.

To properly use this series, X must be a radian angle measurement. Radians (units) is a natural or inherent (to the circle, or rotating points) measurement for angles, but degrees (units) are not totally natural since they are man made units for the measurement for angles, specifically with the near random value of 360.

The symbol that looks like a large S alphabetical letter is called "Sigma" and is to be considered as the symbol and first letter for summation. Here, mathematically related terms are to be summed, and the basic format for each term is shown to its right side. (n) is a variable value that will change in the expression for each next term by an increment of 1. That is, n is the term number, and for the first term n=0 (as indicated immediately below the summation symbol). For the second term, n=1, for the third term, n=2, and so on till (n) is equal to infinity (∞) as indicated at the top side of the summation symbol.

In short, the notation means: "evaluate the terms from n=0 to n=∞, and sum them". Usually, far fewer terms than an infinite number of terms will be utilized for the sum since the value of the "next" terms get significantly smaller and rapidly converges or approaches to being a value close to 0 and meaningless, and the sum of those limited number of terms then rapidly converges or approaches a specific numerical value. The actual number of terms to be utilized for the sum depends on the number of correct (ie., accurate) decimal places (ie., the "precision", preciseness or fineness) desired.

As a basic rule for this series, for (x) decimal places of accuracy, sum (x+2) terms. When X < (π /4), (about 0.79), equivalent to an angle of: ¢<45°, fewer terms are usually required. The value of the factor $(-1)^n$ will produce alternating (positive and negative) signs or values of the terms, hence an "alternating series". **!** (an exclamation point) is called the **factorial** symbol. It is the accepted symbol used to indicate the product of all integers from 1 up to and including the indicated value that precedes it:

      n! = product of all integers from 1 up to and including n   : n! is read as "n factorial"

Ex.   4! = 1 x 2 x 3 x 4 = 24   : "4 factorial"

Ex.   5! = 1 x 2 x 3 x 4 x 5 = 120

$$\text{SIN } X = X^{\wedge}1 - \frac{X^{\wedge}3}{3!} + \frac{X^{\wedge}5}{5!} - \frac{X^{\wedge}7}{!} + \frac{X^{\wedge}9}{9!} - \frac{X^{\wedge}11}{11!} + \frac{X^{\wedge}13}{13!} - \frac{X^{\wedge}15}{15!} \ldots$$

Notice that terms of the expanded series can easily be memorized as having this general form:

$\pm \dfrac{X^{\wedge}\text{odd}}{\text{odd!}} \ldots$: A simplified formula for each term of the SIN X series.
odd = odd numbers starting at number 1.
The first term is positive.

To assist calculations, the factorials were evaluated (simplified) in the series of terms below:

$$\text{SIN } X = X - \frac{X^{\wedge}3}{6} + \frac{X^{\wedge}5}{120} - \frac{X^{\wedge}7}{5{,}040} + \frac{X^{\wedge}9}{362{,}880} - \frac{X^{\wedge}11}{39{,}916{,}800} + \frac{X^{\wedge}13}{6{,}227{,}020{,}800} - \ldots$$

For the 7 terms shown, an accuracy of 9 correct significant decimal digits for angles between 0 and 90 degrees can be achieved by using a computer that can perform computations with 16 digits. Further terms used in the sum will result in the next accurate (correct) significant digit(s) with a descending or less decimal weight of which you may then compare to the previous sum. The smaller the angle, the fewer the terms required for a given precision. It is also possible to use the reciprocals of the denominators as factors (multipliers) in the numerators; (1/denominator), if you prefer multiplication rather than division.

Ex. What are the sine, cosine, and tangent values of 60°.

Converting 60° to its equivalent radian angle measurement required for the SIN X series or formula we find:

60° = about 1.047 (radians or rads)

Since only 3 decimal places are utilized for the angle's approximation, the best accuracy, and practically regardless of the number of terms evaluated, that can be expected for the SIN 60° result using the SIN ¢ series (or even from a calculator) is only about 3 decimal places.

Substituting 1.047 for X in the SIN ¢ formula, and using only the first 4 terms of its expansion (ie., as a sum of terms):

$$\text{SIN } 1.047 = \frac{1.047}{1} - \frac{(1.047)^{\wedge}3}{6} + \frac{(1.047)^{\wedge}5}{120} - \frac{(1.047)^{\wedge}7}{5040} + \ldots$$

Here, it is easier to simplify each fraction rather than trying to add them all using a common denominator.

SIN 1.047 = 1.047 - 0.1913 + 0.0105 - 0.0003 : using approximate values of the terms

SIN 1.047 = 0.8659

Hence, SIN 60° = SIN 1.047 = 0.866

From COS ¢ = $\sqrt{1 - \text{SIN}^{\wedge}2\ ¢}$: finding the COS ¢ using the corresponding SIN ¢

COS 60° = $\sqrt{1 - (\text{SIN } 60°)^{\wedge}2}$

COS 60° = $\sqrt{1 - (0.866)^{\wedge}2}$

TRIGONOMETRY OF ANGLES GREATER THAN 90°

One might logically think that since SIN 90° = 1, that the SIN value of angles greater than 90° is greater than 1. This is however not so since the maximum SIN value, regardless of the angle, is always 1.

SIN ¢ = __side opposite the ¢__   : Trigonometric values are formally defined using a right triangle.
         hypotenuse

And we know that in a right triangle:

hypotenuse > side opposite   and   hypotenuse > side adjacent   ,

that is, the hypotenuse is always the longest side, and using this fact in the SIN ¢ definition above, the numerator will always be smaller than or equal to the denominator, and therefore, this ratio is always less than or equal to 1 mathematically.   Only when the angle is 90°, the side opposite this angle will then be equal to the length of the hypotenuse side.

From 90° to 180°, the SIN values start to decline in a corresponding "reverse" (or "mirror image") type of manner and are equivalent to what would be for the (declining) SIN values of angles from 90° back to 0°. This is indicated in the figure below as the angle of (90°- Q1), which is also the complementary angle value to Q1. That is, each obtuse (>90°) angle has an equivalent SIN value of an (corresponding or trigonometrically equivalent) acute angle between 0° and 90°. After calculating this corresponding acute angle, you can use it as an (trigonometrically) equivalent angle of reference, or reference angle, to calculate the trigonometric values of angles greater than 90°.

In the full range of angles from 0° to 360°, the trigonometric values will cycle or repeat within 90° (but not necessarily every 90, due to the nature of the SIN values or waveform, as will be seen in a graph below) , hence they will repeat a total of four times. Therefore, the trigonometric values for angles in the range of angles from 0° to 90°, can simply be reused (as a reference) for the trigonometric values of all the remaining angles between 90° and 360°.

In this discussion, Qn is used indicate a quadrant of the rectangular coordinate measuring system, where (n) will have a value of 1 to 4. A quadrant angle is an angle measured in reference to the x-axis, 0° reference line or angle. If the ending or terminal side of that angle is in a specific quadrant, the angle is associated and indicated as an angle of, or in that quadrant. For example, a quadrant 2 angle would be indicated or expressed as a Q2 ¢, "a quadrant two angle", or simply as Q2.

Quadrant   Angles

1   0° to 90°   : Q1
2   >90° to 180°   : Q2
3   >180° to 270°   : Q3
4   >270° to 360°   : Q4

Below is a method to find a trigonometrically equivalent (E) or corresponding acute angle that is less than or equal to 90° when given an angle greater than a right-angle (90°). Angles greater than 90° are often called obtuse (wide, large) angles, and angles less than or equal to 90° are often called acute (narrow, thin) angles. As shown below, these obtuse angles greater than 90° have their "ending (after rotating)" side within another quadrant (quarter) other than the first quadrant of a circle or a rectangular co-ordinate measuring system.   By creating a right triangle with the side adjacent to the angle on the horizontal or (x) axis of the quadrant in which the angles ending side is in, and the hypotenuse side of that angle created from the ending side of the (obtuse) quadrant angle, a trigonometrically equivalent or corresponding acute angle is created. The trigonometric functions of this corresponding acute (< 90°) angle are equal to that of the given obtuse (> 90°) angle.

y = sin ¢

Q1   Q2   Q3   Q4

+1

0   90   180   270   360 X = ¢
                                              0

−1
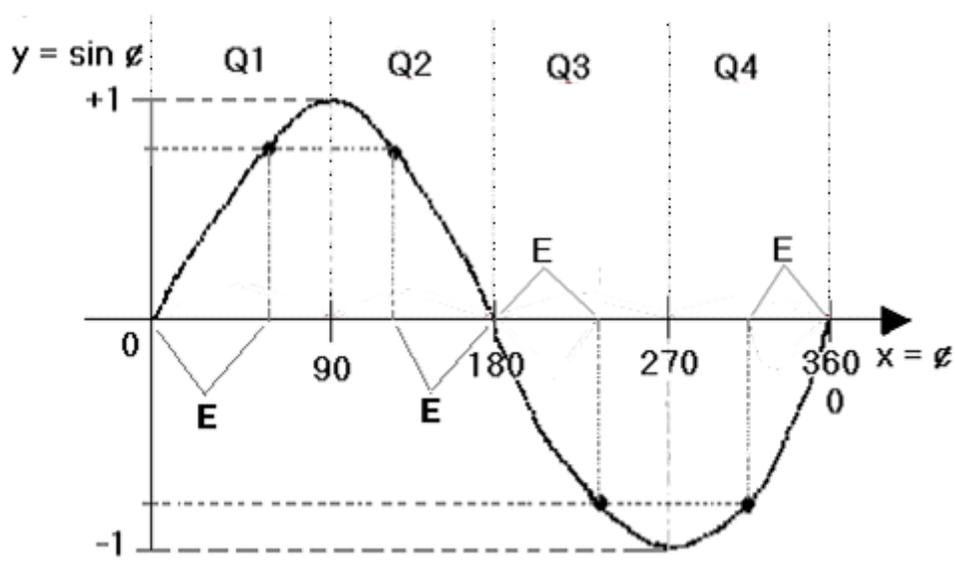
Q = Quadrant

E = Trigonometrically Equivalent Angle (<=90)

Clearly, each trigonometric value (here in this example, SIN ¢ was used) indicated as points on the curve, will occur 4 times in the range of 0 to 360 degrees. Each occurrence will be in one of the 4 quadrants where each quadrant has a range of (360°/4) = 90 degrees.

From 0°, as the angle increases, the corresponding SIN ¢ values will increase up to an absolute value of 1 at the peaks (of the curve or waveform at 90° and 270°), and then decrease in the exact reverse (or mirror-like) manner from this value of 1 to the troughs (of the curve or waveform at 0°=360° and 180°) and have a value of 0.

Another way to state the above graph is that within 0° to 360°, 4 angle values will have the same SIN value as that of an angle (between 0° and 90°, in Q1) that has that same SIN value. This SIN value will occur 4 times during 360°. Notice that the corresponding SIN values are mirror images and symmetrical at and about the 90° mark, and the 270° mark (which is 180° more from 90°). As seen in the image, clearly these angles and their equivalent sine values are not generally separated equally by 90° (ie., not complementary angle values). Take note of the first point indicated, that the amount of angle from here up to 90° is equivalent to its complementary angle (an angle and its complementary angle sum to 90°). 90° plus this same complementary angle value is the corresponding trigonometric angle of the second point and angle with the same trigonometric values. Point 1 and point 2 are equidistant (or "symmetrical") from the 90° mark where SIN 90°=1. For example, they could both be 20° from the 90° mark, when point 1 corresponds to 70°, and point 2 corresponds to 90° + (complementary angle to Q1) = 90°+ (90° - Q1) = 90 + (90° - 70°) = 90° + 20° = 110°. SIN 70° = SIN 110°. The SIN values will decrease on each side of this peak till 0° or 180° is reached where SIN 0°= 0. The sine of 89° is the same as the sine of 91°. The sine of 70° is the same as the sine of 110°. The sine of 1° is the same as the sine of 179°.

Each point, or angle and its corresponding SIN value on the SIN curve, has an angular distance from the peak (ie. high values, where the SIN values are 1) that is a value equal to the complementary angle of the first quadrant angle = Q1 = trigonometrically equivalent angle (E). Just the same, it could be said that each point (angle and its corresponding SIN value) has an angular distance from or to the "troughs" (ie. low values, where SIN values are 0), and this value is equal to the first quadrant angle = Q1 = trigonometrically equivalent reference angle (E).

Here is a method to calculate the trigonometrically equivalent angle given any angle, or angle in any quadrant:

For an angle that is less than or equal to 90°, it is in the first quadrant and the trigonometrically equivalent (E) angle (<90°) is obviously equal in value to that angle:

E = Q1   : trigonometrically equivalent angle for an angle in the first quadrant

For an angle that is greater than 90° and less than or equal to 180°, it is in the second quadrant, and its trigonometrically equivalent angle sum to 180° :

180° = Q2 + E   : supplementary angles sum to 180°, therefore supplementary angles have equivalent trig. values
E = 180° - Q2   : trigonometrically equivalent (E) angle for an angle in the second quadrant,
Q2 = 180° - E

For an angle that is greater than 180° and less than or equal to 270°, it is in the third quadrant and is equal to 180° plus its' trigonometrically equivalent angle:

Q3 = 180° + E   mathematically:
E = Q3 - 180°   : trigonometrically equivalent angle for an angle in the third quadrant

As a special note: A triangle can not contain a single angle equal to or greater than 180° since the interior sum of all three angles within a triangle is always 180°.

For an angle that is greater than 270° and less than or equal to 360°, it is in the fourth quadrant and is equal to 360° less its trigonometrically equivalent angle.

360° = Q4 + E
Q4 = 360° - E
E = 360° - Q4   : trigonometrically equivalent angle for an angle in the fourth quadrant

Because of the facts stated above, the trigonometric (absolute) values of two corresponding supplementary angles (that sum to 180°) are the same value.

Ex. SIN 10° = 0.173648177

    The supplementary angle of 10° is (180° - 10°) = 170°

    SIN 170° = SIN (180° - 170°) = SIN 10° =   0.173648177

Ex.   Q1   Q2   Q3   Q4   : Qn = "Quadrant angles"
       sin 50°   =   sin (180° - 50°)   =   sin (180° + 50°)   =   sin (360° - 50°)
       sin 50°   =   sin 130°   =   sin 230°   =   sin 310°   = about 0.766   : absolute value

    Note above, that for this example, Q1 and Q2 are supplementary angles.   Q1 and Q3 are separated by 180°.
    Q3 and Q4 are separated by 180°

Here is a helpful chart to express these values:

Sinusoidal Oscillations

Many things in nature and science (such as "ac", alternating (back and forth movement or flow, with the measurement values then having an opposite polarity, hence positive and negative) voltage and current electricity) are said have a sinusoidal or sinusoidal-like quality about them because they do something repetitively of which can be mathematically expressed with a sinusoidal equation and curve or waveform that is repetitious. As indicated, a good example of this is with the voltage and current output of an electricity generator.   When its rotating coil of wire effectively becomes perpendicular to the magnetic field of a magnet, then maximum voltage is induced (ie., produced) which causes a corresponding maximum current to be produced. The output voltage and-or current will have a sinusoidal (sine) waveform or shape with respect to the rotation angle and-or time.   The actual amount of voltage produced depends of several factors such as the number of turns of wire to make the coil, and the speed of the rotation of that coil and-or magnet (field), and the magnetic field strength of which also includes how close the coil and magnet can become so as to have maximum electricity generation effect.

Besides the word "sinusoidal", other common words for repetitious behavior are "vibrate", "oscillate", "cyclic", and "periodic" (repeating over the same time period).   All of these words are essentially a form and meaning of the word "repetition" (repeating).

Below is a mechanical oscillation system. A weight is suspended by a spring. If this string is either stretched or compressed, storing that input energy, and then released, its movement and corresponding position will be sinusoidal (sine like).   Note that for the weight to change direction, it must first slow down and then actually stop for an instant of time, before changing direction.   The weight will repeatedly accelerate and decelerate. Its speed or velocity will change, hence on the graph, the slope of the curve is changing and has a value of 0 when the weight stops and changes direction. Because of some (fractional, small) losses (such as due to friction) of energy during each cycle in this system, the maximum height or "amplitude" value will also decrease or decay in value as time increases.

The length or amount of time taken for this system to complete one cycle of this repetitious motion is called the wavelength (as shown in the image below between 0 and w on the time-line axis).   The reciprocal of this wavelength value is the number of repetitions per second, and is usually called the frequency (of repetitions or cycles) and having units of (cycles/second) = hertz. Frequency is "how often" something occurs during some length or period of time. The faster something vibrates, repeats or oscillates, the less time taken for each one of those cycles or waveforms to complete, and therefore frequency and wavelength (time) are mathematically inversely related.

Ex. For a frequency of 1000 hertz or cycles per second. A 1000cps = 1000hz signal has a wavelength of 1/(1000 c/1s) = 0.001s/c   = 0.001 seconds per cycle = 1 millisecond per cycle, or simply 1 millisecond.

[FIG 197]

In the figure below, the motion, particularly the height of the vibrating or oscillating object held by the spring will be sinusoidal in nature. When the spring stretches or compresses, it stores energy and the object will decelerate and loose kinetic energy and the object will eventually be at its maximum, positive or negative height or distance from its rest position where no energy was applied to it.   If there is no energy to replace any energy losses in this system, such as due to mechanical friction and the resulting heat, then the oscillations will decrease in height until they eventually stop. This is sometimes called as the oscillations and the amplitude being dampened over time. [FIG 197]

Extra: 1 revolution = 1 cycle = 360° = 2(pi) radians   : 2(pi) = 2(3.14159265..) =~ 6.2818

Note that: T = 1/F = seconds / cycle = s / **rev**olution = s / **rot**ation

(Also considering the above formula, *):

height = current or instantaneous value = SIN ((t / T) 360°));
height = value = (max. amplitude) SIN ((t) (1 / T) 360°)) = (max. amplitude) SIN ((time)(F) 360°));

Or since: 360° = 2(pi) = about 6.28 radians , and some rearranging of the factors:

value = (amplitude) SIN (2 (pi) F t)

The value of: (2(pi)F) is sometimes called the "angular velocity" per unit of time, and is given a symbol that looks like: W
Angular velocity is the amount of rotation per second   , letting W = angular velocity ,   W = angle / time = ¢ / t

value = amplitude SIN (W t)   ,   if t = T, the angle = 360°   , and:

W = "angular velocity" or "angular speed" = "rotational speed" = _revs_ = _cycles_
                                                                                                                    t   t

W = _360°_ = 2(pi) (1/T) = 2(pi) F   : here we see that F is essentially a multiplier to the angle, cycles or revs, and
        T   determines and is needed to express the actual angular or rotational velocity.

Below is another example of using these types of concepts and equations.

AN EQUATION FOR THE APPARENT TILT OF THE EARTH

        The apparent tilt and-or angle of the Earth with respect to the Sun or solar plane is sinusoidal in nature. The Sun or solar plane is essentially the imaginary disk or plane created between the Earth and the Sun as Earth revolves (orbits, travels, moves, goes) around the Sun. Because of the orbit of Earth about the Sun being a slight ellipse shape, and not exactly circular, the apparent tilt of the Earth with respect to the Sun throughout the year is actually not exactly sinusoidal in nature, but it is close enough to be considered and analyzed as such in terms of practicality.

Earth's axis is tilted, but it does not change. Earth's tilt actually remains constant at about 23.5° with respect to the Solar Plane. The true north to south line, axis or pole line, constantly points to nearly the star called Polaris or the "pole star" that is directly overhead at the north polar axis. This is also due to that the equator line is always tilted by a maximum of 23.5° with respect to the solar plane. What is tilting throughout the year then? It is the angle between the Earth's standardized equator line and the solar plane - say toward the direction of the Sun, that changes throughout the year. As the Earth orbits the Sun throughout the year, in the summer time of the northern hemisphere, the equator line and-or region of the Earth is tilted below the solar plane by a maximum of 23.5°, and in the winter time, it is tilted above the solar plane by a maximum of 23.5°. This apparent axis tilt of the Earth throughout the year is generally said as what actually causes the climate seasons of the year.

Throughout the yearly orbit of Earth about the Sun, the eastward to westward path of the Sun will change slightly every day. The Sun will rise and set in a slightly different location and time every day. The Sun also has an effective or apparent angle with respect to the local horizon throughout the day as Earth revolves on its axis from eastward to westward. The tilt of the Earth, such as at the equator, with respect to the solar plane (or "ecliptic") is at 0° on March 21 and Sept 21 during the year.

Earth's (equatorial) tilt with respect to the solar plane, every day is actually about 23.436° as of the year 2020AD. We can

northern hemisphere of the Earth will experience winter, and the southern hemisphere will experience summer.

The total tilt is then the amount of angle between these two angles: 23.5 - (-23.5) = 23.5 + 23.5 = 47°.

The tilt of the Earth is actually an apparent tilt, or illusion, since the imaginary line extending from Earths daily axis of rotation about itself always extends or "points to" the "north star" which is also called Polaris which is many "light years" (time of light travel) away..   The axis of the Earth is constantly tilted 23.5 degrees with respect to the solar plane (or "ecliptic"), but due to the orbit of Earth around the Sun throughout the year, it will have the effect that the one pole get closer to the Sun and the other pole gets farther away, hence creating the weather changes or seasons throughout the year.

The maximum apparent "height" that the Sun will appear above the horizon (0° reference line) at the observers local noon (true 12:00 pm) will be 90° less the latitude ("lateral" or side location, measured in degrees from the equator plane which is the imaginary line between north and south on the Earth) of the observer plus the current tilt angle. You can find **the latitude of your location** from some maps, or you can measure the angle between your local horizon and the north star (Polaris). The reference latitude at the equator is considered as 0°, and latitude is the angle created from the point at the center of the Earth to a location on the surface of the Earth. When the equator is located at the solar plane during the year, a person at the north pole of the Earth will see Polaris, the North Star, nearly directly overhead, and the angle between the horizon line and the line to Polaris is 90°, and this is the latitude at Earths north pole (axis). A person at the Equator will see Polaris as being at the horizon, and the angle between the horizon and Polaris will be 0°, and which is the latitude at the equator.   The word "latitude" is based on the word lateral and slats (ie., side lateral, location), like the horizontal side shingles on houses that go from the ground level area to the top area of the house. In relationship to Earth, latitude is your location between north and south, and a line of latitude extends around the earth in the east and-or west directions.

Longitude is how far east or west a point on Earth is. The word "longitude" is based on the word "long". In relationship to Earth, longitude is your location eastward and-or westward, and a longitude line actually extends from the north pole to the south pole.

Greenwich, a city near London city in England, is considered as the 0° reference position of longitude. This longitude line is called the **Prime Meridian** and goes in a northern and southern direction from the north pole to the south pole. This line is also the reference meridian and-or time zone of the world. The time of an event anyplace or location in the world can be expressed in terms of the time at this reference longitude, and it is called: Greenwich Mean Time (GMT, mean=average) and-or Universal Coordinated Time (UTC or UT1), and loosely as "World Time" Each of the 24 (longitudinal) and hourly "time zones" is (in theory) approximately 360°/24 time zones = 15°/ timezone.   Each longitude line separated by an angle of 15° can be said as being 1 hour and-or 1 timezone apart. For each and every location along the same longitude line, the true local time will be the same and the Sunlight will be directly overhead at the same time of noon which is 12PM. 15° of Earth's rotation and-or longitude separation corresponds to 1 hour = 60 minutes of time. 5° or Earth's rotation and-or longitude separation will correspond to 20 minutes.   For 1 minute, setting up a equivalent fraction and-or proportion equation:

   15°/60min   as = x°/1min   ,   x° = (15°)(1min) / 60min = 0.25°   , therefore 1 min is to 0.25°, and
                                                            1s is to 0.0041667°

United States decided to keep large areas of the country's population and schedules at the same time, and made 4 time-zones that have roughly a north to south imaginary boundary line separating them. The 4 times-zones in the continental United States are the Eastern, Central, Mountain and Pacific time-zones. Starting from the Eastern boarder of the United States with its Eastern time-zone, and going in a westward in direction, each other time zone is an hour behind its eastern neighbor time-zone.

Many countries have adopted a concept called **Daylight Savings Time (DST)** where all clocks are set an hour "ahead" or later that it actually is during some months so as to effectively or apparently have another hour of daylight each day, particularly in early (AM) waking and start of work hours. When not in the months when DST is applied, it is called

Kansas City, Missouri, USA	39.1°N	94.58°W	: central USA
Los Angeles, California, USA	30.05°N	118.2°W	
Anchorage, Alaska, USA	61.22°N	150°W	
Chicago, Illinois, USA	41.86°N	87.64°W	
Moscow, Russia	55.7°N	37.6°E	
Beijing, China	39.9°N	116.4°E	
Tokyo, Japan	35.7°N	139.8°E	
Sao Luis, Brazil	2.53°S	44.26°W	
Mecca, Saudi-Arabia	21.39°N	39.86°E	
Capetown, South Africa	33.9°S	18.4°E	: S = south of the equator, (-) sign also means south of equator or west of Greenwich

33.9°S , 18.4°E =   -33.9° Lat, -18.4 Lon.

Nagercoil, India's	8.18°N	77.4°E	: India's most southern city:
Rome, Italy	41.9°N	12.5°E	
Honolulu, Hawaii, USA	21.31°N	157.86°W	: near the middle of the Pacific Ocean
Easter Island	27.11°S	109.35°W	: in Pacific ocean, 2180 miles west of Caldera, Chili
Miami, Florida, USA	25.76°N	80.2°W	
Jerusalem, Israel	31.77°N	35.2°E	
Cairo, Egypt	30.06°N	31.25°E	
Perth, Australia	31.95°S	115.86°E	
Mexico City, Mexico	19.43°N	99.13°W	: capital city of Mexico
Brasilia, Brazil	15.79°S	47.88°W	: capital city of Brazil
Ushuaia, Chili	54.8°S	68.3°W	: In Patagonia, South America Region, Chili's most southern city. relatively close to Antarctica
Georgetown Island, or Ascension Island	7.93°S	14.41°W	: In middle of south Atlantic ocean, and between Angola, Africa and central Brazil. Northwest of Tristan Island
Tristan Island	37.11°S	12.28°W	: In middle of the south Atlantic ocean, and between Capetown, South Africa , and Southern Brazil
Mt. Everest, Nepal	27.98°N	86.93°E	: Highest mountain on Earth, in Himalayas Mts, north of India
Manila, Philippines	14.6°S	120.98°W	
Quito, Ecuador	0.18°S	78.47°W	: "ecuador" is a Spanish word for "equator"
Hithadhoo, Maldives	0.61°S	73.1°E	
Islamabad, Pakistan	33.74°N	73.08°E	
Montreal, Canada	45.51°N	73.59°W	
Quebec, Canada	46.81°N	71.21°W	
London, England	51.51°N	0.118°W	: very close to Greenwich, England
Dublin, Ireland	53.35°N	6.266°W	
Stockholm, Sweden	59.33°N	18.06°E	
Gibraltar, Spain	36.14°N	5.354W	: southern extended tip of Spain, a peninsula, a Britain Territory
Accra, Ghana, Africa	5.615°N	0.206°W	: in north-west Africa along the Atlantic ocean to the south

Time Of Year Date	Season	Days From March 21	Tilt
March 21 to June 21	"spring"	0 to 90	0° to +23.5°
June 21 to September 21	"summer"	90 to 180	+23.5 to 0°
September 21 to December 21	"fall"	180 to 270	0° to -23.5°
December 21 to March 21	"winter"	270 to 360	-23.5 to 0°

As was previously indicated, the tilt of the Earth throughout the year is nearly, or not exactly, sinusoidal. This is due to that the orbit of Earth around the Sun is not exactly circular and is rather slightly elliptical or "egg shaped". During the northern hemispheres winter season and a slightly elliptical orbit, Earth is actually slightly closer (about 3 million miles closer than the average of 93 million miles) to the Sun and the increased gravity is causing it to orbit the Sun slightly faster, and during the summer, Earth is slightly farther from the Sun and the lessened gravity is causing it to orbit the Sun slightly slower. This has the effect of changing the (apparent) tilt angle slightly faster (than a yearly average or rate) than expected in the winter, and slightly slower than (the average) expected in the summer. Note that the time of each Earth day or rotation about is axis is still the same time of about 24 hours, and rather the Suns position or (angle of) elevation in the sky is slightly different (more or less) than expected, and this should be considered for (more accurate) Sundial clocks so as to make a correction to the time physically displayed, as a shadow line, and so as to still use the standard, yearly, 24 hour time system. You can look into the topic of: "Equation Of Time" which is the understanding and calculation to account for this apparent "error" or difference from what (ie., average) was to be expected. At the extreme error points, the maximum difference between true solar noon where the Sun is at the highest angle (with respect to your local horizon) overhead and where it will be in the sky throughout the day, and localized (due to time zones, so everyone in a zone has the same time; hour and minute, to avoid confusion) noon time on the clock, will be "off" (in error) by about +- about 15 minutes. Since 1 hour or 60 minutes corresponds to an apparent 15 degrees of Sun movement, the maximum time error of 15 minutes will correspond to an apparent 3.75 degrees of maximum angle error from its commonly expected "average position".

Because of the apparent tilt of the Earth throughout the year, the locations of the sunrise and sunset along the horizon will change in the same (nearly) sinusoidal periodic manner and cycle. The locations of the moonrise and moonset will also be affected in a similar manner. The Moon also has a constant 5° (tilted, not always above Earth's equator) orbit about the Earth, and goes above and below the solar plane during its orbit around the Earth. This is a visible, total apparent change of 10°, and this is the apparent diameter of 20 moons since it appears seen as about 0.5 from an observer on Earth. The apparent tilt of the Earth throughout the year will also make the Moon's angle above the horizon to change. The total apparent tilt of the Earth throughout the year is about 23.5° + 23.5° = 47°.

As the Earth tilts throughout the year, the apparent location of the Sun up in the sky as seen at the same location on Earth and time every day will have a locus or path that has the shape of a "figure eight = 8" throughout the year. As the northern hemisphere of Earth tilts southward and toward the Sun in the summertime, the Sun will appear to travel and set in a slightly more northern direction and results in the Sun setting in a north-west direction along the horizon, and is actually 23.5° northward or clockwise from true west on about July 21. Likewise, in the winter, the Sun will set in the south-western direction along the horizon, and is actually 23.5° southward or counter-clockwise from true west on about December 21. This process and annual (yearly) apparent path of the Sun as seen at the same time of day, such as local noon or 12pm or 0pm on a 24hour time system, is known as the **Analemma** of the Sun. Because the orbit of Earth is closer to the Sun in the Winter and traveling faster, and then farther from the Sun in the Summer, one end of this figure eight (or somewhat sinusoidal or elliptical in shape) position or path up in the sky will be longer than the other.

Equation of a sine wave at particular frequency:

Since some of these concepts were also discussed in the previous topic, you may wish to review it so as to possibly have a greater understanding of these concepts.

For something that is spinning, such as a point on the circumference of a rotating wheel, or a periodic wave (such as a sine wave) that is repeated over and over, what is the basic formula for the height or amplitude (ie. max height) of that waveform?

SIN ¢ = $\sqrt{1 - \text{COS}^2 ¢}$

COS ¢ = $\sqrt{1 - \text{SIN}^2 ¢}$

And from TAN ¢ = $\dfrac{\text{SIN } ¢}{\text{COS } ¢}$: also COS ¢ = SIN ¢ / TAN ¢   and   SIN ¢ = COS ¢ TAN ¢

TAN ¢ = $\dfrac{\text{SIN } ¢}{\sqrt{1 - \text{SIN}^2 ¢}}$ = $\dfrac{\sqrt{1 - \text{COS}^2 ¢}}{\text{COS } ¢}$: calculating TAN ¢ using either the SIN or COS of the angle

Here is the trigonometric identity for cofunctions of complementary angles:

   Given an angle (¢< 90°) its complementary angle is (90° - ¢). The trigonometric cofunctions of these angles have the same value:

   SIN ¢ = COS (90° - ¢)   also: TAN ¢ = COTAN(90° - ¢)

   COS ¢ = SIN (90° - ¢)   Ex. COS 60° = SIN (90° - 60°) = SIN 30°

   TAN ¢ = $\dfrac{\text{SIN } ¢}{\text{COS } ¢}$ = $\dfrac{\text{SIN } ¢}{\text{SIN }(90° - ¢)}$: calculating TAN ¢ using the SIN values of two angles

Here is how to solve for either sin ¢ or cos ¢ using tan ¢.

From: 1 = sin^2¢ + cos^2¢   since tan ¢ = sin ¢/cos ¢, sin ¢ = cos ¢ tan ¢ :

            1 = cos^2 ¢ tan^2 ¢ + cos^2 ¢   factoring out cos^2 ¢ :

            1 = cos^2 ¢ (1 + tan^2 ¢)   solving for cos^2 ¢ :

            cos^2 ¢ = $\dfrac{1}{1 + \text{tan}^2 ¢}$ solving for cos ¢ by taking the square-root of both sides:

            cos ¢ = $\sqrt{\dfrac{1}{1 + \text{tan}^2 ¢}}$ = $\dfrac{1}{\sqrt{1 + \text{tan}^2 ¢}}$

   Since sin ¢ = tan ¢ cos ¢   and substituting the above expression for cos ¢ :

            sin ¢ = (tan ¢)   (cos ¢)

            sin ¢ = (tan ¢) $\left(\dfrac{1}{\sqrt{1 + \text{tan}^2 ¢}} \right)$

            sin ¢ = $\dfrac{\text{tan } ¢}{\sqrt{1 + \text{tan}^2 ¢}}$

Also,   (sin ¢) (cos ¢) = $\dfrac{\text{tan } ¢}{(1 + \text{tan}^2 ¢)^{0.5}}$ $\dfrac{1}{(1 + \text{tan}^2 ¢)^{0.5}}$ = $\dfrac{\text{tan}}{1 + \text{tan}^2 ¢}$

Here are some identities (ie., trigonometric equivalences) of the complementary angles (A and B) of a right triangle :

Ex. $(SIN\ 160°)^2 + (COS\ 20°)^2 = (SIN\ 20°)^2 + (COS\ 20°)^2 = 1$

This leads to a general conclusion that the COS^2 of an angle of a triangle, plus the SIN^2 of the sum of the other two angles (which sum to a supplementary angle of the third angle) of that triangle equals 1.

$SIN^2\ (\cancel{c}_1 + \cancel{c}_2) + COS^2\ \cancel{c}_3 = 1$: Trigonometric Identity Of All Three Angles Of A Triangle

Since $COS\ \cancel{c} = \sqrt{1 - SIN^2\ \cancel{c}}$, after substituting this into the above equation and simplifying:

$SIN^2\ (\cancel{c}_1 + \cancel{c}_2) = SIN^2\ \cancel{c}_3$ taking the square root of both sides and switching sides:

$SIN\ \cancel{c}_3 = SIN\ (\cancel{c}_1 + \cancel{c}_2)$: Trigonometric Identity Of All Three Angles Of A Triangle

Consider that since $\cancel{c}_3$ and the sum of $(\cancel{c}_1 + \cancel{c}_2)$ are supplementary angles which have the same trigonometric values.

Likewise:

$COS^2\ (\cancel{c}_1 + \cancel{c}_2) + SIN^2\ \cancel{c}_3 = 1$: Trigonometric Identity Of All Three Angles

$COS^2\ (\cancel{c}_1 + \cancel{c}_2) = COS^2\ \cancel{c}_3$ taking the square root of both sides:

$COS\ \cancel{c}_3 = COS\ (\cancel{c}_1 + \cancel{c}_2)$: Trigonometric Identity Of All Three Angles Of A Triangle
                                           (use absolute (positive sign) trigonometric values)

$TAN\ \cancel{c}_3 = \dfrac{SIN\ (\cancel{c}_1 + \cancel{c}_2)}{COS\ (\cancel{c}_1 + \cancel{c}_2)}$: Trigonometric Identity Of All Three Angles Of A Triangle,
                                           Angle $\cancel{c}_3$ is supplementary to $(\cancel{c}_1 + \cancel{c}_2)$.

Double And Half Angle Trigonometric Identities

       These trigonometric identities are commonly known today in the mathematics community and they were progressively derived during a span of about two-thousand years starting from the Babylonian, Egyptian, Greek, Indian, Asian, Arab, and European-Western and other mathematicians who took mathematics to new heights. The derivations are typically built upon the other more basic trigonometric identities mentioned in this book. These identities were commonly used to make trigonometric tables and still can be used if you prefer not to calculate them by other means such as the series methods. These equations presented here for are only for basic familiarity purposes if you happen to encounter them or other similar types of equations. These equations are not required for further use in this book and may be skipped over for now and viewed at another time.

$sin\ (2x) = 2\ sin\ x\ cos\ x = \dfrac{2\ tan\ x}{1 + tan^2\ x}$: sine of twice the angle , x = the angle

$cos\ (2x) = cos^2\ x - sin^2\ x = 2\ cos^2\ x - 1 = 1 - 2\ sin^2\ x = \dfrac{1 - tan^2\ x}{1 + tan^2\ x}$

       Note that, from: $2\ cos^2\ x - 1 = 1 - 2\ sin^2\ x$, we have the "Pythagorean-like trigonometric identity" :
       $sin^2\ x + cos^2\ x = 1 = (sin\ x)^2 + (cos\ x)^2$

$tan\ (2x) = \dfrac{2\ tan\ x}{1 - tan^2\ x}$ Note that from: $tan\ x = sin\ x\ /\ cos\ x$, therefore,
                                $tan\ (2x) = sin\ (2x)\ /\ cos\ (2x)$ and $tan\ (x/2) = sin\ (x/2)\ /\ cos\ (x/2)$

FINDING THE ANGLE GIVEN A TRIGONOMETRIC VALUE

The formula for the series expansion of ARCSIN X is given below and its derivation can be found in the appendix. This is one of the possible formulas to keep near a basic "home-use", 4 function calculator. There are similar formulas for ARCCOS X and ARCTAN X but they are not presented right here since it is easier to work with, and possibly remember only one formula. A series for ARCCOS X and ARCTAN X can be found in the appendix.

$$\text{ARC SIN } X = X + \sum_{n=1}^{n=\infty} \frac{(1)\ldots(2n-1)}{(2)\ldots(2n)} \cdot \frac{X^{(2n+1)}}{(2n+1)}$$

: ARC SIN Series expansion of the function SIN x. This expansion, and some others, were developed by **Colin Maclaurin** when he used a (Brook) Taylor series with the functions derivatives evaluated at x=0.

The angle calculated by this series is a radian angle. Notice that some initial factors (on the left) are indicated along with the common expression for those factors. This indicates that all the previously calculated factors of:

$\frac{(2n-1)}{(2n)}$ up to and including this terms value are to be also included in each new term.

n is an integer (regular counting number) from 1 to infinity (∞) for this series: 1, 2, 3, 4, 5, 6, 7, . . .

Expanding the first eight terms of the series:

$$\text{ARCSIN } X = X^1 + \frac{(1)X^3}{(2)(3)} + \frac{(1)(3)X^5}{(2)(4)(5)} + \frac{(1)(3)(5)X^7}{(2)(4)(6)(7)} +$$

$$+ \frac{(1)(3)(5)(7)X^9}{(2)(4)(6)(8)(9)} + \frac{(1)(3)(5)(7)(9)X^{11}}{(2)(4)(6)(8)(10)(11)} + \frac{(1)(3)(5)(7)(9)(11)X^{13}}{(2)(4)(6)(8)(10)(12)(13)} +$$

$$+ \frac{(1)(3)(5)(7)(9)(11)(13)X^{15}}{(2)(4)(6)(8)(10)(12)(14)(15)} + \frac{(1)(3)(5)(7)(9)(11)(13)(15)X^{17}}{(2)(4)(6)(8)(10)(12)(14)(16)(17)} + \ldots$$

Notice the pattern of which can be memorized as: The coefficients of X start with one in the numerator and increase by one alternately in every numerator and denominator until it equals the exponent of X that increases by 2 for each term.

As a memory aid, the general format for each term can be memorized as:

$\frac{\text{(products of odd series up to n-2)}}{\text{(products of even series up to n-1)}} \frac{X^n}{n}$: where the exponent of X is the odd series: 1, 3, 5, 7, 9, . . . You can view the topic of: Products Of First Terms Of The Odd and Even Series for some formulas that could be used to find and-or express these indicated products.

Simplifying most of the terms given above:

$$\text{ARCSIN } X = X + \frac{X^3}{6} + \frac{3X^5}{40} + \frac{15X^7}{336} + \frac{105X^9}{3,456} + \frac{945X^{11}}{42,240} + \frac{10,395X^{13}}{599,040} + \frac{135,135\, X^{15}}{9,676,800} + \ldots$$

The smaller the sine value (generally due to a smaller trigonometrically equivalent reference angle (<=90°)), the faster (ie. fewer terms needed to be evaluated) the sum of the terms converges to a given value since the value of each next term is

If you want to use all radian angles instead of degree angles:
Since 90° = 1.570796327 rads:

1.570796327 = A + B   : here, A and B are complementary angles with units of radians, that sum to 1.5708 radians
   mathematically:
A = 1.570796327 - B
B = 1.570796327 - A

Modifying the general formula so as to correspond to radian angles:

$\text{ARCSIN (SIN A)} = 1.570796327 - \text{ARCSIN} (\sqrt{1 - \text{SIN}^2 A})$: using radian angles

If given an angles COS or TAN value and you want to use the ARC SIN series, convert the given value to the angles corresponding SIN value.

Ex.   Given COS ¢ = 0.82 find the angle.

First convert the value given to the (same) angles corresponding SIN value.

$\text{SIN } ¢ = \sqrt{1 - \text{COS}^2 ¢}$

$\text{SIN } ¢ = \sqrt{1 - (0.82)^2}$

SIN ¢ = 0.57236352

Placing this SIN ¢ value into the ARC SIN expansion, we find that the angle is 34.91520625°.

Hence: ARCCOS 0.82 = ARCSIN 0.57236352 = 34.91520625°

Ex.   Given TAN ¢ = 1.732 find the angle.

First convert the value given to the corresponding SIN value of the (same) angle.

$\text{SIN } ¢ = \dfrac{\text{TAN } ¢}{\sqrt{1 + \text{TAN}^2 ¢}} = \dfrac{1.732}{\sqrt{1 + (1.732)^2}} = \dfrac{1.732}{1.999956} = 0.866019052$

Placing this SIN value into the ARCSIN series, we find that the angle is about 60°.

Hence: ARCTAN 1.732 = ARCSIN 0.866 = 60°

Here are some other trigonometric identity methods to evaluate ARCSIN X:

From: $\text{COS } ¢ = \sqrt{1 - \text{SIN}^2 ¢}$ and letting:

ARCSIN X = ¢ = ARCCOS (COS ¢)   : their trigonometric results is the same angle, therefore:

$\text{ARCSIN X} = \text{ARCCOS} \sqrt{1 - \text{SIN}^2 ¢}$ Since X = SIN ¢ :

$\text{ARCSIN X} = \text{ARCCOS} \sqrt{1 - X^2}$: X = SIN ¢

EPSILON (e)

The symbol (e) represents a constant that is an irrational value, like pi=3.14159...is, and is approximately 2.718281828. This value is used to represent the limiting (ie. maximum) factor value of constantly (ie. infinitely) compounded growth (growth with respect to the current size, or "growth upon (previous) growth - which includes any growth due to interest growth") over a (one) period of predetermined time. It is mathematically defined to be a very natural number and is frequently used when representing growth or decay (loss) processes in nature, therefore, the constant (e) is often found in many mathematical formulas. You could say that (e) and (pi) are natural constants. (e) is equivalent to the expression below, and a derivation of it is given in the appendix section of this book.

$$e = \left(\frac{1}{1} + \frac{1}{n}\right)^{\wedge}n = \text{about } 2.71828 \text{ as } n \to \infty$$

: (n -->∞) means: "as n approaches infinity".
 Remember, infinity has no specific value or end, and therefore, it cannot be mathematically represented, but only "approached", or approximated for practical purposes. In a way, infinity is an irrational value. Infinity is a concept and general description for a non-ending or unending process.   "As n approaches infinity" could be thought of as: "as (n) gets (unendingly) higher in value".
(e) is also called Euler's (pronounced as "Oiler's") Number.

Mathematically, combining the fractions:

$$e = \left(\frac{n+1}{n}\right)^{\wedge}n = \frac{(n+1)^{\wedge}n}{n^{\wedge}n} : \sim 2.71828 \text{ as } n \to \infty$$

As (n) approaches infinity, the entire base of this indicated power approaches: 1.0000....1 which is practically 1. It is as if the added 1 in the numerator becomes less and less meaningful as (n) increases, and it is a mathematical fact when you consider that 1 part of an increasing number of total parts, becomes less of a portion or fraction of the whole part or entire parts. The result will always be slightly over 1 as (n) becomes high n value:

For example, consider these fractions where (n) increases in value:

$$\frac{(n+1)}{n} , \frac{8}{7} = 1.142857143 , \frac{15}{14} \sim 1.071428571 , \frac{1500}{1499} = 1.000667111 , \frac{999999}{999998} = 1.000001$$

Also, using the mathematically equivalent form of the base value:

$$\left(\frac{1}{1} + \frac{1}{n}\right) , \left(\frac{1}{1} + \frac{1}{1499}\right) = 1 + 0.000667111 , \left(\frac{1}{1} + \frac{1}{999999}\right) = 1 + 0.000001 = 1.000001$$

Raising this value to a very high power (such as with an exponent that is high value of (n)):

$$(1.0000...1)^{\wedge}\infty = e \sim 2.71828...$$

Now it will be shown in a simple manner where this value of 2.71... , and not some other value, comes from.

If we let x = $\frac{1}{n}$, and substituting this value into the above power of a binomial, and expanding the indicated power:

$$\left(\frac{1}{1} + \frac{1}{n}\right)^{\wedge}n = (1 + x)^{\wedge}n$$

also notice that since x = 1/n, n = 1/x, hence: $(1+x)^{\wedge}n = (1+x)^{\wedge}(1/x) = {}^{x}\sqrt{1+x}$
:when n is high, as n ----> infinity,   x---->0,   and the result approaches 2.71828.... = e

A negative power, such as a negative power of e, of can be calculated by using the reciprocal of the corresponding positive power:

$b\text{^-}x = \dfrac{1}{b\text{^}x}$ If the base of the power is (e):

$e\text{^-}x = \dfrac{1}{e\text{^}x} =\sim \dfrac{1}{2.7182818\text{^}x}$

$e\text{^-}2 = (e\text{^-}1)\text{^}2 = \left(\dfrac{1}{2.7182818} \right)\text{^}2 = 0.367879441\text{^}2 = 0.135335283$

And generalizing the above, **we can solve for a negative power of a base, here (e), using the same indicated power (here, x) of the reciprocal of that base.**

Since $(1/e) = (1/2.718281828) = 0.367879441 = e\text{^-}1$ and raising each side to the (x) power, and switching sides:

$(e\text{^-}1)\text{^}x = 0.367879441\text{^}x$ after using the power to a power rule on the left side:

$e\text{^-}x = 0.367879441\text{^}x$ and   **$e\text{^}x = 2.718281828\text{^}x$**

$b\text{^-}x = (1/b)\text{^}x$ and-or   **$b\text{^-}x = \dfrac{1}{b\text{^}x} = (b\text{^-}1)\text{^}x = (1/b)\text{^}x$**   **: A negative power of a base is equal to the same positive power of the reciprocal of that base.**

As discussed previously in this book, (e) is the base for natural logarithms. Fortunately, a series for calculating powers of (e) has been developed. It will provide great computational use. This series for (e) is shown next, and its derivation is shown in the appendix.

Ex. e^3.74 = e^(2 + 1.74)
     e^3.74 = (e^2)(e^1.74)
     e^3.74 = (e^2)(e^1)(e^0.74)
     e^3.74 = (7.389056098)(2.718281828)(2.095935514)
     e^3.74 = 42.09799016

Ex. e^-1.47 = (e^-2)(e^0.53)   : from -2 + x = -1.47 , therefore x = 2 + (-1.47) = 2 - 1.47 = 0.53
     e^-1.47 = (0.135335283)(1.698932309)
     e^-1.47 = 1/e^1.47 = 0.229925485**...**

Ex. Evaluate e^21

N = e^21 = (e^20)(e^1)
N = (e^10)^2 (e^1)
N = (22,026.46579)^2 (2.718281828)
N = (485165195.4)(2.718281828)   : Using a 10-digit calculator
N = 1,318,815,734

Ex. Evaluate 5^2.3 using a power of (e). To do this, lets first convert the base of 5, to its equivalent power of (e):

   Let 5 = (e^x)   raising each side to the 2.3 power: 5^2.3 = (e^x)^2.3

   Given: 5 = e^x   let's solve for the necessary and proper exponent of (e) by taking the logarithm of both sides:

   ln 5 = ln e^x   using the log of an indicated power rule, and this can be expressed as:
   ln 5 = x ln e   dividing both sides by ln e:

   x = ln 5   = ln 5 = 1.609437912   therefore:
          ln e

   5^2.3 = (e^1.609437912)^2.3   using the power to a power rule; multiply the exponents:

   5^2.3 = e^3.701707199 = ~ 40.5164

   For any power of the same base, here 5, the expression to evaluate would then have the form of:

   5^x = (e^1.609437912)^x =   e^(1.609437912 x)   : using a power of (e) to evaluate any indicated power of 5.

Using the above expressions, a general formula for evaluating any indicated power of a value by using a power of (e), is:

   First we would arrive at:

   b^x = (e^(ln b))^x   which can be expressed using the power to a power rule as:

   b^x = e^(x ln b)   : **A GENERAL FORMULA FOR EVALUATING ANY INDICATED**
                                                   POWER OF A NUMBER WITH A POWER OF (e).
                                                   Since indicated roots can be expressed as an indicated power
                                                   of a number, this same formula can be used to solve for roots.
                                                   Consider this derivation also:

   b^a = e^x   ,   therefore: ln b^a = ln e^x   therefore:
   a ln b = x ln e ,   therefore: a ln b = x   therefore, substituting this value of (x) in e^x, we have:

Now, rather than perform repeated multiplication of the above constants, you can make a small, but powerful table of all the integer powers, from 1 to 9, of these constants .   For example:

x	e^x	e^0.x	e^0.0x	e^0.00x	e^0.000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1	2.718281828459045	1.105170918075648	1.010050167084168	1.001000500166708	1.000100005000167
2	7.38905609893065	1.22140275816017	1.020201340026756	1.002002001334	1.000200020001334
3	20.08553692318767	1.349858807576003	1.030454533953517	1.003004504503377	1.0003000450045
4	54.59815003314424	1.49182469764127	1.040810774192388	1.004008010677342	1.000400080010668
5	148.4131591025766	1.648721270700128	1.051271096376024	1.005012520859401	1.000500125020836
6	403.4287934927351	1.822118800390509	1.06183654654536	1.006018036054065	1.000600180036005
7	1096.633158428459	2.013752707470477	1.072508181254217	1.007024557266849	1.000700245057177
8	2980.957987041728	2.225540928492468	1.083287067674959	1.008032085504274	1.00080032008535
9	8103.083927575384	2.45960311115695	1.09417428370521	1.009040621773868	1.000900405121527

x	e^0.0000x	e^0.00000x	e^0.000000x	e^0.0000000x	e^0.00000000x
0	1.000000000000000	1.0000000000000	1.000000000000000	1.000000000000000	1.000000000
1	1.00001000005	1.0000010000005	1.000000100000005	1.00000001	1.000000001
2	1.000020000200001	1.000002000002	1.00000020000002	1.00000002	1.000000002
3	1.000030000450005	1.0000030000045	1.000000300000045	1.000000030000001	1.000000003
4	1.000040000800011	1.000004000008	1.00000040000008	1.000000040000001	1.000000004
5	1.000050001250021	1.0000050000125	1.000000500000125	1.000000050000001	1.000000005
6	1.00006000180036	1.000006000018	1.00000060000018	1.000000060000002	1.000000006
7	1.000070002450057	1.0000070000245	1.000000700000245	1.000000070000003	1.000000007
8	1.00008000320085	1.000008000032	1.00000080000032	1.000000080000003	1.000000008
9	1.000090004050122	1.0000090000405	1.000000900000405	1.000000090000004	1.000000009

e^0.35 = e^0.3   e^0.05   : e^x factored to indicated powers that have one digit greater than 0.
                                             For using the table, this can be expressed as:
e^0.35 = e^((0.1)3)   e^((0.01)5)

In the table above, use:

     (e^0.x) where x = 0.1 , and choose the corresponding row 3 which is the 3rd multiple of e^0.1
     (e^0.0x) where x = 0.01, and choose the corresponding row 5 which is the 5th multiple of e^0.01

e^0.35   =   (1.349858808)(1.051271096)
e^0.35   =   1.419067549

If the indicated power of (e) is negative, the same method can be used if you place it into a form with a positive exponent and then take its reciprocal:

e^ - 0.35   =   ___1___   =   ____1_____   =   0.704688089
                     e^0.35   1.419067549

By using a reverse type of manner to that described above you can solve for the (indicated exponent) power of (e) that will be equal to the power value presented. In essence, you will be solving for a logarithm (an exponent), and here, with this table with powers of (e), it will be a natural logarithm. To find the next digit of the logarithm, divide the remainder (ie.

Hence x is approximately 1.609 , and e^1.609 = about 5   : note also that 1.609 = ln 5 = log $_e$ 5

The table of powers of (e) can also be used to find the natural logarithm (base e) of a number.

The above concepts are powerful, and can even be applied to powers of any other base, for example, powers of ten =
= 10^x as shown here:

Evaluating Powers Of 10 Using A Small Table Of Constants

Ex. 10^0.752 = 10^(0.7   +   0.05   +   0.002   ) =
                   = (10^0.7)   (10^0.05)   (10^0.002) ,   Converting these exponents to integers we have:
                   = (10^0.1)^7 (10^0.01)^5 (10^0.001)^2

10^0.1   = 1.258925411794
10^0.01   = 1.023292992281
10^0.001   = 1.002305238078
10^0.0001 = 1.000230285021

Again, rather than perform repeated multiplication, a small table is presented here:

x	10^x	10^0.x	10^0.0x	10^0.00x	10^0.000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.00000000000000	1.000000000000000
1	10.00000000000000	1.258925411794167	1.023292992280754	1.0023052380779	1.000230285020825
2	100.0000000000000	1.584893192461114	1.0471285480509	1.004615790278395	1.00046062307284
3	1000.000000000000	1.99526231496888	1.071519305237606	1.006931668851804	1.000691014168259
4	10000.00000000000	2.51188643150958	1.096478196143185	1.009252886076685	1.000921458319296
5	100000.0000000000	3.162277660168379	1.122018454301963	1.011579454259899	1.001151955538169
6	1000000.000000000	3.981071705534972	1.148153621496883	1.01391138573668	1.001382505837099
7	10000000.00000000	5.011872336272722	1.17489755493953	1.016248692870696	1.001613109228309
8	100000000.0000000	6.309573444801933	1.202264434617413	1.018591388054117	1.001843765724026
9	1000000000.000000	7.943282347242816	1.230268770812382	1.02093948370768	1.002074475336479

x	10^0.0000x	10^0.00000x	10^0.000000x	10^0.0000000x	10^0.00000000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.00000000000000	1.000000000000000
1	1.000023026116027	1.000002302587744	1.000000230258536	1.000000023025851	1.000000002302585
2	1.000046052762256	1.00000460518079	1.000000460517125	1.000000046051703	1.00000000460517
3	1.000069079938699	1.000006907779138	1.000000690775767	1.000000069077555	1.000000006907755
4	1.000092107645368	1.000009210382787	1.000000921034461	1.000000092103408	1.000000009210341
5	1.000115135882277	1.000011512991739	1.000001151293209	1.000000115129261	1.000000011512926
6	1.000138164649436	1.000013815605993	1.00000138155201	1.000000138155115	1.000000013815511
7	1.000161193946858	1.000016118225548	1.000001611810864	1.00000016118097	1.000000016118096
8	1.000184223774555	1.000018420850406	1.000001842069771	1.000000184206825	1.000000018420681
9	1.00020725413254	1.000020723480565	1.000002072328731	1.00000020723268	1.000000020723266

Ex. 10^1.79 = (10^1) (10^0.7)   (10^0.09)
      10^1.79 = (10) (5.011872336) (1.230268771)
      10^1.79 = 61.65950019

Ex, 10^-3.7 = 10^(-4 + 0.3) =
      10^-3.7 = (10^0.3)(10^-4)
      10^-3.7 = (1.995262315)(10^-4)   By observation; moving the decimal point 4 places leftward:

NATURAL LOGARITHMS

Natural logarithms are logarithms where the base is the ("natural") constant of (e) = 2.71828... , and which has been discussed previously in this book. Several series for natural logarithms have been developed and most are similar, and they are of great and necessary computational aid, especially before the age of electronic calculators. The following discussion about logarithms is fairly extensive, and the reader may skip over most of the material till some other time as long as they have a basic understanding of what a logarithm is.

Here is a simple explanation of what a logarithm, and specifically what a natural logarithm means:

N = e^x   : e^x will calculate the growth (numerical) factor that a value will effectively have after continuously (with infinitely compounded growth) growing after x periods of time, where after each 1 period of time, it grows by a factor of e^1 = about 2.718 times more.

With x=2, e^2 = (e^1)(e^1)   , a value will grow by a factor of (e^1) twice,   There will be two periods of natural or infinitely compounded growth. After the first period it will grow by a factor of (e^1) and produce a resulting growth, say growth1, and then after another period of this type of growth, the growth1 value will also grow by a factor of (e^1) resulting in the value of growth2.   The initial value will grow or increase by the factor of: (e^1)(e^1) = e^2 times more.

Time: 0   ,   period 1 of growth   ,   period 2 of growth
Value: value   ,   value (e^1) = value2   ,   value2(e^1) = value3   by substitution:

value3 = value2(e^1) = (value1(e^1))(e^1) = value1(e^2)

After 2 periods of infinitely compounded growth, value 1 will grow by a factor of (e^2). In general, after x periods of infinitely compounded growth, an initial starting value will grow by a factor of (e^x).

ending value = initial value (e^x)   : after x periods of infinitely compounded growth

x = Ln (N)   : will solve for how many periods of time (ie. the exponent x, in e^x) it will take a value to grow continuously (with infinitely compounded growth) so as to be equal to the value of N.

If you know the starting value of growth (c), it will grow to N after x periods of infinitely compounded growth:

N = c e^x   : after x periods of growth.   If you divide both sides by (c) we can still find the number of time periods of infinitely compounded growth for (c) so as to be equal to N. Note, (c) can equal 1 also, and then the equation is simply: N = 1 e^x = e^x, but you you understand that 1 is now the starting value that the natural growth will be applied to.

$\dfrac{N}{c}$ = e^x = (1)e^x   solving for x, by taking the natural logarithm (ln) of both sides:

ln (N/c) = ln e^x = x ln e = x (1) = x
                                        e
x = ln (N/c)

Ex.   36.945 = 5e^x

We see that 5 is the initial value that will undergo continuous or natural growth, and that this value of 5 has, or will, grow to 36.954 after some time (ie., some number of time periods) of growth.   We can find the number of time periods of growth by solving for x:

ln 0.7 = ln (1 - 0.3)   use x = 0.7 - 1 = - 0.3
ln 1.3 = ln (1 + 0.3)   use x = 1.3 - 1 = + 0.3   : note that this has the same absolute value as when N=0.7

Here are the corresponding values of (x) for ln 0.2 and ln 1.8:

ln 0.2 = ln (1 - 0.8)   : x = 0.2 - 1 = - 0.8
ln 1.8 = ln (1 + 0,8)   : x = 1.8 - 1 = 0.8   ; the same in absolute value as above

By observing the two results above, the closer the value of N is to 1, as the first instance is, the smaller the absolute value of (x). When N is less than 1, (x) will be negative and all terms of the series, and the resulting sum, will be negative in sign. Logarithms of values between 0 and 1 are negative in value.

The natural logarithm of 2 can be useful when calculating some natural logarithms by first factoring the number and taking the sum of their logarithms so as to find the actual logarithm.   Other methods will be presented below for calculating any natural logarithm.

ln 2 = 0.6931471805599451. . .

Ex. ln 5 = ln $\left(\dfrac{10}{2}\right)$ = ln 10 - ln 2 = 2.302585093 - 0.693147181 = 1.609437912

Here's another example that shows what to do when the value (ie.,N, the number of the logarithm) is greater than or equal to 2 when trying to find its logarithm with the series:

Ex.   Evaluate ln 21

   ln 21 = ln (10^2)(0.21)   : expressing the number (here 21) in scientific notation
   ln 21 = ln (10^2) + ln (0.21)
   ln 21 = 2 ln 10 + ln 0.21
   ln 21 = 2 (2.302585093) + ln 0.21   : the important point to remember (or at
                                                                     least have available) here is that (ln 10)
                                                                     is always equal to the constant of
                                                                     2.302585092994..., and that the natural
                                                                     logarithm to be found is in the required
                                                                     range (0<N<2) for the logarithm series
                                                                     formula given.

   ln 21 = 4.605170186 + (-1.560647748)
   ln 21 = 4.605170186 - 1.560647748
   ln 21 = 3.044522438

Or if you want slightly better accuracy using the same number of terms, you can use a different logarithm factor that is a higher power of 10, for example, here is a small reference table that is very useful:

ln 10 = 2.302585092994)
ln 100 = ln 10^2 = 2 ln 10 = 2 (2.302585092994) = 4.605170185988
ln 1,000 = ln 10^3 = 3 ln 10 = 3 (2.302585092994) = 6.907755278982
ln 10,000 = ln 10^4 = 4 ln 10 = 9.210340371976
ln 100,000 = ln 10^5 = 5 ln 10 = 11.51292546497
ln 1,000,000 = ln 10^6 = 6 ln 10 = 13.81551055796

Ex. Using the last example: ln 21

Ex. ln 9

ln 9 = ln (10 - 1)   : since we already know ln 10. Factoring 10 from each term:

ln 9 = ln 10 (1 - $\frac{1}{10}$)

ln 9 = ln 10 (1 - 0.1)

ln 9 = ln 10 (0.9)   using the log product rule:

ln 9 = ln 10 + ln 0.9   : Note, since ln 10 is obviously greater than ln 9, there must be
                                            something to reduce this, and this value is the negative value of
                                            logarithms of values less than 1, such as ln 0.9 shown.

A perhaps simpler method to get the same results as above is to factor the number by the highest positional product (ie., the weight of that digit position, times the digit value) of that number. For example:

ln 537

$\frac{537}{500}$ = 1.074   , hence: 537 = (500)(1.074)   using substitution:

ln 537
ln (500)(1.074)
ln (5)(100)(1.074)
ln 5 + ln 100 + ln 1.074

If 5 is factored, you would have to evaluate the logarithm of another number, hence with this type of factorization method, it is handy to have the logarithms of the first 9 integers available:

ln 1 = 0
ln 2 = 0.693147180559
ln 3 = 1.098612288668
ln 4 = 1.386294361119
ln 5 = 1.609437912434
ln 6 = 1.791759469228
ln 7 = 1.945910149055
ln 8 = 2.079441541679
ln 9 = 2.197224577336

ln 10 = 2.302585092994
ln 100 = 4.6051701859881   : same as ln 10^2 = 2 ln 10
ln 1,000 = 6.907755278982   : same as ln 10^3 = 3 ln 10
ln 10,000 = 9.210340371976   : same as ln 10^4 = 4 ln 10

Here is another method to evaluate logarithms:

ln $(\frac{1}{x})$ =   ln $(\frac{1 x^0}{x^1})$ =   ln $(\frac{1 x^{0-1}}{1})$ =   ln 1x^-1   =   -1 ln x

N = e^((ln 5) / 2.3)
N = e^(1.609437912 / 2.3)
N = e^0.699755614
N = 2.013260635

Ex. Show that N^x = e^(x ln N)   : a good way to calculate powers of N using the series for powers of e.

     Taking the log of both sides:

     ln N^x   =   ln e^(x ln N)   Using the log rules:

      x ln N   = (x ln N) ln e   since ln e = 1:

      x ln N = x ln N   : since the results are the same, the values were the same and therefore equal

Ex. Given 10^x = e^1   solve for x.

     Taking the natural logarithm of both sides:

      ln e^1   =   ln 10^x   using the log rules:
      1 ln e   =   x ln 10
          1 = x ln 10   solving for x and switching sides:

         x = ____1____   : = about 0.4342944819
                 ln 10   : ln 10 = about 2.302585093

Ex. Given 10^x = e^2   solve for x.

     Taking the natural logarithm of both sides:

      ln e^2   =   ln 10^x   using the log rules:

      2 ln e   = x ln 10

        2 =   x ln 10   solving for x and switching sides:

        x = ___2___   : = approximately (2) (0.4342944819) = 0.8685889638
                ln 10   Or: two times the reciprocal of ln 10

Ex. Given   10^3 = e^x   solve for x:

     Taking the natural logarithm of both sides:

      ln e^x = ln 10^3
      x ln e = 3 ln 10
         x = 3 ln 10   : about 3 (2.302585093) = 6.90775527898
                                        Or: three times (ln 10)

CALCULATING LOGARITHMS USING A SMALL TABLE OF CONSTANTS

In general, this is the reverse process of finding powers of (e) using a table of constants as shown previously in this book.

Ex. Find   x = ln 3.2

The general working method, or formula in relation to the table (shown previously) is:

(Note, (x) below represents a place holder or digit position in the number, and is not a regular variable):

ln N   =   ln(e^x.xxxx...)   : or perhaps for more clarity as: ln (e^a.bcde...)
ln N   =   ln(e^x.0 e^0.x e^0.0x e^0.00x e^0.000x . . .)
ln N   =   ln(e^x.0)   +   ln(e^0.x)   +   ln(e^0.0x)   +   ln(e^0.00x)   +   ln(e^0.000x)   +   . . .
ln N   =   x   +   0.x   +   0.0x   +   0.00x   +   0.000x   +   . . .

Here is a helpful description of this process:

ln N = ln N   factoring N:
ln N = ln (factor1)(factor2)   factoring factor2 into factor3 and factor4:
ln N = ln (factor1)(factor3)(factor4)   factoring factor4 into factor5 and factor6:
ln N = ln (factor1)(factor3)(factor5)(factor6)   : and so on, the factorization can continue with the "remaining factor"

ln N = ln (factor1)(factor3)(factor5)(factor6) = ln factor1 + ln factor2 + ln factor3 + ln factor4 + ln factor5 + . . .

Using the table, when x=2 , e^x = e^2 and this value is larger than the given value of 3.2 , so therefore, x=1 is where to start the factorization process:

$\frac{3.2}{e^1}$ = $\frac{3.2}{2.718281828}$ =   1.177214212   : factoring 3.2, here by using a power of (e)
                                                                    Therefore:

ln 3.2   =   ln ((2.718281828)(1.177214212))   : ln N = ln (factors of N)
ln 3.2   =   ln (2.718281828) + ln (1.177214212)
ln 3.2   =   ln (e^1) + ln(e^0.?)   : now we are to find the value of the exponent in e^0.x

From the table for the e^0.x constants, e^0.2 is larger than the new value of "N" we are considering, specifically the remaining factor of: (1.177214212) that we are now comparing powers of (e) to, so use e^0.1 to factor this new "N" factor in question:

$\frac{1.177214212}{e^0.1}$ = $\frac{1.177214212}{1.105170918}$ =   1.065187468

Now we have these factors of N=3.2::

ln 3.2   =   ln (2.718281828) + ln (1.177214212)   continuing the factorization process:
ln 3.2   =   ln (2.718281828) + ln (1.105170918) + ln(1.065187468)   expressing these factors as a power of (e):
ln 3.2   =   ln(e^1)   +   ln(e^0.1)   +   ln(e^0.0?)   A ln results in the indicated power (ie. exponent):
ln 3.2   =   1.0   +   0.1   +   ?

From the table, the greatest value of (x) for e^0.0x is 6 for the new "N" (1.065187468), hence the updated (more correct) logarithm is:

Here is an example of using the above table to calculate negative powers of (e):

Ex. Evaluate e^-5.03

      e^-5.03 = e^(-5 (+) -0.0 (+) -0.03) = e^-5 e^-0.0 e^-0.03
      e^-5.03 = (0.006737946)(1.0)(0.970445533)
      e^-5.03 = 0.006538809

You could always calculate the result using the table for positive powers of (e) when (e) is first expressed with a positive exponent as:

    e^-5.03 = $\dfrac{1}{e^{5.03}}$ = (value) ,   and since (value) is the reciprocal of the actual result, we simply take
                                              the reciprocal of (value):

    e^-5.03 = $\dfrac{1}{(value)}$ = $\dfrac{1}{152.933}$ = 0.006538811

N = b^x = e^(x ln b)   **: A GENERAL FORMULA TO EVALUATE ANY INDICATED POWER**
                                 USING A POWER OF (e)

Ex. If the base of the power is: b=10:

N = 10^x = e^(x ln 10) = x e^((x)(2.302585093)) = e^(2.302585093 x)   : as just shown above

Using almost the same derivation as above, a formula can be derived for evaluating powers of any number via powers of 10. Evaluated powers of 10 are often found in common antilogarithm tables:

N = b^x   taking the (common) logarithm of both sides:
log N = log b^x   : USING base 10 so as the result is a power of 10
log N = x log b   hence by the definition of logarithms:
N = 10^(x log b)   which can be expressed as:
N = b^x = 10^(x log b)   :A formula to evaluate any power via a power of 10.

Ex. 30^2 = 10^(2 log 30) = 10^((2)(1.4771<u>2</u>)) = 10^2.9542<u>4</u> = 900   : for comparison, 10^3 = 1000

If the base of an indicated power is (e) and you want to calculate or express that power value using powers of 10, and using the formula above:

N = b^x = e^x = 10^(x log e) = 10^(0.4342944819033 x)   : POWERS OF (e) VIA POWERS OF 10

Ex. e^1 = 10^(0.4342944819033 (1)) = 10^(0.4342944819033) = 2.718281828**...**

Here is a more general or algebraic method for calculating any power using any base:

  N = b^x   taking the log (using any base) of both sides:
  Log N = Log b^x   : this can be expressed as:
  Log N = x Log b   from the definition of logarithms:

  N = b^x = logbase^(x log   b)   **: A GENERAL FORMULA FOR CALCULATING ANY POWER**
                       logbase   **BY USING A DIFFERENT BASE.**
                                      b = base of the given power, x = exponent or indicated power,
                                      use a common logbase for calculations, N=power value

  Ex. N=5^2 , calculate this power using a power of (e) and then a power of (10):

     N = 5^2 = e^(2 ln 5)   : here, base (e) was used as expected for a natural logarithm
               e

     Or by using any other base for the logarithms, such as base 10:

     N = 5^2 = 10^(2 log 5)   : e^(2 ln 5) = 10^(2 log 5)   : both equal b^x = 5^2
                 10   e   10

     N = 25

For example: e^(ln 7)   =   e^1.945910149   =   7
                         e

Hence: 7^x   =   (e^ln 7)^x   =   e^(x ln 7)   : using the power to a power rule of multiplying exponents

The same can also be said whenever the base of the power and the base of the logarithm are the same:

   N2   =   log N1   : Choose any base (b) for the logarithm. By the log definition:

   b^N2   =   b^(logb N1)   =   N1

   For example: 10^log 100   =   10^2   =   100   : same as first taking a log on N, and then taking the antilog
                                                                     of that exponent that equals the log of N.

CALCULATING ROOTS USING (e)

        When solving for a root of a value using the constant (e), first place the radical into its equivalent exponential form (ie. an indicated power form), and then equate it to a power of (e).

Ex. $\sqrt{10}$ =   10^(1/2)   =   10^0.5

from: 10^x   =   e^(2.302585093 x)
         10^0.5   =   e^(2.302585093(0.5))
         10^0.5   =   e^1.151292546
         10^0.5   =   3.16227766

          $\sqrt{10}$ =   3.16227766   : the square root of a value calculated by its equivalent power value of (e)

Ex. $3\sqrt{7}$

ln 7 = 1.945910149   , hence   e^1.945910149 = 7,   therefore, raising each side to the same indicated power:

7^x   =   (e^1.945910149)^x   =   e^(1.945910149x)   : any power of 7 converted to a power of (e).

$3\sqrt{7}$ =   7^(1/3)   =   7^0.333333333...

7^0.333333333   = $3\sqrt{7}$ =   e^(1.945910149(0.333333333))
                    $3\sqrt{7}$ =   e^0.648636715
                    $3\sqrt{7}$ =   1.912931181

Here is another general formula for calculating square-roots using (e):

Find $\sqrt{N}$

Let $\sqrt{N}$ = e^x   : converting the radical to its equivalent exponential form:

N^0.5 = e^x   taking the natural logarithm of both sides of the equation:

LN N^0.5   =   LN e^x   using the log exponential rule:

A generalization of the above, using any base (b), can be expressed as:

let N = (factor1)(factor2)   : or as many factors used,

N = b^(logb ((factor1)(factor2))) = b^ ((logb factor1) + b^(logb factor2)) = b^(logb factor1) b^(logb factor2)

SOLVING A LOGARITHM USING ANOTHER BASE

Since a series for logarithms which have a base other than (e) have not been developed, the natural logarithm of the same number can be found and then adjusted so that it equals the actual logarithm in question by using the constant of proportionality (c) between the two logarithms. This constant between two log systems is often called a modulus.

Ex. What is the constant of proportionality between common logarithms (base 10) and natural logarithms (base e) ?

First choose a value for N. A value of 10 was chosen for N in the example given below since the common logarithm of 10 is easily found to be 1. Then divide the two logarithms to find the quotient which is the constant of proportionality (ratio, factor) between the systems (ie., with a different log base) of logarithms in question.

$$\frac{\log N}{\ln N} = c1 \qquad or \qquad \frac{\ln N}{\log N} = c2$$

Notice that c1 and c2 are algebraically reciprocals (c1 = 1/c2 and c2 = 1/c1), and that log N or ln are logarithms having a different base which can be found from either equation:

$$\log N = c1 \ln N = \frac{\ln N}{c2} \qquad and \qquad \ln N = c2 \log N = \frac{\log N}{c1}$$

$$\frac{\log 10}{\ln 10} = \frac{1}{2.302585093} = 0.4342944819033 = c1 = (1/c2)$$

$$\frac{\ln 10}{\log 10} = \frac{2.302585093}{1} = 2.302585092994 = c2 = (1/c1)$$

Expressing the above common and natural logarithms in a general or basic terms for any value:

log N = (0.4342944819033) ln N   : SOLVING COMMON LOGS USING NATURAL LOGS

ln N = (2.302585092994) log N   : SOLVING NATURAL LOGS USING COMMON LOGS

Both of the above equations have a linear equation form of : y = m x. A discussion and graph of this will be given at the end of this current discussion.

Ex. Find log 15 using the natural log base (e).

We already found that the constant of proportionality when evaluating a common log with a natural log is always: (log N / ln N) = (0.4342944819033) :

Given: $N = b2^x$ by the definition of logarithms:

    $x = \log_{b2} N$ taking logs of both sides of the above equation using another base (b1):

$\log_{b1} N = \log_{b1} b2^x$ using the log power rule:

$\log_{b1} N = x \log_{b1} b2$ since $x = \log_{b2} N$:

$\log_{b1} N = \log_{b1} b2 \; \log_{b2} N$: **A GENERAL FORMULA OF EVALUATING LOGS WITH A DIFFERENT BASE**
                               Mathematically:

$$\frac{\log_{b1} N}{\log_{b2} N} = \log_{b1} b2 = r$$

: (constant) ratio of logarithms of N using different bases
Here, the second equivalence can be thought of as:
$$\log_{b1} b2 = \log_{b1} \text{new base} \atop \text{old base}$$

Since b1 and b2 are both constants, $\log_{b1} b2$ can only be a constant, and the ratio above is a constant value.

When a base of a log is set to a lower value, the value of this log expression will naturally be higher, therefore in order to equal the original log of the number, this higher value must be reduced. It gets reduced to a fraction of itself by multiplying it by a value less than 1 (ie., a fraction), and this will be the value of the constant.

Ex. Evaluate $\log_{10} 20$ using natural logs.   Since base (e) is a lower value than base 10, the result will be a higher value than it would be using a base of 10. This higher value will be reduced by the constant known between the two bases. Using the above general formula:

$\log_{10} 20 = \log_{10} e \; \ln 20$

$\log 20 = 0.434294481 \, (2.995732274)$
$\log 20 = 1.301029996$

Using the above concepts we can also derive another formula for equating powers:

Given: $b1^x = N = b2^y$ Solving for y: By the definition of logarithms:

    $x = \log_{b1} N$ and $y = \log_{b2} N$ from the concepts discussed above:

      $\log_{b1} N = \log_{b1} b2 \; \log_{b2} N$ using algebraic substitution:

        $x = \log_{b1} b2 \; y$ solving for y:

$$y = \frac{x}{\log_{b1} b2} = \frac{x}{r}$$

equating this to the previous derivation given in the GENERAL FORMULA FOR EQUATING POWERS**:**
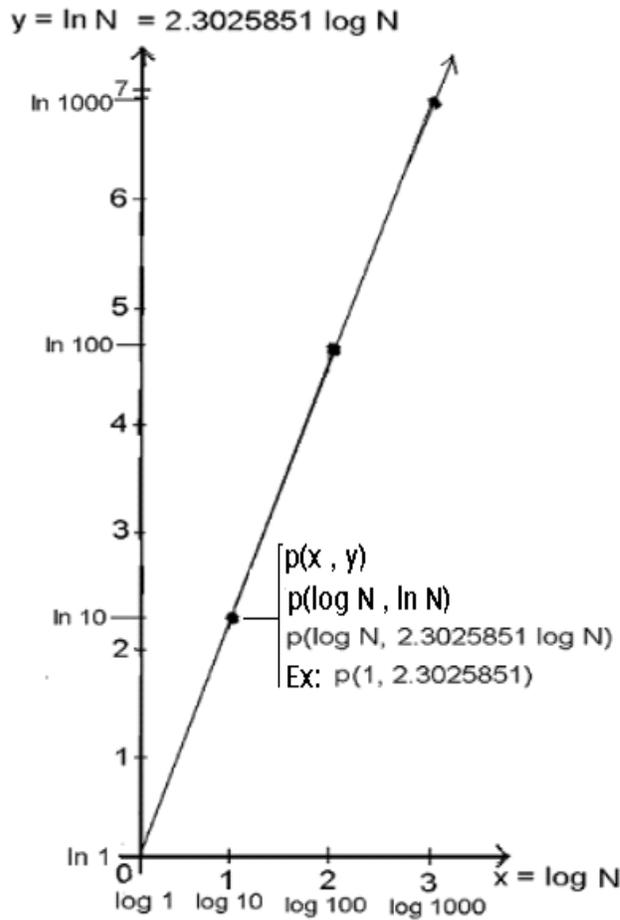
$y = \ln N = 2.3025851 \log N$

$$\ln N = (2.302585092994) \log N$$: The relationship between natural and common logarithms (or any other two logarithms having different bases) of N is a linear or constant mathematical relationship. You can also write an equation and make a graph of their mathematical relationships and the main difference in the graphs of the lines would be the value or steepness or slope of the line.

This has the form of a linear equation:

$y = m\ x$: where $y = mx = \ln N = (2.302585092994) \log N$, where $m = 2.302585092994$ and $x = \log N$

The mathematical relationship between the log of a power of a value, and the indicated exponent is linear if N, the base of the indicated power, is a constant value:

$y = \log N^x = x \log N = \log N\ x$: has the general linear equation form of: $y = mx$
We see that (y) is simply a multiple of the log of the base of the indicated power, and this multiple or factor is the value of (x).

Before moving on, here is a related thing to know about logarithms. It is somewhat surprising, and It may help fill in some gaps when trying to understand logarithms to the fullest.

If b1^x = b2, then b1 = x√ b2 = b2^(1/x) = b2^y : after assigning variable (y_ as being equal (1/x)

Ex. b1^5 = b2 then b2^(1/5) = 5√b2

Ex. 100^0.5 = 2√100 : 0.5 and 2 are reciprocals of each other: 0.5 = 1/2 and 2 = 1/0.5

 10 = 10 : checks

 b1^x = b2 and b2^y = b1 : this condition is true when x and y are reciprocals

 100^0.5 = 10 10^(1/0.5)=10^2 = 100

 Here is the mathematical relationship between b1 and b2 expressed algebraically:

From: b1^x = b2 and b2^y = b1

Dividing both sides of the first equation by the same or equivalent value of b1:

 $\frac{b1^x}{b1^1}$ = $\frac{b2^1}{b2^y}$ by using the exponential rules:

 b1^(x-1) = $\frac{1}{b2^{(y-1)}}$ or=:

 b1^(x-1) = b2^(0 - (y-1)) = b2^(1-y) : The mathematical relationship between b1 and b2. As shown above, the relationship between (x) and (y) is that they are reciprocals and where: x=log b2 and y=log b1
  b1 b2

Ex. Given log 10 = 2.302585093 = x
  e

 We now know that the reciprocal of the above expression is:

 log e = 0.434294481 = y checking: 2.302585093 = 1 / 0.434294481 , or= x = (1/y)
  10

 If we let b1=e and b2=10 their mathematical relationship is:

 b1^(x-1) = b2^(1-y)

 e^(2.3026 - 1) ~ 10^(1 - 0.4343)
  e^1.3026 ~ 10^0.5657
  3.6788 = 3.6788 : checks

Ex. This is a basic example: **1.4142136 = 2^x** , by the log definition: x = log2 1.4142136 = log 1.4142136
  2

 Taking logs of both sides: log 1.4142136 = log 2^x = x log 2 , therefore, after solving for x:

 x = log 1.4142136 = $\frac{log\ 1.4142136}{log\ 2}$ = $\frac{log\ Number}{log\ Base}$: using any consistent base,
  2 say 10 or e, we find: x = 0.5

EQUATING POWERS OF (e) AND NATURAL LOGARITHM EXPRESSIONS

        Sometimes you may prefer to convert between expressions containing a power of (e) and a natural logarithm for various reasons. One reason for the conversion may be to use a series that you are more comfortable with. Below, the results are interesting and easy to remember, but the result is such that a series must be used twice.

Let $e^x = \ln N$

N can be solved simply by the definition of logarithms:

$$N = e^{\left(e^x\right)}$$: Note, this is not a "power to a power",
that is, it is not: $N = (e^e)^x = e^{(ex)}$

Ex. $e^2 = \ln N$ Solving for N:

$$N = e^{\left(e^x\right)}$$

$$N = e^{\left(e^2\right)}$$

$N = e^{7.389056099}$
$N = 1618.177992$ Hence:

$e^2 = \ln 1618.177992$: both sides are equal to 7.389056<u>1</u>

Solving for (x) given $e^x = \ln N$

Taking logarithms of both sides of the equation:

$\ln e^x = \ln(\ln N)$ Using the log exponential rule:

$x \ln e = \ln(\ln N)$ Dividing both sides by (ln e) to isolate (x):

$x = \dfrac{\ln(\ln N)}{\ln e}$ Since (ln e) = 1:

$x = \ln(\ln N)$

Ex. $e^x = \ln 1.5$

    $x = \ln(\ln 1.5)$
    $x = \ln(0.405465108)$
    $x = -0.902720455$

Hence $e^{-0.902720455} = \dfrac{1}{e^{0.902720455}} = \ln 1.5$: both sides equal 0.405465108

As like all other equations, if either expression is multiplied by some value (known, such as a constant, or unknown, such as a variable), simply multiply the other side of the equation by the same value so as to keep the expressions and-or values on both sides of the equations equivalent:

SOLVING FOR RECIPROCALS USING (e) AND NATURAL LOGARITHMS

      Solving for reciprocals is easily done on a calculator or computer, otherwise, you must solve for it using "long" division or some other method. One reason to solve for a reciprocal is that division of a dividend (or numerator) value can effectively be performed by multiplying that dividend value to the reciprocal value of the divisor (or denominator). For example, multiplication is usually performed in les time (ie., "faster") than division in the realm of computers, and especially if many calculations by "pen and hand" have to be made which could essentially slow the performance or results. Even though it may be overly complex and the long way of doing things, it is still helpful to know how to manipulate equations of all sorts when needed and-or to express them in a certain way. Here is one possible method to solve for a reciprocal:

let $y = \dfrac{1}{x}$ taking the natural logarithm of both sides:

$\ln y = \ln\left(\dfrac{1}{x}\right)$ using the "log reciprocal rule":

$\ln y = -\ln x$ by the definition of logarithms:

$y = e^{\wedge}(-\ln x)$ and since $(1/x) = y$, we have:

$\dfrac{1}{x} = e^{\wedge}(-\ln x) = \dfrac{1}{e^{\wedge}(\ln x)}$: A RECIPROCAL FORMULA USING (e)

Ex. $\dfrac{1}{53} = e^{\wedge} - \ln 53 = \dfrac{1}{e^{\wedge}\ln 53} = e^{\wedge} -3.970291914 = 0.018867924$

Clearly, by taking the reciprocal of each fraction above, or by simply equating numerators and denominators visually, $e^{\wedge}\ln x = x$. That is, this power of (e) simply equals (x) since it is the general form of the natural antilog notation for the number (here it's x). Also, as a check:

    If $x = \ln 53 = \dfrac{\ln 53}{e}$, then $e^{\wedge}x = 53 = e^{\wedge}\ln 53$

Ex. Find log 1.5   : N = 1.5

log 1.5 = log (10^-1) 15 = log (_15_)   : the right side shows the equivalent of N, and it's being divided by 10^1
                                            (10^1)   From the log quotient rule, this is equal to: log 15 - log 10^1 = log 15 - 1

log 1.5 = log (10^-1)15 = log 10^-1 + log 15   =   -1 + log 15   or=   log 15 - 1

log 1.5 = -1 + 1.176091259   : (or: 1.176091259 -1)

log 1.5 = 0.176091259

This concept can also be verified directly from the log "quotient rule":

log (N)(10^-x)   =   log (__N__)   =   log N - log 10^x   =   log N - x   : Since N was increased by a factor of
                            (10^x)   10^1, from 1.5 to 15, the logarithm
                                                                                                will then need to be reduced by that
                                                                                                same power of 10, which is 1 here.

Note, as (x) approaches a value of infinity (and much lower for practical purposes of the curve), the reciprocal of e^x approaches a value of 0, and (y) approaches a value of: (e^x)/2. When x=0, y=1. This is also the minimum point of the catenary curve.

The denominator of 2 simply shifts the curve vertically downward by dividing each (y) value by 2, and even if this denominator was not present, the curve still has the same shape. This value of 2 is essentially due to that the derivation of this equation considered that each side supports half the weight (or force) of the chain or material used. That is, the (y) value corresponded to the force needed to supporting each side of the suspended chain at that value of (x).

The equation of this curve rather seems to be relative in nature for values between 0 and 1. Given the same length of chain or rope, and the same distance between the vertical supports, the height of the poles and the curve above the ground plays no actual role in the shape of the catenary curve. Even if the poles are at different heights, the curve is still a catenary curve, and that all catenary curves are similar, but are effectively, magnified versions of one another.

Hyperbolic Trigonometric Functions

The right-hand side of the equation above is formally called the hyperbolic cosine (here, of x) function. The words hyperbolic cosine are often expressed or noted in the simple form of: cosh. Expressing this with an arbitrary numerical coefficient (a) for plotting on a graph:

$$y \; = \; a \; \cosh \left(\frac{x}{a} \right)$$

: a catenary curve, the greater (a) is, the "steeper" the sides of the curve
If a=1, the equation reduces to simply: y = cosh x. The lower (a) is, the more "narrow" the curve, and its sides do not expand or open wider as fast when (x) increases.

Remember, this equation is symmetrical (-x can be substituted for x without any problems) about the y-axis and in actual physical reproductions of the basic form of the defined catenary curve where both ends of the material used to construct the curve are suspended at the same height and under no pull, tension or force which would then distort, alter or "straighten" the natural catenary curve. The less the distance between the points of suspension given a fixed length of material, the "steeper" the sides of the catenary curve. The minimum distance possible between the points of suspension is practically 0, and the maximum distance is the length of the material used. Regardless of the material used to construct the curve, it has no effect on the basic equation of the catenary curve. Here is the basic shape of a catenary curve, and it could be describe as having a shape between that of a circle and parabola: [FIG 199]
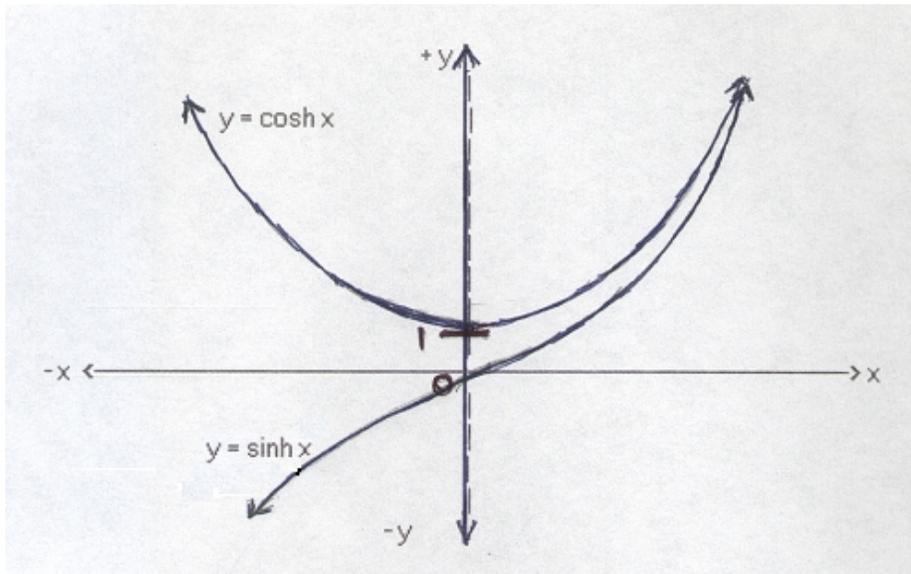
The three common hyperbolic trigonometric functions are given below: [Note, these not actually trigonometric.]

$$\cosh x = \sum_{n=0}^{n=\infty} \frac{x^{2n}}{(2n)!}$$

(x) is a radian angle measurement. For both the (sinh x) and (cosh x) series above, you can expect about 8 decimal places of accuracy when (x) is near 1, and 8 terms are summed. The lower (x) is, the greater the accuracy for a given number of terms evaluated. Actually, the series for sinh x and cosh are almost identical to the (sin x) and (cos x) series, except that the terms of these hyperbolic series do not have the $(-1)^n$ factor that produces an alternating (signed terms) series, hence all the terms are formally defined as being positive in sign.

Ex.   sinh 1 =   1.175201194
Ex.   cosh 1 =   1.543080635

Since TAN x = $\frac{SIN\ x}{COS\ x}$, and applying the above hyperbolic expressions to this, tanh is defined as:

$$\tanh\ x\ =\ \frac{\sinh x}{\cosh x}\ =\ \frac{\frac{e^x\ -\ e^{-x}}{2}}{\frac{e^x\ +\ e^{-x}}{2}}\ =\ \frac{e^x\ -\ e^{-x}}{e^x\ +\ e^{-x}}$$

The greater (x) is, the values of sinh (x) and cosh (x) become nearer and nearer, e^{-x} becomes very small, and therefore, the value of tanh (x) approaches $e^x / e^x\ = 1$.

Below, some generalities about the cosh equation will discussed.

$$y\ =\ \cosh u\ =\ \frac{e^x\ +\ e^{-x}}{2}$$

Why is the denominator 2? Like a numerical coefficient, the denominator for this equation shifts the curve of:
($e^x\ +\ e^{-x}$) along the y-axis. To further simplify this analysis, we will let the denominator equal 1, and let $N = e^x$. It is clear to see that the equation reduces to that of a number plus the reciprocal of that number:

$$y\ =\ e^x\ +\ e^{-x}\ =\ e^x\ +\ \frac{1}{e^x}\ =\ N\ +\ \frac{1}{N}\ =\ \frac{N^2 + 1}{N}$$: the numerator shows that the catenary curve is related to a quadratic or parabola curve.

When (x) is negative in value, the terms effectively switch position, resulting in the same general equation:

$$y\ =\ e^{-x}\ +\ e^{-(-x)}\ =\ \frac{1}{e^x}\ +\ e^x\ =\ \frac{1}{N}\ +\ N$$

This simply indicates that the curve of this equation is symmetrical about the y-axis. It also indicates that (y) can never be negative in value (assuming N is always a positive value).   When x=0, we will find the minimum value of (y):

$$y\ =\ e^0\ +\ \frac{1}{e^0}\ =\ 1 + 1\ =\ 2$$: though with the full equation above, the min. is this value divided by 2, hence: 2/2 = 1

To verify this, first consider y = 0:

catenary curve will have the same identical shape on both the Moon and Earth. (g) of the (local) mass or body is not included as a part of the catenary curve equation. The forces acting upon the chain or catenary curve will be proportional to the amount of (local) gravity upon the chain. In short, all catenary curves are similar in shape and one is simply a magnification of the other. The catenary curve equation is expressed as a very natural mathematical representation of a very natural effect of the (downward) force of gravity upon the entire curve and (changing, force) tension needed to support that curve at points along it. The shape of a catenary curve is somewhat like an "upside-down" bell curve.

Regardless of the height of the curve above the surface of the ground, the catenary curve will still have the same shape and equation describing it. Excluding the 2 in the denominator, here is an analysis to consider of the sum of a power value of a value (N) and the reciprocal of that power value:

$$N^x + N^{-x} = N^x + \frac{1}{N^x}$$ combining these fractions:

$\frac{N^{(2x)} + 1}{N^x}$, when N and-or x is high, the added 1 becomes more meaningless and the equation reduces to:

$N^{(2x)} / N^x = N^{(2x - x)} = N^x$, and this can also be seen above when $(1 / N^x)$ becomes a small value.

since $\dfrac{6}{4}$ = 1 with a remainder of 2

Ex. 7 MOD 4 = 3

since $\dfrac{7}{4}$ = 1 with a remainder of 3

Ex. 8 MOD 4 = 0

since $\dfrac{8}{4}$ = 2 with a remainder of 0

Whenever the numerator is less than the denominator, the result of the MOD operation will always be equal to the that numerator.

Ex. 6 MOD 8 = 6

since $\dfrac{6}{8}$ = 0 with a remainder of 6

Here is how you can perform a MOD operation by hand or with a calculator:

1. Divide the two operands and obtain the whole portion of the quotient.
2. Subtract the product of the whole portion of the quotient and the denominator from the numerator.

Ex. Evaluate: 17 MOD 8

```
       2.___
  8 | 17.0
    -  16
        1          : The division can stop here since the entire whole portion
                      of the quotient has been found.
```

(17 MOD 8)   =   17 - (8)(2)   =   17 - 16 = 1   : 1 is the remainder, hence (17 MOD 8) = 1

In computer programming, the bits of a byte generally do not have a unique (ie. integer) memory address (location) assigned to them. Still, you can find out the status (set=1, not set=0) or numerical value of a bit by using a small amount of computer code or a program. This results in accessing (reading or writing) the entire (8-bit) byte a bit is part of, and then use various forms of logical or "bitwise (bit for corresponding bit)" operations or bit-shifting (such as with the C-programming language, binary value (bit) shift operators: << and >> which can be used to essentially rid or remove unwanted bits from a byte), etc. to find the specific value (0 or 1) of a bit. These methods are forms of "software addressing". Given the bit address value of a certain bit in memory, how do you find which actual byte address in memory that bit is within or part of? We can use the MOD operation to find what the byte number or byte address is. For example, and optionally using logical (you can think of this as referring to offset values from the start location of 0) rather than physical address values which start at 1 as for counting things.

Byte address: 0   1   2   3 . . .   : each byte has 8 bits
Bit address: 0 1 2 3 4 5 6 7,   8 9 10 11 12 13 14 15, 16 17 18 19 20 21 22 23, 24 . . .   "8 bits wide"

Below, the corresponding byte of which a bit is in is being found. An operation, somewhat like the concepts of a MOD operation is to be used, but here the quotient, and not the remainder, is to be used. Essentially then, the MOD

SOME ADVANCED FUNCTION CONCEPTS

COMPOSITE FUNCTIONS

A function that is a function of another function is called a composite function. A composite function is then composed of at least two functions.

If function (g) is a function of variable (y), this can be expressed with the notation of: g(y). If $g(y) = 7y^2 + 2$, and if (y) is a function (ie. depends upon) of another function (f) where (y) = f(x) = 3x, this composition can be expressed as the composite function gf. This composite function, for example, if set equal to (z), might be used to plot points: P(x, y, z), or perhaps better expressed as: P(x, y(x), z(y)) in a three dimensional coordinate system. Note that gf is the (function) notation for a composite function, and it does not mean the sum or product of function (g) and function (f). Here is an example:

y = f(x) = 3x   : y is set equal to a function, and here, it is a function of x.
                                             y is then simply described as a function of x.

z = g(y) = 7y^2 + 2   : z is set equal to function g of which is a function of y.   And since y = f(x):

z = g(y) = 7(f(x))^2 + 2   : function g is now explicitly shown as a function of x on the right hand side:

z = g(f(x)) = 7(3x)^2 + 2   Using simplified notation:

z = gf(x) = 7(9x^2) + 2   : gf is a composite function, a function composed of two or more functions.
                                             gf can be noted as a "function that contains another function".

z = gf(x) = 63x^2 + 2   : variable z is a function of the composite function gf

INVERSE FUNCTIONS

If (y) is a function of (x), you may want to solve for the corresponding (x) values which determined those (y) values. This can be accomplished by the substitution of the (y) value into the inverse equation and then solving for (x). However, here we are interested in a general or algebraic method in the form of a formula. The method to do this is to create an inverse function from the known function to essentially undo or "work backwards" to find the value of the variable (x). A given function and its corresponding inverse function are both inverse functions of each other. It should also be stated that these inverse functions are also not generally the mathematical reciprocal of each other.

If (g) is the inverse function of y=f(x), then evaluating (g) with this specific value of (y) will produce its corresponding value of (x):

y = f(x)   : f(x), (f) is a function, here evaluated using or containing (x), hence f(x) is called a function of (x).
                    The actual function is an expression containing variable (x).

x = g(y)   : g(y), (g) is a function, here evaluated using or containing (y), hence (g) would be called a function of (y).
                    If (g) is the inverse function of (f), and if (g) evaluates the output of function (f). which is the (y) value, it
                    it will find its corresponding value of (x).

The relationship of f(x) and g(y) can be considered as like the corresponding relationship of the (x) and (y)
values of a point, except that (f) and (g) are used to find all the specific corresponding points. That is, (f)
and (g) actually represent all the possible corresponding pairs of two values, rather than just a single specific
pair. This can be expressed as:

Switching the axis, we have: [FIG 201]
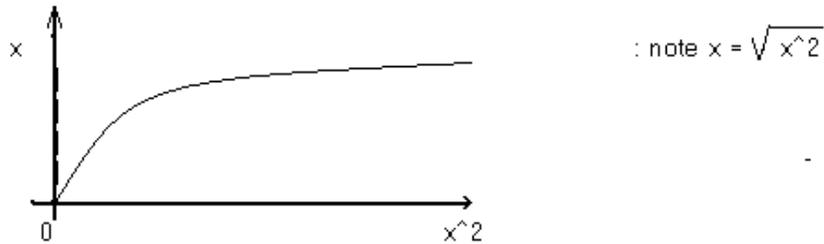
However labeled, still the expressions on the horizontal axis represent the horizontal distance (ie. "x") from 0, and the expressions on the vertical axis represent the vertical distance (ie. "y") from 0.

To find the inverse function of f(x), solve for x given that function:

$y = x^2$ taking the square root

$\sqrt{x^2}$ simplifying:

$g = x$

Since the process only involved taking a square root, the inverse function is therefore a square root:

$g(x) \;=\; \sqrt{x^2} \;=\; x$: that is, if x was squared, you would need to take the square root of that resulting value so as to find what x was

Since: $y = f(x)$ and $x = g(y)$, $f(x)$ and $g(y)$ are inverse functions

Ex. If $f(x) = n^x$, find the inverse of this function where base (n) is a constant value.

n^x solving for x by taking a logarithm:
$\log_b n^x$

Or something as the following where y or z is set equal to the given function:

$y \;=\; n^x$ taking the log of both sides:
$\log y \;=\; \log n^x$ using the log exponent rule:
$\log y \;=\; x \log n$ using a log with a base of n: $\log_n n \;=\; 1$, and switching sides:
$x \;=\; \log_n y$ since $y = n^x$:
$x \;=\; \log_n n^x$

Hence to solve for x, only a logarithm needs be taken. For example:

$y \;=\; n^x$: here, n is a constant, not a variable
$y \;=\; 2^3$: if n=2
$y \;=\; 8$ expressing these steps with a general formula:

$x \;=\; g(y) \;=\; \log_2 y$: Note the base is expressed as written right next to the word log, here it's 2, rather than as a subscript or other form such as log_2. It's a simple text alternative form of expressing logarithm expressions.

INVERSE SQUARE LAW

       The inverse square law is a common name given to a relationship when values or variables are not only related to the reciprocal or inverse (ie. inversely related) of another value or variable, but related to the mathematical square (ie. value^2) of that reciprocal value.

$A = \dfrac{1}{B}$: Here, A is related to the inverse or reciprocal of B, and if given:

$A = \dfrac{(1)(1)}{(B)(B)} = \left[\dfrac{1}{B}\right]^{\wedge}2 = \dfrac{1}{B^{\wedge}2}$: Here, A is related to the square of the inverse or reciprocal of B, and may be commonly spoken as: "A is related to the inverse square of B", or "A is the inverse square of B".

Common examples of this relationship are found when dealing with energy (or the measured intensity of it) as it expands outward in all directions from the (point) source. For example, light, sound, radiation, gravity fields and so on, often expand this way and their corresponding intensity or strength reduces dramatically by the square of the distance from that source. Below in the drawing we see a "point-source" (infinitely small and uniform source or distributor) of energy at the center of the (imaginary) concentric spheres drawn to be used for this analysis. This point could actually be an infinitesimally small object or a large object so far away as to be considered practically as a point (such as a star in the sky). Since the energy is uniformly distributed outward from the source, a surrounding "shell" or (surface of a) sphere at various distances or radius values from that source will be used for the analysis. [FIG 202]
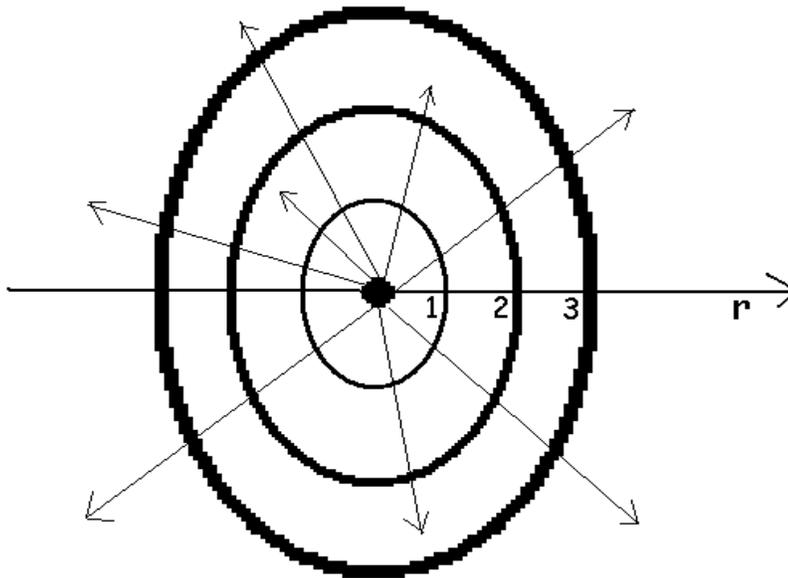
$Ss = 4\pi r^{\wedge}2 \approx 12.566371\ r^{\wedge}2$: Formula for the **SURFACE AREA OF A SPHERE** (essentially a ball) as derived by Archimedes is approximately: $12.57\ r^{\wedge}2$. Units are those used for r.

$Vs = \dfrac{4\pi r^{\wedge}3}{3} \approx 4.1887902\ r^{\wedge}3$: **VOLUME OF A SPHERE (with units being the same as those used with r)** is approximately: $Vs \approx 1.333...\ (pi)\ r^{\wedge}3 = 4.18879\ r^{\wedge}3 \approx 4.1888\ r^{\wedge}3$ This formula is credited to Archimedes.

At r=2 from the source, for example, even though the intensity decreases by r^2 = 2^2 = 4, the surface area actually increases by 4, and the total energy passing through or received by each entire shell or sphere from the source is constant:

Total Energy In = Total Energy Out   : considering no losses along the way
        Intensity In = Intensity Out

From $D = \dfrac{I}{A}$: energy or intensity per unit area, we have $I = DA$: total energy at that area

Ex. If the density (D) or intensity per unit area was measured to be 100, and if A increases by 4, D will also decrease by a factor of 4, such as when r=2:

I_1 = I_2 : total intensity or energy at surfaces at r=1 and r=2 respectively
D1A1 = D2A2   : You may sometimes see proportions like: D1/A2 = D2/A1 when dealing with
                                     inverse square relationships. an alternate expression of these values is:
                                     DnAn is Dn Rn^2   : R = radius , n = corresponding values of D and A

100(1) = 25(4)

100 = 100   : showing that, the total energy at each (increased in size) shell or sphere is the same.

Ex. If r is now 3, what is the calculated density or intensity per unit area?

From: D1A1 = D2 A2   mathematically:

D2 = $\dfrac{D1A1}{A2}$ or= $\dfrac{D1R1^2}{R2^2}$ = (100)(1) / 3^3 = 100/9 =~ 11.1

You probably have noticed that the intensity or level of sound and-or light changes as the distance between you and that source changes.   A sound can be very loud if you are close to it, or barely heard if you are far from the source of that sound. As your distance doubles (at r=2) from that source (or possibly a reflection [and redirection]), the intensity of the energy received will be reduced (ie. divided by the factor) by r^2 = 4 , or multiplied by factor of (1/4) = 0.25. Just the same, in a reverse type of manner, if the distance between you and the source is halved (r/2 = 0.5r), the intensity of the energy will increase or multiplied by the factor of 4.

An article in the appendix section of this book will contain another verification, ideas and examples; see
More About The Inverse Square Law

If the intensity (I) and-or measurement is inversely related to the square of the distance (r) from the source or a measurement location, we can express this as:

$I = \dfrac{1}{r^2} = r^{-2}$ solving for r:

$r = \sqrt{\dfrac{1}{I}}$

If the distance (r) from the source has increased, the intensity (I) will decrease due to the inverse real (physical) and mathematical relationship. and if the intensity is measured to be a certain percentage (n) as that of the first measurement, this can be expressed as: nI , and for example, if I is now at its 50% level, n = 0.5 and nI = 0.5I. The corresponding distance (r) can be found by using this value in the above equation. Considering I as a relative value of 100%=1 value:

$$I = \frac{1}{r^2}$$
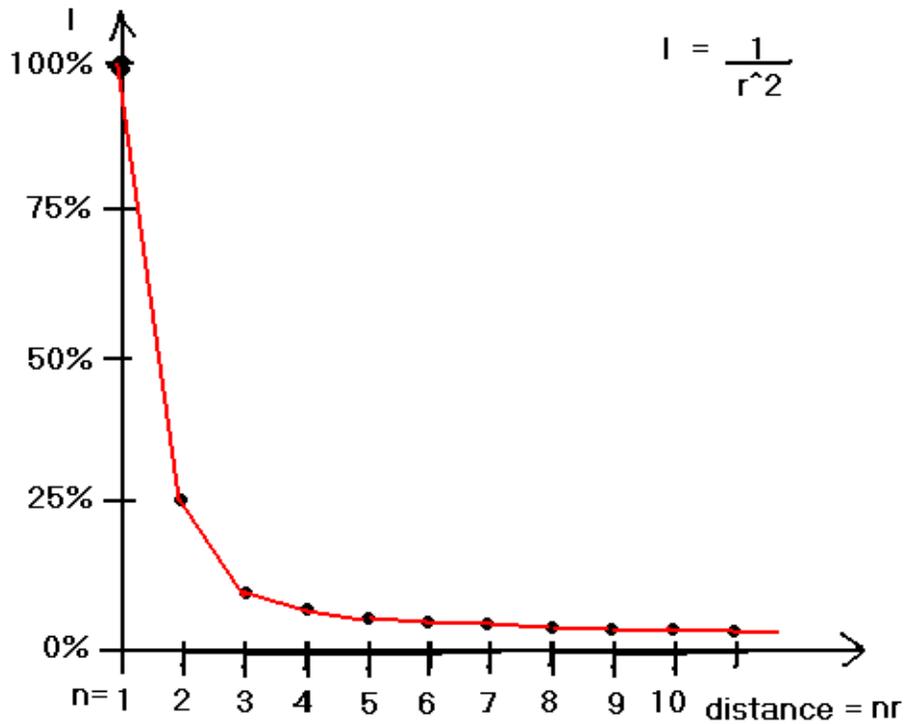
(ie., multiples of (r))

Notice that at high values of (r), say past r=5, or 5r, that the intensity is a low value, but it remains relatively stable in value after that for very high values of (r), and this allows us to still receive and see enough light from a very distant star. A telescope with a large main (or "objective") lens or mirror can gather very small amounts of light and collect and concentrate them (into a smaller area or image) so as to effectively increase the intensity of that received light per unit area, and therefore, produce a brighter and more visible image.

THE BINARY NUMBER SYSTEM

      The reader may skip the section below about binary numbers and read it at a later time if they become interested in this topic. The binary number system will not be discussed further in this book, except that it may possibly help with the included computer programs and with further study about some methods of computer programming.

Before this discussion begins, some people may be worried as to what great of an extent they may need to understand binary numbers as to be a computer programmer. With most modern computer languages, working directly with binary bits and bytes and-or their their signed value computer representation is often not needed as rather their equivalent and common decimal values are used directly in computer programs.

      The word "binary" has a "bi" prefix which means double, two or 2. "Unary" is a word meaning single, unit or 1. The binary number system uses only two counting symbols (0 and 1). Hence, a "highest count" of only one (1) can be reached before positional notation, just like that used with the decimal number system, is required to represent larger quantities or numeric values. That is, in binary addition, 1+1 = 0 with a carry of 1. This can be shown as 1+1=10. As an aid to avoid ambiguity and to be certain which system is being used, letter b for binary (base 2), or letter d for decimal (base 10) can be explicitly expressed or indicated. For example: 1b+1b = 10b. 10 ("one, zero") is the binary representation or equivalent of the decimal notation for 2:   10b = 2d.

    Lets increment a binary value of 0 by 1, using the binary system and positional notation:

```
        0
      + 0
        0                 : 0b = 0d

        0b      0d          : Same as:   1b     : using the commutative law.
      + 1b    + 1d                     + 0b
        1b  =   1d                       1b    ,   1b = 1d


        1 <------------------: This is a carry of 1 from the previous column or position sum.
        1b      1d          : As in the decimal system, leading zeros are also possible, for
      + 1b    + 1d            example:  1 + 1  =   0001 + 0001  =  0010
        10b     2d          : "1 plus 1, equals zero with a carry of 1".  The result is "one, zero" in
                              binary, or 2 in decimal.  10b = 2d

        10b           10b      2d
      + 1b    or:  + 01b    + 1d
        11b           11b   =   3d          : 11b = 3d

       11            11
        11           011       3d
      + 1    or:  + 001      + 1d
       100b          100b  =   4d          : 100b = 4d
```

requested, are automatically read from the disk and stored in this cache for possible immediate use.

Here are the binary number system weights of the bits of a byte which is a binary encoded number or value, having a "width" (digits or bit count) of 8 bits:

most significant part of the number   least significant part of the number

       2^7 , 2^6 , 2^5 , 2^4 , 2^3 , 2^2 , 2^1 , 2^0 : binary weights in powers of 2 (binary)
       128 , 64 , 32 , 16 , 8 , 4 , 2 , 1   : equivalent decimal values

Notice that in the binary number system that each weight is twice (2) that of the previous bit position weight, rather than ten (10) times that of the previous position weight as with the decimal number system.

For either the decimal or binary number system, a positional product is the product of the digit or bit value and the weight of the position it is in. This is so since each digit or bit value is a multiple of the weight value of that position.

Since a bit can only be 0 or 1, each positional product of a binary number can only be either 0, since (0 x any_weight) = 0, or (1 x weight) = weight. This makes it easy to find the decimal equivalent of a binary value since no multiplication needs to be done, and essentially, just the addition of the binary weights (ie., positional significance, value) is needed.

Ex. What is the decimal equivalent of the byte: 00100011 , or without the leading 0's as: 100011 ?

    0   0   1   0   0   0   1   1   : 00100011b

    0 + 0 + 32 + 0 + 0 + 0 + 2 + 1   : position products =
                                  bit value x weight value   or=:
                                    position value x position weight

                                  By simply "skipping over the 0 bits", that have no value:
      32 + 2 + 1 = 35b   : positional sum
                                    00100011b = 35d

Ex. What is the binary equivalent of 35 = 35 decimal = 35d ?

   First, since the value is less than or equal to 255d, only one byte, or= 8 bits, is needed to represent this value in binary. Actually, only 6 bits are needed to represent this value since the leading two bits are 0.

   Since the binary weights of 128 and 64 are larger than 35, they will not be included or set (to 1) in the binary representation. If building a (8-bit) full byte they should rather be indicated as 0, and since leading 0's wont add to the value. It is possible in some electronic systems that if they are not deliberately set to zero (or "cleared"), that they may still contain a previous ("garbage", incorrect) value that could result in an error if those bits are possibly considered and included as part of the value.

   0 0 . . . . . .   : first two bits of the byte set to 0. Since these bits actually represent a larger part of the number, they are called most significant bits. The logical bit numbers of these are bit #7 and bit #6 from left to right. The first physical bit has a logical bit number of bit #0.

   The bit corresponding to the weight of 32 can be set to 1 since 32 is less than or equal to 35, that is 32 is part of, or within 35, and therefore it needs to be included or represented in the binary number:

   0 0 1 . . . . .

   The remainder of decimal value must now be considered so as to keep building the equivalent binary value.

NEGATIVE BINARY VALUES

When not explicitly specified, binary values are assumed as being positive values. It is possible to have a system of signed (positive and-or negative) binary values. The usual way to accomplish this is to assign the first or most significant bit as the "sign bit". If the bit is set to, or has the value of 0, the entire binary value is a positive value. If the bit is set to, or has the value of 1, the entire binary value is a positive value. This following discussion about how negative binary values are expressed is generally only for technical, near circuit level, or for computer programmers only, and may be skipped over if the reader does not need it.

Since a bit of the byte is used for the sign, there is now a "trade off" or loss, and that is that the signed binary value can now only have an absolute value of about half of what would normally be possible with that given number of bits in the binary number. For an unsigned 8 bit binary number, the maximum value is 255d:

According to the above discussion, if the first bit is the sign bit:

```
10000001  =  -1
10000010  =  -2
 11111111  =  -127    : 127 is nearly half of 255 which is the decimal value of a full byte where each bit is set to 1.
                        -127 is the maximum negative value of a 8 bit byte.  The maximum positive value of a
                        8 bit byte is :  01111111 = +127.
```

Though the above is the practical method to express negative numbers, the actual way a computer does it is slightly different, and it is to encode the negative value using a well defined manner so as it can then be used with the computers (preexisting) addition (or "combine") circuitry and produce the correct result of a signed number addition operation. Essentially then, the (coded) negative value will be considered as an operand in a subtraction operation of which is then performed using the addition circuitry of the computer.

Here is how is a negative value encoded so as to be used by the computers addition circuitry:

```
11111111b   =   255d        : maximum value for an 8 bit unsigned binary value
```

If you were to then add 1 to this binary value, all of the 8 bits will "roll over" (as if on a wheel, such in a cars mechanical mileage counter mechanism) to 0, and the (final) carry will be discarded ("dropped", or "lost" if its unused).

```
   11111111b  =    255d
 + 00000001b  =  +   1d
  100000000b       256d      if this last or final carry is dropped (unused) we have this value remaining:

   00000000b  =  0d   when the carry bit is dropped.
```

A decimal system analogy to this would be for example: if given a 4 decimal digit system, when the max. value of 9999d is reached, and if 1 is then added to it, the value of 9999 will "roll over" to all 0's, and the (final) carry is dropped:

```
   111 <--------- carrys from the previous column or digit position.  Each carry is a value that is 10 times the previous
   9999d                weight value.
 + 0001d
  10000d  =  0000d  =  0d   when the last or final carry is dropped (unused)
```

Now consider, given 0000000b, if you were to "go back" one or subtract 1 from this given value of 0bso as to have the previous value before 1 was added to it, the result would be the value just before this "roll over" value of 0, and that value is: 11111111b. You can also consider this verification:

number representation of -3d:

```
00000011b  = +3d      Inverting each bit:
11111100b             adding 1:
11111101b  =  -3d
```

On a technical or computer programming note, for various reasons and usually other than the concepts of a computers representation of negative binary numbers, each bit of a binary value can be inverted by XORing (with the XOR binary "bitwise" operation; bit by bit, or bit to corresponding bit position) it with a bit set to 1. The resulting value is sometimes called the (ones) complement of the given binary value. An entire 8 bit binary value can also be complemented in just one step by XORing it with an 8 bit binary value with all of its bits set to 1. 255d = 11111111b which has all 8 bits set to 1.

Ex.   10000001b XOR   11111111b = 01111110b

The C-computer language also has a special bitwise operator to easily complement all the bits of a byte value, and this uses the "tilde" symbol of (~). If variable (a) was set equal to 1d = 00000001b , then if variable (b) is set equal to the complement of variable (a): b = ~a , then b = ~1d = ~00000001b = 11111110b = 254d when non-signed or unsigned values are only being considered.

FRACTIONAL BINARY VALUES

The concepts of expressing fractional (<1) binary values is nearly identical to that of expressing fractional decimal values, except that for the binary system which is based on 2, will use negative integer powers of 2 instead of negative integer powers of 10 as it is for the decimal number system. Here are the binary weights and their equivalent decimal equivalents, and note for example that $2^{-1}d = 1/2d = 0.5d$, that is a negative indicated power is equivalent the reciprocal of the corresponding positive indicated power. Here are the decimal equivalents of the bit positions of a binary number:

```
2^7  , 2^6 , 2^5 , 2^4 , 2^3 , 2^2 , 2^1 , 2^0 . 2^-1 , 2^-2  , 2^-3  , 2^-4   , 2^-5    , 2^-6
128  , 64  , 32  , 16  , 8   , 4   , 2   , 1   . 0.5  , 0.250 , 0.125 , 0.0625 , 0.03125 , 0.015625
```

Notice that going from left to right, or from the most significant weights to the least significant weights, that each is half of the previous. Going in the other direction, each is double (2 times, twice) the previous.

Convert 1.1b and 1.01b to their corresponding decimal equivalent value and representation:

Showing many intermediary steps:

1.1b   = $1(2^0)d + 1(2^{-1})d$ =   1d + $(1/2^1)d$ = 1d + 0.5d = 1.5d

1.01b = $1(2^0)d + 0(2^{-1})d + 1(2^{-2})d$ = 1d + 0$(1/2^1)d$ + 1$(1/2^2)$1d = 1d + 0(1/2)d + 1(1/4)d   =
                                                                                    = 1d + 0(0.5)d + 1(0.25)d =
                                                                                    = 1d + 0.0d + 0.25d
                                                                                    = 1.25d

To convert a binary value that has a both a whole part and a fractional part, simply convert the whole part and then the fractional part to their corresponding and equivalent decimal values, and place the (.) symbol between them. This same method can be used to convert a decimal value to its corresponding and equivalent binary value.

can communicate this image by letting number 1 mean to fill a box, and 0 to keep it unfilled. A longer binary number say for example: 10101010 might numerically represent the upper row of a text character (ie., the image of a letter) or a picture being drawn. We see that a binary, two state system and-or numbers is the simplest way to represent information, and it can be transmitted as very fast electronic pulses through a wire of an electric circuit, including the telegraph or radio communication system. It should be also of note that each letter and-or text character is usually given a unique 8 bit code or binary number such as with the ASCII code system. A computerized, "8 by 8, dot-matrix" printer will effectively have in its memory storage a matrix or array of picture elements for each ASCII text character to be printed, say printed or displayed using 8 horizontal bits in each of 8 rows. Each row of pixel element information can be stored as if it is an 8-bit binary number or information. A pixel of an image is also the smallest discernible or resolvable (ie., resolution) piece of an image.

With binary picture elements being either 1 or 0, how can colors or gray scales be used since we can not have half a bit of information and we can only have "full bits", each with just two possible states only? The answer for example would be to use a binary number coded (gray or "color intensity") scale of values. We can create a 256 shades of gray using an 8 bit binary number. We can let 00000000b represent no gray in the box, hence clear or white, and 11111111b represent full black. Values in between 0 and 255 will represent a particular shade of gray, say increasing in darkness as the binary value increases. This system can display what is called a "black and white" picture, but the image or picture is technically seen as a "**gray scale**" picture that has been printed with just one available color, usually black. Now that an image can become better and-or more accurate or representative, we see that more information must be used, and here we see that each picture element now requires 8 bits now instead of just 1 bit. The entire image will require 8 times the information or data and its storage, and 8 times longer to transmit it.

As mentioned previously, 1's and 0's are numeric and-or logic values, that can be thought of as either a signal or presence, and a lack of signal or presence. For example the signal or presence can be a voltage value say +5 volts to represent a "high" or 1 value, and 0 volts to represent a "low" or 0 value. The signal could be a charge storage or lack of it such as in a capacitor, or it could be a magnetized spot or location on a metallic hard-disk drive, or it could be a transistor setting considered either on or off. Here is a basic graphical image of 1's and 0's such as that which could also be visualized using an electronic oscilloscope that can display a visualization of a signal and-or waveform: [FIG 202B]
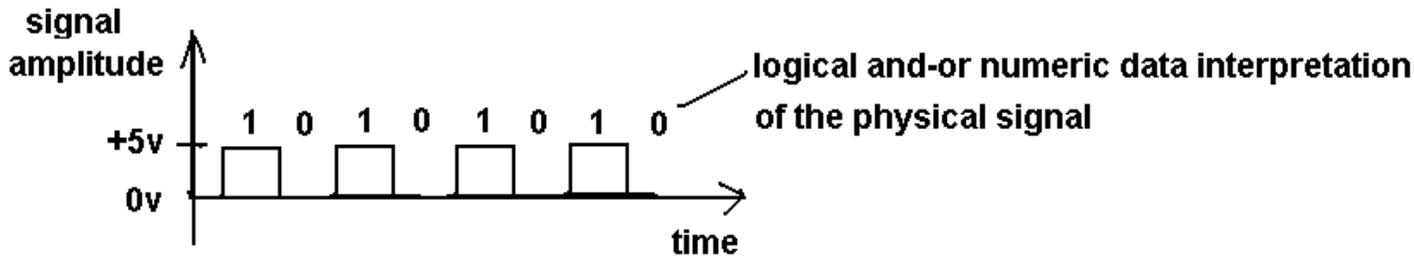
When a computer program receives and senses 8 bits of data at its communication port or input, it will convert this to a byte of binary data that the computer can store it in a file and-or process it further, such as displaying it as a text character on the screen.

If there is a set of 10 values labeled from 0 through 9 in a bag, and if only one is being picked out from, the probability of each different number, or picking a certain number, is: 1/10 = 0.1   : "one in ten" = 1 in 10 = 1 out of 10

If a second set of the same 10 values: 0 to 9 was initially combined into the first set of 10 values, the probability of each different number, or picking a certain number, is: 2/20 = 1/10 = 0.1 , hence the same probability as before

If an event or outcome is 10% probable, then it could be said that it is (100% - 10%) = 90% "not likely" or "not probable":

  "odds" of something happening = "probability" odds   = $\dfrac{\text{probable}}{\text{not probable}}$,   ex: $\dfrac{10\%}{90\%} = \dfrac{0.1}{0.9}$ = 0.1111 = 11.11%

12

12

12 x 12 = 144

r = 1
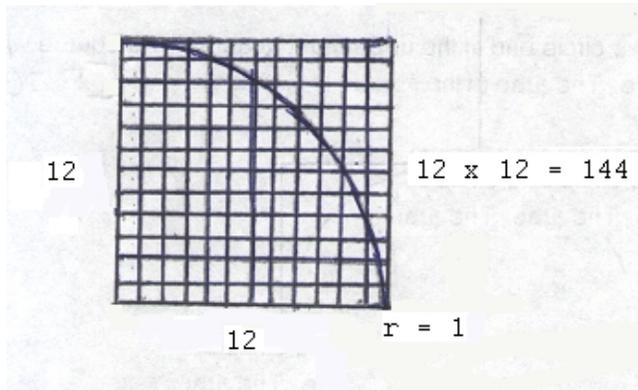
Obviously, in the figure directly above, the portion of the area of the circle that is within the square (ie. a quadrant, or quarter of the circle) is only some fraction (or percentage) of that square's area. The area of this portion of the circle is less than 1 r^2, but greater than half of it or $r^2/2 = 0.5\ r^2$, as can be seen by drawing a diagonal line from the upper-left to the lower-right corner in the drawing above. Letting variable (x) represent this portion, we algebraically get:

Ac = xr^2 + xr^2 + xr^2 + xr^2 : x<1, combining like terms:
Ac = 4 x r^2 : Since x<1, therefore for Ac, the actual numerical coefficient (n) of r^2
                                         is a fraction of this value of 4, and can be expressed as n<4:

Ac = nr^2 : where (n<4)

(x) can be calculated from:

x = $\dfrac{\text{Area of the circle portion in the square}}{\text{Area of the square}}$ = $\dfrac{Aq}{As}$: x is a ratio, Aq = area of a quadrant of the circle

Using the above graph, x can be geometrically (approximately) calculated without using a specialized formula as:

x = $\dfrac{Aq}{As}$ =~ $\dfrac{\text{number of small squares within the circle's portion of the square}}{\text{total number of small squares in the square}}$

This ratio value can also be quickly estimated by observation. Obviously, the area of the circle that is within the square is clearly greater than 1/2 = 50% of the square (can check this by drawing a diagonal within the square from upper-left to lower-right), and less than 1 = 100% of the square. It appears to be a value about midway between these two values, and this is three-quarters of the square = 3/4 = 0.75 = 75%. Taking the average of these two values we have:

x = $\dfrac{50\% + 100\%}{2}$ = $\dfrac{150\%}{2}$ = 75% = 0.75 in decimal form

Substituting this value of (x) into Ac:

Ac = nr^2 < 4 x r^2
Ac = nr^2 ~= 4 (0.75) r^2
Ac = nr^2 ~= 3 r^2 : approximation, and this is similar to the approximation of the length of a circles circumference
                                     being about 3 times its diameter length.

If you solve for (x) by the geometric methods, you will see that this ratio value is slightly greater than 0.75 and therefore, the numerical coefficient (3) of r^2 shown is slightly lower than it should be. For a simple example, using the drawing

but it is sill possible, or in theory, to have a triangle and circle having the same exact area.

$\dfrac{C\,r}{2}$ = **π r^2** : **AREA OF A CIRCLE** , r = radius of the circle = half the circles diameter (d or D).
r = distance from center point of the circle to its (curved-line) edge or circumference (C).

Solving for C:

C = $\dfrac{2\pi r^2}{r}$ canceling or dividing out common factors, here it's r^1:

C = 2π r : **CIRCUMFERENCE OF A CIRCLE**

Since 2r = D: : D = diameter = longest length, or width, across ("dia-") or through and measurement ("meter")
of the circle. r = D / 2 , D / r = 2 ,

C = π D : hence, the circumference of a circle is about 3.14 times larger or longer than its diameter.

π = $\dfrac{C}{D}$: **"pi"** = **π**, is usually defined as the (constant) ratio of any circles circumference to its diameter
(pi) =~ 3.13159265... , r = C / 2π = C / 6.2832... , C / r = 2π = 6.2832...

We could have found π if we knew before-hand that it was equal to the circumference of a circle divided by its corresponding diameter. Logically, or "truly", Pi is mathematically expressed as, and equal to exactly: C/D, however the value of Pi calculated by any means possible will always be an approximation no matter how "close" it is to the true value of C/D. Because of this, the value of Pi is said to be an irrational value or number. With the calculation of π using small squares as shown previously, the result will always an approximation. With advanced mathematical formulas, π can be calculated to a great and accurate precision ("preciseness", fineness, fraction) but it is still an approximation since Pi will always be an irrational value with endless digits.

It is not easy to measure the circumference of a circle using a typical and inflexible ruler. One method to measure the circumference of a circle is to mark a point on the circumference of the circle where it is tangent (ie. coincides, touches) to a flat surface, and also mark a corresponding point on the flat surface. Roll the circle along a straight, line-like path on that flat surface for one complete rotation and then mark another point on that flat surface where the point on its circumference is again tangent to the flat surface. The measure of this distance between these two points on that flat surface will equal the circumference (distance is equivalent to corresponding "straight" linear distance when "flattened") of that circle.

If the concept of Pi is already known, a basic verification to the above derivation(s) can be made by considering a more well known derivation for the area of a circle, and that is to divide the circle into an infinite number of similar isosceles triangles that have the same height and base side values. The height of each triangle will be equal to the radius (r) of that circle, and the base side of each triangle will be an infinitely small segment portion of the circumference. As the number of triangles gets larger in value, the base side of each will approach a value of 0, but will never actually be zero. Ac = (area of the sum of each triangle), and since all the triangles are the same, their areas are the same, and the circle area can then be expressed with multiplication of the area of just one similar triangle by the total number (n)of triangles. Ac=n(area of triangle). one might quickly say that the area of that circle will be infinite if the number of triangles to be summed is infinite, and this is a good thought, but we must also consider that the area of each triangle is infinitely small in value or close to 0. A practical version of this derivation is to then construct a rectangle from all these triangles by alternately alternately placing upside-down next to each other in a row. The area of this rectangle construction will be:
Ar = (length)(width), and where the (length) = (circumference / 2), and (width) = (r). Ar = (length) (width) = (2(pi)r / 2) (r) = (pi) r^2 = Ac

$Vp = \dfrac{Vc}{6} = \dfrac{LWH}{6} = \dfrac{\text{(base area of each pyramid) H}}{6}$: Vc = volume of cube,   Vp = volume of pyramid.

base area of each pyramid = area of each side of cube

The height of each pyramid is only half of that of the cube. h = H/2. Therefore, we cannot use just H in the formula for the volume of each internal pyramid.   We can create an equivalent fraction by dividing both the numerator and denominator by 2:

$$Vp = \dfrac{\dfrac{\text{(base area of each pyramid) H}}{2}}{\dfrac{6}{2}}$$

Expressing (H/2) as the height of the internal pyramid:

$Vp = \dfrac{\text{(base area of pyramid)(height of pyramid)}}{3}$:VOLUME OF A PYRAMID OR CONE

This formula also works for the volume of a cone since it can be considered as a unique pyramid with an infinite number of sides extending up from its polygon base area. The word "cone" is rooted in the word of "point", "combine" and "concentric" which means all having the same center or point of reference. A cone can also be thought of as an infinite number of concentric circles or disks that get smaller and smaller until the radius is just the size of a point. The rate at which the diameters or radius get smaller and smaller as the height or base value changes (ie., increases) will determine the specific shape (ie., side slant or "slope") of that cone.

For the surface area of a cylinder or rod shape, imagine taking the curved surface area portion of a cylinder or rod and making a straight cut along its length, and then flattening that surface into a rectangle plane shape.

A = lateral or side area of the rod + area at both ends of the rod
A = area of curved portion + area of both ends

A = area of rectangle created from the curved portion   +   area of both circles at the ends

A = (length) (height)   +   2(pi)(r^2)   : r = d/2 = (outer width or thickness of the cylinder or rod) / 2

The height of this rectangle is essentially that of the circumference of the cylinder or rod:

A = (length) (2)(pi)(r)   +   2(pi)(r^2)   factoring 2(pi)r from each term:

A = (2)(pi)(r) (length + r) = (circumference)(length + r)   : SURFACE AREA OF A CYLINDER OR ROD SHAPE

length of the rectangle = height of the cylinder or rod

The figure in the particular example above clearly shows that a quarter (here A2) of the circle is not present, and it will help verify that the ratio or fraction of the arc length (s2) to the circumference (Ct) of the circle, is equal to the ratio of the corresponding sector area (A2) to that of the total area of the circle (At = A1 + A2). Clearly, s2 is a quarter ("one-fourth" = 1/4) of the circle's circumference (Ct), and A2 is a quarter of the circle's area (At):

s2 = Ct / 4 = 0.25Ct   and   A2 = At / 4 = 0.25At   therefore:

S2 / Ct = 0.25   and   A2 / A t = 0.25

* Extra: s2 / A2 = 0.25Ct / 0.25At = Ct / At   : the ratio of the arc length of a (area) sector, to the corresponding area
                                                          of that sector, is equal to the same ratio of the circumference and
                                                          corresponding area of the circle. The circumference of a circle is also
                                                          the largest arc length of a circle. Therefore, when the ratio of s1 and
                                                          A1 are also considered:

$$\frac{s2}{A2} = \frac{s1}{A1}$$ therefore: $$\frac{s1}{s2} = \frac{A1}{A2}$$: the ratio of are lengths of a sector of a circle is equal
                                                  to the corresponding ratio of their sector areas

   A1 = (s1 / s2) A2   : sector area A1, this is the cones lateral (or "side") surface area.
   A2 = (s2 / s1) A1   : sector area A2

Expressing the above (*) into a more generalized formula:

From: s2 / A2 = Ct / At   : At = total area of a circle = Ac, mathematically:

$$\frac{\text{arc length of circle segment}}{\text{circumference of circle}} = \frac{\text{area of circle segment}}{\text{area of the circle}} = \frac{s2}{Ct} = \frac{A2}{At} = \frac{\text{sector angle}°}{360°}$$

We find that the ratios of arc lengths (s) is equal to the ratio of their corresponding sector areas (As). Arc lengths and sector areas are proportional.

From these equations, other helpful equations can be derived:

s2/A2 = Ct/At

A2 = (s2/Ct) At   : A2 is a sector area of a circle , and in general for any circle where Ac = At :

sector area = As = (S / C) Ac   : As = sector area and S = arc length of the corresponding sector area,
                                              Ac = total area of the circle , C = circumference or arc length of the entire circle.

or=: As = S (Ac / C) , substituting the known formulas for these variables:

sector area = S (((pi)r^2) / ((2)(pi)(r))) = S (r/2) = S (0.5r) =

sector area = 0.5 S r   : SECTOR AREA OF A CIRCLE , with units of a square unit of the length units used.
                                         (r) = radius of the circle , S = arc length of the sector area
                                         Another derivation here that is easy to remember is: 2As = Sr

As a check on this formula, if the are length is equal to the circumference:

sector area = 0.5 C r = $\frac{C r}{2}$ = $\frac{2(pi) r r}{2}$ = (Pi) r^2   : as expected since the corresponding sector area is that of the
                                                                                   entire circle

A METHOD TO REMEMBER π AND EPSILON (e)

Below is a non-mathematical method to obtain the first ten digits of π = "Pi". To do so, you must first know that it is approximately equal to a value of 3.0 and that it is slightly higher in value, say someplace between 3.0 and 4.0, but closer to 3. The next closest integer value greater than 0 is 1, hence we now have:

3.1 , For the third digit, add the first two digits to produce 4:
3.14 , The next integer value greater than the digits of 14 is 15:
3.1415 , Summing the last digits of 14 and 15 we have 9:
3.14159 , Summing the first two digits of the 14 and 15 we have 2:
3.141592

These last two digits above could have also been found by summing 14 and 15, producing 29, and then reversing these digits to have 92.

So far, we have used the integers: 1, 2, 3, 4, and 5, but have not used 6 yet, and this is also the sum of the 1, 4, and 1 digits, or the next number after 5:

3.1415926 , The next two descending integers after 6 are 5 and then 4:
3.141592654 π = (pi) rounded to ten digits (or 9 decimal places)

As a check, the sum of the first ten digits of π is 40. In this close approximation, there was no digit used here with a value of 7 or 8.

In a similar manner, below is a method to obtain epsilon (e) to 10 digits. First you must know that epsilon is approximately equal to 2.7 and is slightly larger in value. Given 2.70, the next higher integer value greater than 0 is 1:

2.71 , Summing the digits of 7 and 1, we have 8:
2.718 , The first and the fourth digits are 28:
2.71828 , Repeating the last four digits of 1828 we have:

2.718281828 : Epsilon (e) to ten digits (or 9 decimal places)

As a check, the sum of the first 10 digits of (e) is 47. The digit values used are: 1, 2, 7 an 8.

Here is a mnemonic (a shorthand or a way to memorized) where the number of letters for each word corresponds to a digit value of the number (e):

2 . 7 1 8 2 8 1 8 2 8 4
"If natural e constant is pleasing, a circular Pi inspires math"

Some other options or choices: 8 : pleasing , tasteful, fruitful, delivers, flattens, embraces, inspires
4 : food, good, love, math

$\sqrt{0.5}$ = $\sqrt{50}$ $\sqrt{10\text{^-}2}$ And the estimate of the root is therefore:

$\sqrt{0.5}$ ~= 7 (10^-1) = 0.7 : checking, (7)(7) = 49 , 49 (10^-2) = 0.49
or: (0.7)(0.7) = 0.49 , (almost 0.5)

Ex. $\sqrt{0.00267}$ = $\sqrt{26.7}$ $\sqrt{10\text{^-}4}$

And the estimate of the root is therefore:

$\sqrt{0.00267}$ = about: 5 (10^-2) = 0.05 : (5)(5) = 25 which is almost 26 , and
(0.05)(0.05) = 0.0025

Iteration	Result = Rn+1
1	3.166666666
2	3.162280702
3	3.16227766
4	3.16227766

For the fourth iteration, since there were no new changes in any of the digits, we can assume that this is the accurate or correct (at least for the limited number of digits, or precision shown) square root.

Using a similar method to finding square roots, cube and even higher integer roots can be found.

A description of the cube root successive approximation formula is: The coefficient of the first term in the numerator is 1 less than the indicated root (x), and the denominator of the second term in the numerator is also raised to an indicated power of 1 less than the indicated root.   The entire numerator is divided by a value equal to the indicated root. Summarizing this into a general formula for iteration:

The (x) root of N is expressed as $R = \sqrt[x]{N}$, and R can be calculated using:

$$R_{n+1} = \frac{(x-1)R_n + \frac{N}{(R_n)^{(x-1)}}}{x}$$

: **A FORMULA FOR CALCULATING ANY INTEGER ROOT**
Here x = indicated root , such as 2, 3, etc.
The appendix section of this book contains a computer program that utilizes this formula to find any root or power.

Ex. For the 3rd, or cube root of N, use:

$$R_{n+1} = \frac{2R_n + \frac{N}{(R_n)^2}}{3}$$

Not only can this general formula be applied for solving integer roots, but it can just as easily be applied for solving for a root or power of any real number. The appendix section of this book contains a computer program using this method to solve for a root. Real is a general mathematical name given to all the integers and mixed number values which are values with a possible whole portion and-or a possible fractional ("decimal") portion whose digits terminate at some point, or where some or all of its digits repeat endlessly to be classified as a "rational" (real) number. Any rational (real) number can be mathematically expressed as the division of two integers. "Irrational" (real) numbers are values that are not rational, such as "pi" and roots where the decimal portion is non-ending and non-repeating. Real numbers or values are sometimes called "floating (adjustable, position) point" values in computer programming, where point essentially means the decimal point that separates the whole portion from the fractional portion that is less than 1.

Ex. Here is how to find the **2.7**th root of N:

$$R_{n+1} = \frac{1.7R_n + \frac{N}{(R_n)^{1.7}}}{2.7}$$: For this example: (x-1) = (2.7 - 1) = 1.7

However, as you can now see, you must raise Rn to a non-integer power. Computer languages and machines, or scientific calculators, can handle such computations easily. It appears almost impossible to calculate by hand, but there are several methods. Here is a simple method besides the others presented in this book:

operand of the last operation entered. Integer powers can be easily found this way,and without the need for entering a factor (here the base of the power) repeatedly. For example, to calculate N^3, you would first enter N x N = to get N^2 (actually on many calculators you can also simply enter: N [x] [=], that is, first enter the number, then press the multiplication or "times" button, and then press the "equals" buttons to get the square of N, and this is of great aid when N has many awkward digits). Pressing the [enter] key or [=] again will yield N^3, pressing [enter] again will yield N^4, and finally pressing [enter] again will yield N^5 and so on.

Here is another note on solving for integer powers using repeated multiplication. The (portable or "hand-held") calculator is a limited precision device, that is, it is built to display only a specific number of maximum digits (usually 8 or 10). Any least significant values that cannot be entered or displayed are lost. However insignificant those digits are, it can eventually result in less accuracy in the displayed results. For example, if you have an 8-digit calculator, we know that multiplying two 8-digit operands usually yields a 16 or 17 digit product. Since the calculator can only enter, process and display 8 digits, the other 8 or 9 least significant digits are essentially lost and not displayed on the calculator. If this result is used "as is" for an operand of further operations, errors or differences from the actual or true values can possibly start to "show up" beginning in the least significant digits of the result(s) displayed. These errors can grow in value, and become a significant error or difference from the true value, when these values are used for operands of further operations. For most practical cases, where say being accurate to 3 to 6 decimal places or a "thousandth", this will not be of a concern. In general, when the calculator returns (ie., the result or output) a value of one of the function key operations such as for the square root of the operand, that value is usually accurate for all of the digits displayed except perhaps the last digit if it was rounded. The calculator generally does not know if any entered and-or reused is accurate or not, and essentially considers any entered and-or reused value as a user desired operand.

The same formula for roots can be applied for calculating any power of N. You might ask what root of N results in a value equal to a certain indicated power of N? The answer is easily determined if you express this question in a mathematical form as:

$$x\sqrt{N} \;=\; N^y$$

Solving for (x), the indicated root, by placing the radical into exponential form:

$$N^{(1/x)} \;=\; N^y \qquad \text{equating the exponents by observation (or by taking}$$
$$\text{the log of both sides and dividing by (log N)):}$$

$$\frac{1}{x} \;=\; y \qquad \text{solving for x:}$$

$$x \;=\; \frac{1}{y} \qquad \text{: we see that the index and exponent of N are reciprocals of each other}$$

For example, to calculate the second power of N using the root formula shown, you would calculate the 1/2 = 0.5 root of N since $0.5\sqrt{N}$ = N^1/0.5 = N^2 .

It is possible to calculate odd roots, such as cube roots, by using square roots. Here is a method:

$$R \;=\; 3\sqrt{N} \qquad \text{: R = cube root of N, therefore (raising each side to the 3rd power):}$$
$$R^3 \;=\; N \qquad \text{after dividing each side by R:}$$
$$R^2 \;=\; \frac{N}{R} \qquad \text{taking the square root of both sides:}$$

$$R \;=\; \sqrt{\frac{N}{R}} \qquad \text{expressing this with successive approximation notation:}$$

A GENERAL METHOD FOR FINDING ANY POWER

Previously, a method was shown to calculate a power of a base value where the exponent had only one digit in the fractional portion. Here is a similar method which is generalized to calculate any possible power. Though you may generally not use these types of methods to find a power or a root, the knowledge or "know-how" aspect could have potential use in various mathematical situations and understanding.

Consider this:

$N = X^{a.bcd}$ "factoring" the right hand side:

$N = X^{(a + 0.b + 0.0c + 0.00d)} = X^a \quad X^{0.b} \quad X^{0.0c} \quad X^{0.00d}$ OR:

$N = X^a \quad (X^{0.1})^b \quad (X^{0.01})^c \quad (X^{0.001})^d$

$N = X^a \quad X^{0.1b} \quad X^{0.01c} \quad X^{0.001d}$

$N = X^a \quad X^{(b/10)} \quad X^{(c/100)} \quad X^{(d/1000)}$ expressed in radical notation:

$N = X^a \quad (\sqrt[10]{X^b}) \quad (\sqrt[100]{X^c}) \quad (\sqrt[1000]{X^d})$ or for the purpose here:

$N = X^a \quad (\sqrt[10]{X})^b \quad (\sqrt[100]{X})^c \quad (\sqrt[1000]{X})^d$:GENERAL METHOD

$N = X^a \quad (\sqrt[10]{X})^b \quad (\sqrt[10]{\sqrt[10]{X}})^c \quad (\sqrt[10]{\sqrt[10]{\sqrt[10]{X}}})^d$: a working method

The best way to solve the last equation is to first find the 10th root of X and then you can reuse this result to find the 100th root of X and so on since:

$$\sqrt[10]{\sqrt[10]{X}} = \sqrt[100]{X}$$

$$\sqrt[10]{\sqrt[100]{X}} = \sqrt[1000]{X}$$

If the exponent is negative, the simplest method to evaluate the power is to first find the power with an equivalent positive exponent and then take the reciprocal of this power since:

$N^{-x} = \dfrac{1}{N^x}$, also taking the square root of both sides: $\sqrt{N^{-x}} = \sqrt{\dfrac{1}{N^x}}$ or$= \dfrac{1}{\sqrt{N^x}}$

If the base of the indicated power is negative, things can get somewhat cloudy. For example:

$(-2)^2 = (-2)(-2) = +4$

$(-2)^3 = (-2)(-2)(-2) = -8$

So far so good, you can check these with a (scientific) calculator, and get the same results. By observing the above examples, it seems quite logical that if you make the exponent equal to a value anywhere between 2 and 3 then you should also be able to have an answer (the evaluated power of the base -2). After all, for example, $+2^2=4$, $+2^{2.5}=5.6585...$, and $+2^3=8$. Now take notice:

from +1 on the real number line.   The square root of -1 is called the **imaginary unit** (which we can imagine as equal to some constant value) and is often given the identifying symbol of: (**i** or j) so that you can work with **i**maginary numbers in a manner similar to that with working with typical algebraic variables. Therefore:

$2 \sqrt{-1}$ = 2i   : sometimes the letter j is used instead of i, for example 2i = 2j

What is the connection between real numbers and imaginary numbers? As already shown above, a real number can be a coefficient of an imaginary number. Another important fact is that the product of two imaginary numbers is a real number. First consider the square roots of -1:

$\sqrt{-1} \ \sqrt{-1}$ = -1   or:

i   i   = i^2 = -1   : -1 is a real number. Negative numbers are 180° from the
                                  positive side of the real axis, and this could be thought of
                                  as being in the opposite direction. Imaginary numbers on
                                  the imaginary number line could be thought of as being in
                                  a sideways (90°, right angle) direction from the real number
                                  line.

Ex. (2i)(5i) = (2)(5)(i^2) = 10(-1) = -10   : multiplying by i^2 is the same as multiplying by (-1)

Also note: i^3 = i^2 i^1 = -1i = - i   : Numbers 180° from positive side of imaginary axis.
                                                                      This is equivalent to a 270° rotation of a number
                                                                      from the positive real number axis.

          i^4 = (i^2)^2 = (i^2)(i^2) = (-1)(-1) = +1   : Numbers on the positive side of the real axis.

A **complex number** is a number that is part real and part imaginary. You might have already encountered a root of a quadratic equation that is a complex number. The pseudo and formal algebraic forms of a complex number are:

    (real + imaginary)   =   (a + bi)   : Basic form of a complex number. (a) represents the real
                                                          part, and (b) is a real number numerical coefficient of the
                                                          entire imaginary number part. We see that a complex
                                                          number is expressed as the sum or union of its real and
                                                          imaginary part,

Ex.   5 + 2i   : 5 is the real number, and 2i is the imaginary number. Commonly said as 5 is the real part,
                        and 2 is the imaginary part.

The sum or product of a complex number and its corresponding conjugate (ie., with a "sign change") complex number is a real number.

Ex. (5 + 2i) + (5 - 2i)   removing parentheses or grouping symbols:
       5 + 2i + 5 - 2i   after combining like terms:
            5 + 5
              10

Ex. (3 + 5i)(3 - 5i)   : the product of a complex number and its conjugate.   Distributing:
     (3)(3) + (3)(-5i) + (5i)(3) + (5i)(-5i)
      3^2 + (-15i) + (15i) + (-25i^2)   since i^2 = -1:
        3^2 - 15i + 15i + (-25(-1))   combining like terms:

A METHOD TO FIND A LINEAR (LINE) EQUATION GIVEN A SET OF POINTS

There are many methods to find a (representative) linear equation given a set of points or data that appear to be linear and could be graphed using a single (representative or approximate) line . A simple method is given below that will provide very reasonable results. This method is what can be called an "upper and lower average method". Many "scientific" calculators have some statistical (the key is often labeled as "stat") functions such as averaging a series of input data values.

If there is an excessive amount of data values or points to consider, then simply choose a manageable and representative sample amount to use, and in particular, discard any extreme values that appear as very high or low as compared to most of the data values since these "extreme [too high,or too low] and-or erroneous values" can affect the accuracy and effectiveness of the entire representative line being created. Sometimes it may help to visually plot the points so as to determine if there area any "extreme" values.

Another quick test to determine if there is a linear or linear-like relationship is to take some of the data values (to be plotted or graphed as points) and find a rough or basic slope (m) value by dividing a few dependent (y) values by its corresponding independent (x) value of that point, and then using this (roughly constant) value for comparing and determining if a data value or point is "out of (the linear or line) range", possibly due to some anomaly or measurement error. These "extreme values" can then be discarded and-or eliminated from this line analysis.

1. Given a set of acceptable points (that are graphed and appear to be linear in nature), find the average of the x coordinates of all the points, which can be expressed as x with a short line over it, and then find the average of all the y coordinates of all the points. An **average value** of a set of values can be found by summing up all those values and then dividing by the total number of the values used or included in that sum.

$$Pa = (\ \bar{x}\ ,\ \bar{y}\ ) \qquad : \text{main average point, at the average x value and}$$
average y value of the given points, the more the better the line equation will represent and-or predict the data.

On a point plotting graph and-or coordinate (location) system of points, you may wish to draw a horizontal and vertical line at this point (Pa). All points to the right of Pa will have greater x values and will be used to find the upper-x average value. All points to the left of Pa will have lower (x) values and will be used to find the lower-x average value. This same reasoning will be used to find the upper and lower y average values.

2. Given the set of all points whose coordinates are less than the average, find the lower-x and lower-y average point:

$$Pl = (\ \bar{x_l}\ ,\ \bar{y_l}\ ) \qquad : \text{average point of points lower than the main average}$$

3. Given the set of all the points whose coordinates are greater than the average, find the upper-x and the upper-y average point:

$$Pu = (\ \bar{x_u}\ ,\ \bar{y_u}\ ) \qquad : \text{average point of points greater than the main average}$$

We now have three points on the line: Pa , Pl , Pu These points can be connected to display the average line. To make an equation of this line so as to plot any general point on that line given an (x) value, we need to write the equation for that specific line, and we will need to find the slope value (m) of that line, and the y-axis intercept point: Pi (0 , b)

4. With the upper and lower average points, these two points can be used to determine the slope (m) of the corresponding line of which those points are on:

A DERIVATION OF SIN ¢ SERIES

In order to give you validity and confidence in using the SIN ¢ and other similar series, the fundamental steps of its' derivation are presented below. It is recommended to first view the previous topic of: Understanding Basic Calculus.

In theory, many functions (the mathematical relationship of variables) can be expressed as a series of terms. For a very simple example, consider this:

$y = f(x) = 5x$ Expressing the function 5x using repeated addition which is a series of terms:
$y = f(x) = 1x + 1x + 1x + 1x + 1x$: You can solve for the value of y = 5x by using this sum of terms.
                       Expressing this series of terms using "summing notation":

$$y = f(x) = \overset{5}{\underset{1}{S}} \; 1x$$: sum the terms, where each is term is 1x, and for a total of 5 terms of these terms

Here is the general format for a **Power Series** representation of a function:

$$y = \text{a function of } x = f(x) = \sum_{n=0}^{n=\infty} Cn \; x^{\wedge}n$$

    : General format for terms of a power series.
     C or c are constants that are specific to the function.

     Sometimes "inf." is used in place of the infinity symbol: ∞ ,
     n is a variable, here starting at 0, and is to be incremented by 1 for each and every successive term. For Cn, n is a subscript corresponding to the term number less 1. For example: c0 = "constant zero", which is actually the first constant in the series.   c1 = "constant 1".

Expanding some of the terms of the general power-series format shown above:

$y = f(x) = c0 \; x^{\wedge}0 + c1 \; x^{\wedge}1 + c2 \; x^{\wedge}2 + c3 \; x^{\wedge}3 + \ldots$: Power Series (in, or of variable x)
                             general format. $c0 \; x^{\wedge}0 = c0 \; (1) = c0$

Here is a simple verification of the power series, however, the terms used are finite (not infinite) in number since you should expect a definite result for an algebraic function (as opposed to the non-algebraic functions such as the trigonometric functions):

$y = f(x) = (x + 2)^{\wedge}3$

$y = f(x) = (x + 2)^{\wedge}3 = (x + 2)^{\wedge}2 \; (x+2)^{\wedge}1 = (x^{\wedge}2 + 4x + 4)(x + 2) = x^{\wedge}3 + 6x^{\wedge}2 + 12x^{\wedge}1 + 8$

Arranging these terms into an ascending series of expressed or indicated power of x:

$y = f(x) = 8 + 12x^{\wedge}1 + 6x^{\wedge}2 + x^{\wedge}3$: note the ascending exponents or powers of the variable

Note that the first constant (c0) can be evaluated by setting the value of the variable (x) to 0, effectively eliminating all the terms containing that variable (x), after all, 0 is often a possible value for any variable in a function. In the above expression, C0 = 8. In mathematical notation:

$c0 = f(0)$: f(0) = evaluate the function after setting the value of the variable (here, x) to 0.
         This value is also the "y-axis intercept" of the curve of the equation, and where the corresponding value of (x) is 0.   y intercept point = point(x, y) = p (0 , 8).

$y = f(x) = ax + b$: basic linear equation and function

$y(0) = f(0) = a(0) + b = b$: evaluating the function with (x) set equal to 0, and this is actually the y-axis
                                       intercept where x=0. point(x , y) = p(0 , b)

The first derivative is:

$y' = f'(x) = (1)ax^0 = a(1) = a$

The second and all further derivatives are 0, that is, the derivative of any constant (such as (a), or even 0) is equal to 0. If you consider the line y=a, where (a) is a constant, it is a horizontal line which has no slope, hence 0. After substituting this into the general Maclaurin Series:

$y = f(x) = b + ax^1 + 0 + 0 \ . \ . \ .$

$y = f(x) = b + ax$: This is actually the same original function
                                     except that the terms are rearranged.

Now consider the basic form of a quadratic equation:

$y = f(x) = ax^2 + bx + c$: basic quadratic equation, or by rearranging terms for an ascending
                                         order of increasing powers of the variable (x):
$y = f(x) = cx^0 + bx^1 + ax^2$: in relation to the general power series, we see
                                         that constant c0 = c, c1 = b and c2 = a
$y(0) = f(0) = a(0^2) + b(0) + c = c$

The first derivative of this quadratic equation is:

$y' = f'(x) = 2ax^1 + b$: this is a linear equation

The second derivative of the quadratic equation, and which is the derivative of this linear equation is:

$y'' = 2ax^0 = 2a$: this is a constant, and this is common for the derivative of linear or line equations.

The first derivative where x is set equal to 0 is:

$y'(0) = f'(0) = 2a(0^1) + b = b$: $0^1 = 0$

The second derivative where x is set equal to 0 is:

$y''(0) = f''(0) = 2a(0^0) = 2a(1) = 2a$: Note that: $0^0 = 1$

$0^0 = 1$, and to verify this, remember that any value raised to the 0 power is equal to 1. The value can be very huge or infinitely so small that it is practically considered as equal to 0. Be sure to also remember $0^1 = 0$, that is, any value raised to the first (1) power is equal to that value. It does seem natural or intuitive to think that a value with a higher exponent should result in a greater value than the base value (here 0). 0^1 might be thought of as being greater than 0^0, but it is not. Consider that a number very close to 0, or infinitely small, raised to the 0 power is still equal to 1. For example: $(0.000,000,000,001)^0 = 1$. One point argued about this is "nothing (0) raised to the nothing (0), or no (0) power should still be equal that nothing (0)". But then with that same mindset, in division, a value that is divided by nothing (0) should still be equal to that undivided value since no (0) division has actually taken place. Formally, division by 0 is undefined (not defined, having no definite or specific result, hence it is considered an error to even try to divide by 0).

COS ¢ SERIES

$$\text{COS } X = \sum_{n=0}^{n=\infty} (-1)^n \frac{X^{(2n)}}{(2n)!} \qquad : X \text{ is a radian angle value}$$

Expanding the terms:

$$\text{COS } X = \frac{1}{0!} - \frac{X^2}{2!} + \frac{X^4}{4!} - \frac{X^6}{6!} + \frac{X^8}{8!} - \frac{X^{10}}{10!} + \frac{X^{12}}{12!} - \ldots$$

Simplifying the factorials in the denominators:

$$\text{COS } X = 1 - \frac{X^2}{2} + \frac{X^4}{24} - \frac{X^6}{720} + \frac{X^8}{40,320} - \frac{X^{10}}{3,628,800} + \frac{X^{12}}{479,001,600} - \ldots$$

The results to be expected are similar to those of the SIN X series. When the angle decreases, the greater the accuracy achieved for a given number of terms. When X=1 (about 57°) you can expect an accuracy of 6 digits when using only 6 of the terms shown. Note that when the angle X is 0, that sin 0 = 0, and cos 0 = 1, and hence as to why the cos x series has the initial "1 term". The main difference between the sin and cos series is in the use of all odd or even powers of (x).

Notice the last term given for the series, if you only have an 8 digit calculator, you cannot enter the 9 digit denominator of this fraction. The method to overcome this limitation is to factor the denominator, resulting in numbers (factors) with less digits. For a quick method to simplify or process the resulting fraction, the numerator will be divided by one of the factors in the denominator, and then the resulting quotient will be divided by the other factor in the denominator. Perform the division as shown below:

$$\frac{X^{12}}{479,001,600} = \frac{X^{12}}{(10)(47,900,160)} = \frac{\frac{X^{12}}{47,900,160}}{\frac{10}{1}} \qquad \text{or:} \qquad \frac{\frac{X^{12}}{10}}{\frac{47,900,160}{1}}$$

You can also evaluate COS X from one of the other trigonometric identities previously shown, for example:

$$\text{COS } X = \frac{\text{SIN } X}{\text{TAN } X} = \sqrt{1 - \text{SIN}^2 X}$$

The series above for COS X is derived from the SIN X series since the derivative of SIN X and its equivalent series is the COS X series. Taking the derivative of the expansion of the SIN X series term by term will yield the COS X series:

$$\text{COS } x = \frac{d (\text{SIN } x)}{dx} = \frac{1 x^{(1-1)}}{1} - \frac{3x^{(3-1)}}{3!} + \frac{5x^{(5-1)}}{5!} - \ldots$$

$$\text{COS } x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

As previously indicated, taking the derivative of this series, term by term, will yield the negative of the SIN series:

man-made values, whereas radian units are essentially defined naturally with the dimensions of any circle.

$$\text{TAN } X = \cfrac{X}{1 - \cfrac{X^2}{3 - \cfrac{X^2}{5 - \cfrac{X^2}{7 - \cfrac{X^2}{9 - \cfrac{X^2}{11}}}}}}$$: TAN X continued fraction

. . . (and so on)

The simplified form of all the 6 "tiers", "levels" or divisions shown is:

$$\text{TAN } X = \frac{10395X - 1260X^3 + 21X^5}{10395 - 4725X^2 + 210X^4 - X^6}$$: **A FORMULA FOR EVALUATING TAN ¢**

With the expression above, you can expect almost 7 decimal places for an angle of 70° (about 1.2217 radians). Expect 9 decimal places for an angle of 57.3° (about 1 radian), and more for lower angles when using the same number of "tiers" or "levels" (ie. denominators or divisions).

Note for example that you can quickly calculate powers of a value (here, such as X) by "reusing" precalculated values:

$5^4 = (5^2)(5^2) = (25)(25) = 625$ then:
$5^5 = (5^4)(5^1) = (625)(5) = 3{,}125$

This can be algebraically expressed as: $X^n = (X^{(n-1)})(X^1) = X^{(n-1)} X$

For angles greater than 45°, in particular those closer to 90°, you can use the derivation below for improved accuracy.

From the trigonometric co-functions of angles A and B of a right triangle:

$$\text{TAN } A = \frac{\text{SIN } A}{\text{COS } A} = \frac{\text{COS } B}{\text{COS } A} = \frac{\text{COS } B}{\text{SIN } B} = \frac{\text{SIN } A}{\text{SIN } B} = \text{COT } B = \text{COT } (90° - A) = \frac{1}{\text{TAN } B} = \frac{1}{\text{TAN } (90° - A)}$$

, and 90° = 1.570796327 radians = π/2 radians

$$\text{TAN } X = \frac{1}{\text{TAN } (\pi/2 - X)} = \frac{1}{\text{TAN } (1.570796327 - X)}$$: X is a radian angle

Here is an identity and a short table which will allow you to calculate TAN X by evaluating the tangent of a smaller angle, in particular as shown below, an angle of 10° or less where you can expect an accuracy of 9 or more significant digits when using the simplified continued fraction shown above:

$$\text{TAN } (¢1 + ¢2) = \frac{\text{TAN } ¢1 + \text{TAN } ¢2}{1 - (\text{TAN } ¢1)(\text{TAN } ¢2)}$$: TANGENT OF THE SUM OF TWO ANGLES FORMULA

A small table is presented below to aid your calculations. The second column contains the radian equivalent of the degrees angle. The appendix contains a table with more values entered.

A DERIVATION OF THE ARCSIN SERIES

To give you validity and confidence using the ARCSIN X and other similar series, the fundamental steps of the derivation of the ACRSIN X series are presented below.

First, the Binomial Series is a series expansion for expressions of the form: (x+1)^a , or its equivalent: (1+x)^a. We see that this expression is a power of a binomial or two terms expression. The Binomial Series can be derived from a Maclaurin Series expansion of (1+x)^a, or from the Binomial Formula that is discussed ahead. Remember, binomial means two terms, and for the binomial series below, one term must be the value of 1. The binomial series will be used further ahead in this derivation of the series for ARCSIN X.

Binomial Series general format:

$$(1 + x)^a = 1 + \frac{ax^1}{1!} + \frac{a(a-1)x^2}{2!} + \frac{a(a-1)(a-2)x^3}{3!} + \frac{a(a-1)(a-2)(a-3)x^4}{4!} + \ldots \qquad : \text{Binomial Series}$$

Here is a simple example to verify the Binomial Series by solving (1 + 4)^3. First, this could easily be solved by using the order of operations or by "extending" the binomial by using distribution. The result is 5^3 = 125. Letting a=3 and x=4:

$$(1 + 4)^3 = 1 + \frac{(3)4^1}{1} + \frac{(3)(2)4^2}{2} + \frac{(3)(2)(1)4^3}{6} + \frac{(3)(2)(1)(0)4^4}{24} + \quad \text{(all further terms will also include the factor of 0)}.$$

$$(1 + 4)^3 = 1 + 12 + \frac{96}{2} + \frac{384}{6} + 0 + 0 + \ldots$$

$$(1 + 4)^3 = 1 + 12 + 48 + 64$$

$$(1 + 4)^3 = 125$$

The derivative of ARCSIN X is: $\frac{1}{\sqrt{1 - X^2}}$ or= $(1 - X^2)^{(-1/2)}$, and this is derived below:

Since ARCSIN X is the corresponding angle (¢) associated with the sine value that is equal to X, the derivative of ARCSIN X is the instantaneous rate of change of the angle (¢) with respect to the SIN value = X, of that angle. It was previously shown that the derivative of SIN ¢ with respect to the ¢ is equal to COS ¢. If we let X represent the angle here, this can be expressed as:

$$\frac{d(SIN\ X)}{d(X)} = COS\ X$$

There is a simple way to verify this: notice on the graphing of the SIN X curve, that at 0°, the slope of the curve, and hence derivative, appears to be 1 (such as for the tangent value of 45°), and that is what COS 0° equals. At 90°, the slope of the (now horizontal) curve is obviously 0, and that is what COS 90° equals. This leads to the fact that the COS and SIN curves are essentially a derivative curve of each other. The actually numeric values of the SIN and COS derivatives are also negative in value to each other, and this can bee seen on the curves where as one curve is sloping upward with a positive value, the other curve is sloping downward with a negative value for a given angle in question.

Given ¢ = Y = ARCSIN X = SIN^-1 X, therefore, X = SIN Y

We are to find $\frac{dY}{dX}$: the derivative of Y with respect (in reference) to X.
Or in common words: The derivative of the angle with
respect to the SIN of that angle: d¢ / d SIN ¢

$$1 = \sqrt{1 - \sin^2 \phi} \; \frac{(d \arcsin x)}{(dx)} \qquad \text{since } x = \sin \phi:$$

$$1 = \sqrt{1 - x^2} \; \frac{(d \arcsin x)}{(dx)} \qquad \text{which can be expressed mathematically as:}$$

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1 - x^2}} = \frac{1}{(1 - x^2)^{(1/2)}} = (1 - x^2)^{(-1/2)}$$

Note also that the derivative of this trigonometric-transcendental function is a definite algebraic function.

Expanding the ARCSIN X derivative using the Binomial Series, letting $a = -\frac{1}{2}$, and $x = -X^2$:

$$1 + \frac{(-\frac{1}{2})(\frac{1}{1})(-X^2)^1}{1!} + \frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{1}{1})(-X^2)^2}{2!} + \ldots$$

Note: $(-X^2)^2 = (-1X^2)(-1X^2) =$
$(-1)(X^2)(-1)(X^2) = +1 X^{(2+2)} = X^4$

$$1 + \frac{X^2}{(2)} + \frac{3X^4}{(2)(4)} + \ldots$$

Integrating (essentially the reverse of the derivative process, that is, finding the anti-derivative or function that the derivative was derived from) this term by term and keeping the products in their factored form so that a mathematical pattern can be seen, we get the actual function from which the derivative given above was derived:

$$ARCSIN\ X = \frac{X}{1} + \frac{1X^3}{(2)(3)} + \frac{3X^5}{(2)(4)(5)} + \ldots$$

$$ARCSIN\ X = X + \frac{1X^3}{3!} + \frac{3X^5}{5!} + \ldots \qquad : ARCSIN\ X\ SERIES$$

Since X = COS

ARCCOS X = ARCSIN $\sqrt{1 - X^2}$

If you want to use the ARCTAN X series to evaluate ARCCOS X:

From: TAN ¢ = $\dfrac{\sqrt{1 - COS^2\ ¢}}{COS\ ¢}$

Therefore: ARCCOS X = ¢ = ARCTAN (TAN ¢) = ARCTAN $\dfrac{\sqrt{1 - COS^2\ ¢}}{COS\ ¢}$

Since X = COS ¢

ARCCOS X = ARCTAN $\dfrac{\sqrt{1 - X^2}}{X}$

Here is a continued fraction developed by a mathematician named **Johann Lambert** (1728-1777), from Switzerland-Germany-France**,** so as to evaluate ARCTAN X. The result is a radian angle. The result here is much easier to obtain than by using the series previously shown:

$$\text{ARCTAN X} = \cfrac{X}{1 + \cfrac{1X^2}{3 + \cfrac{(2X)^2}{5 + \cfrac{(3X)^2}{7 + \cfrac{(4X)^2}{9 + \cfrac{(5X)^2}{11}}}}}}$$: **ARCTAN continued fraction**

This continued fraction is similar to the one for TAN X. This fraction also converges slower, but it is still practical. The pattern is not too difficult to recognize. Each denominator is the next odd integer plus a term that contains the square of the product of the next integer and X. This method is good for values of X less than or equal to 1 (X<=1). The lower X is, the less "tiers" or "levels" (essentially divisions) of the continued fraction required for a specific number of correct digits.

Here is a simplified form of 7 levels of the continued fraction:

$$\text{ARC TAN X} = \frac{387660\ X + 300680\ X^3 + 50332\ X^5}{387660 + 429900\ X^2 + 116100\ X^4 + 8100\ X^6}$$

To give you an idea of the results using the above expression, when X is close in value to 1, you can expect 3 decimal places of accuracy. When X = 0.5, you can expect 5 decimal digits of accuracy, and when X = 0.2, you can expect almost 9 decimal digits of accuracy. Using this expression, all results will be slightly lower than the true value.

Here is a method you can use when x>1:

From the trigonometric co-functions of the complementary angles being equal:

$$\text{TAN A} = \text{COT B} = \frac{1}{\text{TAN B}}$$: where (TAN B) has a value > 1, then COT B and TAN A will be < 1

And from the sum of complementary angles:

B = 90° - A   in terms of radians:

$$B = \frac{\pi}{2} - A$$

B = 1.570796327 - A
B = 1.570796327 - ARCTAN(TAN A)
B = 1.570796327 - ARCTAN(1/ TAN B)   or in a more general form:

¢ = 1.570796327 - ARCTAN(1/ TAN ¢)

THE BASIC FORMAT OF A C LANGUAGE COMPUTER PROGRAM

Below is a basic format of a c language program. The next program after this one contains some more notes about the C programming language. This book is not about C programming, but contains some helpful math computer programs written in the C programming language, and so some minimal discussion about it is needed for people who have not seen and-or used it before. The reader is also encouraged to obtain some entry level C programming books if they wish to know more and to create and-or edit their own programs which can do something useful such as calculate a value. The program addnumbers.c is shown further below and it utilizes this basic and common format discussed next.

```
/*----------------------------------------------------------------------------------------
    myprogram.c

    C source and text code, and saved as myprogram.c

    (c) date,  name    Ex.  (c)  June 10, 2021 by J.P.A.

    Compiler and-or version used.

    Program description and or notes, edits and updates.

    Use a plain text editor to write a C program. This source
    code which is quite readable with practice, will be then
    converted to an .exe or executable code that the computer
    can utilize.  A C language compile program will do this conversion.

  This program below shows a basic format of a C programming
   language program.  Every C program must start with the default
   starting or entry point that is called: main and this is actually
   the name of an instance of a C programming structure called a
   function.  C can have many functions that are called by their
   uniquely assigned  name such as:  addnumbers()  or
   get_text_string(), etc.  A function is usually terminated from
   being run or executed when it encounters the return keyword
   and program flow command.  The program will resume after the
   point where the function  was called.  Data can be sent to a
   function to work with, and a function can also create a value
   within it and return it to the calling point in a program.  Once a
   function is written, it can be used over and over millions of
   times and with different data values sent to it, accessed by it,
   calculated by it, and-or returned from it.  When a function and-or
   its program code is called to be  ran or processed, it will have a
   format of:

   function_name(arguments to be sent to the function separated by a comma);
   or simply:

   function();  if the function is void (ie., absent) of any arguments
   such as the value(s) of the variable(s) to be sent to it, and so as to be
   used by it.

   For the actual function program itself, its syntax will also include the
   data type of each of the arguments or variables sent to it, with each
   separated by a comma:
```

REM CurrentDiretory.bat to goto the current directory
cmd.exe
EXIT
Type the above batch commands into a text editor as a plain, pure textfile and save
and-or rename it with a .bat filename extension and not the .txt filename extension.

To see a list of files in a directory, enter DIR. To see a list of some or most
DOS commands, enter HELP . To get information on using a DOS command
enter the command name followed by /?. Books are available about creating
batchfiles of DOS commands. A batchfile is essentially a program written
with the DOS language (ie., commands).

Here is an actual basic C program example below that you may compile:
/*--
addnumbers.c

A program to add two user input numbers and displays the result or sum.

 (c) JPA 2021
  Version 1.0 , Compiled with the tinyC compiler.
---*/
void main(void)   /* : Required or default starting function of a C program. */
{   /* : The function body is placed within braces */
double number1=0.0;   /* : Declares a variable to hold a data type that
                                    is a double precision floating-point data value
                                    or number, and initialize or assign it with 0.0 */
double number2=0.0;   /* A float means floating-point, adjustable or movable */
                                /* decimal point location within the number */
double result=0.0;

printf("Input Number 1 : ");   /* : A statement to display a line of text.   */
scanf("%lf",&number1);   /* : Let the user input the value of number 1. */
                                  /*   Statements and other C programming
                                      structures are ended with a semicolon. */

printf("\nInput Number 2 : ");   /* : Display a line of text on the next line */
scanf("%lf",&number2);   /* : Let the user input the value of number 2 */
                                  /*   %f is a format specifier of the type of data
                                      that will be entered, and it will be stored
                                      in the address assigned to hold the data of
                                      the variable identified as number2.   */
/* To add explicit numbers or "hard-coded numbers" and-or variables with
   any value they may be, use the + = add operator. Use - to subtract.
   Use * to multiply. Use / to divide.   */

result = number1 + number2;   /* Add the value of number1 and number2
                                  and store the result in variable called
                                  result. */

/* Display the values and the result. Note below that %lf format specifier
can be used but %g is used to eliminate any possible trailing 0's from being
displayed. Each format specifier must have an argument associated with it,
and those are expressed one after the other with a comma. Format specifiers

A COMPUTER PROGRAM TO MAKE A TRIGONOMETRIC TABLE

```
/* -----------------------------------------------------------------------------------------------------------------------*/
TrigonometricTable.c

   This program displays the trigonometric values of angles from 0 to 90 degrees using the C-language
   (input) source code compiler's "built in", or prewritten mathematics trigonometric functions.  Programs
   are shown in this book as to how to create and-or modify your own "homemade" trigonometric and
   other basic math functions for use with a computer.  You may use a text editor to copy the source code
   of these programs, so as to enter it into a compiler program that converts it into to a machine or
   computer code ("running" or "executable" [.exe]) program which is basically a list of numbers which
   represent the data, instructions, and memory addresses that will be needed for that program.

   With a DOS compatible (PC) computer, you can redirect the screen output to a file for some programs.
   This method uses something such as shown here in symbolic form :  program_name.exe  >  textfile.txt

   (c) JPA July 20, 2017, March 2019

   The next program in this book briefly discusses some of the basics of the C computer programming
   language.
/*-----------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include "math.h"    /* Used to access C's built in trigonometric functions that were prewritten. */


/*-----------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
int n=0;

double degreeangle=0.0;
double radianangle=0.0;

long double sin1=0.0;    /* long double float or floating point values have more precision available than a double or float */
long double cos1=0.0;
long double tan1=0.0;

printf("\nDeg      Rad           Sin           Cos           Tan");

printf("\n\n");                                              /* display two blank spaces */

for(;n<=90;){

   if(n==90){  degreeangle=89.99999999999999; };  /* not generally needed, but keeps tan 90° = undefined,  from
                                         rolling over to some negative value with the programming language,
                                         here C, and remains a high value  */

   radianangle=degreeangle * 0.0174532925199433;  /* convert the degree angle to the equivalent radian
                                         angle measurement needed for the std. sin() in the C language  */

   sin1=sin(radianangle);  /* Calculate the sine of a radian angle using the ANSI standard C's built in math function sin() */
   cos1=cos(radianangle);  if(n==90){ cos1=0.0; };      /* to correct a slight error, garbage; a very small negative value */
   tan1=tan(radianangle);
```

33	0.57595865315813	0.54463903501503	0.83867056794542	0.64940759319751
34	0.59341194567807	0.55919290347075	0.82903757255504	0.67450851684243
35	0.61086523819802	0.57357643635105	0.81915204428899	0.70020753820971
36	0.62831853071796	0.58778525229247	0.80901699437495	0.72654252800536
37	0.6457718232379	0.60181502315205	0.79863551004729	0.75355405010279
38	0.66322511575785	0.61566147532566	0.78801075360672	0.78128562650672
39	0.68067840827779	0.62932039104984	0.77714596145697	0.80978403319501
40	0.69813170079773	0.64278760968654	0.76604444311898	0.83909963117728
41	0.71558499331768	0.65605902899051	0.75470958022277	0.86928673781623
42	0.73303828583762	0.66913060635886	0.74314482547739	0.90040404429784
43	0.75049157835756	0.6819983600625	0.73135370161917	0.93251508613766
44	0.76794487087751	0.694658370459	0.71933980033865	0.96568877480707
45	0.78539816339745	0.70710678118655	0.70710678118655	1
46	0.80285145591739	0.71933980033865	0.694658370459	1.0355303137906
47	0.82030474843734	0.73135370161917	0.6819983600625	1.0723687100247
48	0.83775804095728	0.74314482547739	0.66913060635886	1.1106125148292
49	0.85521133347722	0.75470958022277	0.65605902899051	1.150368407221
50	0.87266462599716	0.76604444311898	0.64278760968654	1.1917535925942
51	0.89011791851711	0.77714596145697	0.62932039104984	1.2348971565351
52	0.90757121103705	0.78801075360672	0.61566147532566	1.2799416321931
53	0.92502450355699	0.79863551004729	0.60181502315205	1.3270448216204
54	0.94247779607694	0.80901699437495	0.58778525229247	1.3763819204712
55	0.95993108859688	0.81915204428899	0.57357643635105	1.4281480067421
56	0.97738438111682	0.82903757255504	0.55919290347075	1.4825609685127
57	0.99483767363677	0.83867056794542	0.54463903501503	1.5398649638146
58	1.0122909661567	0.84804809615643	0.5299192642332	1.6003345290411
59	1.0297442586767	0.85716730070211	0.51503807491005	1.6642794823505
60	1.0471975511966	0.86602540378444	0.5	1.7320508075689
61	1.0646508437165	0.8746197071394	0.48480962024634	1.8040477552714
62	1.0821041362365	0.88294759285893	0.46947156278589	1.8807264653463
63	1.0995574287564	0.89100652418837	0.45399049973955	1.9626105055052
64	1.1170107212764	0.89879404629917	0.43837114678908	2.0503038415793
65	1.1344640137963	0.90630778703665	0.4226182617407	2.1445069205096
66	1.1519173063163	0.9135454576426	0.4067366430758	2.2460367739042
67	1.1693705988362	0.92050485345244	0.39073112848927	2.3558523658238
68	1.1868238913561	0.92718385456679	0.37460659341591	2.4750868534163
69	1.2042771838761	0.9335804264972	0.3583679495453	2.6050890646938
70	1.221730476396	0.93969262078591	0.34202014332567	2.7474774194546
71	1.239183768916	0.94551857559932	0.32556815445716	2.9042108776758
72	1.2566370614359	0.95105651629515	0.30901699437495	3.0776835371753
73	1.2740903539559	0.95630475596304	0.29237170472274	3.2708526184841
74	1.2915436464758	0.96126169593832	0.275637355817	3.4874144438409
75	1.3089969389957	0.96592582628907	0.25881904510252	3.7320508075689
76	1.3264502315157	0.970295726276	0.24192189559967	4.0107809335358
77	1.3439035240356	0.97437006478524	0.22495105434386	4.3314758742842
78	1.3613568165556	0.97814760073381	0.20791169081776	4.7046301094785
79	1.3788101090755	0.98162718344766	0.19080899537654	5.1445540159703
80	1.3962634015955	0.98480775301221	0.17364817766693	5.6712818196177
81	1.4137166941154	0.98768834059514	0.15643446504023	6.3137515146751
82	1.4311699866354	0.99026806874157	0.13917310096007	7.1153697223842
83	1.4486232791553	0.99254615164132	0.12186934340515	8.1443464279746
84	1.4660765716752	0.99452189536827	0.10452846326765	9.5143644542226

A COMPUTER PROGRAM TO CALCULATE THE SINE OF AN ANGLE

Below is a computer program to calculate the sine value of a degrees angle. The computer program is shown below and is written using the popular C programming language syntax (essentially the required grammar of the programming language used). By observation, it should not be too difficult to understand and convert it to the BASIC or any other computer language syntax. Some explanation about the program and its syntax will also be given. You may skip over this section if you have no need of learning or understanding any computer language that is used to program a computer to help do things.

For the following computer programs as written here, the user must be use a "plain" (ie. basic, using the ASCII text codes only in the source or origin text file) or unformatted (ie., not a [formatted] word processor) input and output text editor and a C language compiler. A compiler is a computer program program that will transform or translates the input text, "source code", computer language program to its equivalent or effective "low level" machine-electronic numeric codes for the computers electronic circuitry. In short, the compiler program creates the executable (working, run time) program for the computer.

In program below, many extra notes about C programming and its syntax are given for clarification, however, a new computer programmer is invited to obtain some programming books about the subject of learning computer programming. This book here focuses upon writing some mathematical source code programs with the C Language syntax of computer programming, and specifically for user-made math functions often found on a scientific calculator or computer. These functions are built into the ANSI C language, but its of mathematical importance to show them here. Here is the format or syntax of a programmers comment or note within programs in the C language: /* comment(s) */

```
/*****************************************************************************************************************************
SINE.C

Two examples of  functions to calculate the trigonometric sine of a
degrees angle value(s) input from the user.  The function implements
the SIN X series.

This function can be improved upon for use with more modern
computers that allow simplicity obtaining greater precision.

Comments are enclosed within these delimiters (bounding, set apart):
   Comments are placed between or inside the forward slash and the star
   character and then the comment is terminated by the  star character
   and a forward slash character.  Comments are not processed by the
   computer, but are notes for the writer or reader of the program to
   review later to help understand the computer program, parts or
   functions within it.  Comment syntax can also be placed around
   program code to omit it, such as for temporary test code.

This text or readable, "high-level" computer code or program requires
a C-compiler to convert it to a "low-level" executable or usable electronic
machine or computer code program consisting of only binary numbers
where each 0 or 1 bit will be represented by a high or low voltage level
to be used by the specific computer processor circuitry within the
computer.

The specific C programming language compiler used was the free (ANSI
compatible) TinyC Compiler which can be obtained or downloaded to your
computer from some websites.  The basic format of the command to compile
a c programming language file (ie. with a input filename like:  myfilename.c)
```

```
      if(angle<=0.0){ printf("\a");  /* a beep sound */ exit(0);  };
      printf("\nThe sine of %.16g degrees is %.9g  and  %.9g",angle,SINE(angle),SINE2(angle));
   };   /* End of the defined loop, now go back to the start of the loop and let the user enter another value */
```

/* This computer code, with a use of the printf() function, above will display the result (the angle and the sine of that
     angle) on the screen. To "print" (ie., send) the std. output to a printer, disk or memory file, use the fprintf() function.
     %g is a display or output text format specifier that indicates how the corresponding function argument is
     to be displayed on the computer screen.
     g means to display the corresponding data argument as a floating point value. 16 and 11 are the
     max. number of decimal digits to display, and these values can be changed to be more or less by the
     programmer. When SINE(angle) is encountered (ie. the SINE function is called), the program code
     to be then run or processed will then take place at this SINE function (below), until a return (to the
     previous or calling code position) statement or other code is encountered. */

 return;   /* Essentially ends the computer program. Sends the (computer processing) control back to the
               "calling program" that called this SINE.EXE program. Often, the calling program is the computer
                 operating system or program. */
};
/*--*/
/*   SINE FUNCTION , SINE()

     This computer function returns a result that is a double float sine value of the double float angle argument. The angle
     must be a degrees angle. This function uses the sin x series, in hard coded form, to calculate sin x.

     (c) J.A. 2014
--*/
double SINE(double angle)
{
double z=(angle * 3.1415926535897932384626)/180.0;   /* converts the input degrees angle to its
                                                            corresponding radian angle */

/* Using this series of terms calculation for Sin z, the lower the value of z, the input angle, will result in greater
     accuracy, so it is best to keep the input angle less than 180 , and generally 90° or less for the best accuracy.
     This is mainly due to only using a few or limited number of terms of an infinite series of terms which expresses
     the true result. */

/* The above expression declares the identifier (or variable) called z
    and initializes (sets) it to an equivalent radian angle value of the degrees
    angle value that was sent to this function. This radian angle value is what
    the sine series below works with. 16 digits of values, such as (pi) shown,
    are usually accepted with many computers and/or programming languages.
     * = the "star" text character, is the syntax for the multiplication symbol for
    man computer languages. If your computer or electronic machine is capable,
    here is Pi to 31 decimal places: pi=3.1415926535897932384626433832795 */

double sine;

/* Lets progressively add each next term of the sin series to the running total called sine. */
/* It is also possible to make a loop to create and add each new term.   */

sine = z;
sine = sine - pow(z,3.0)/6.0;   /* for more of a homemade style, can use, for ex., z*z*z = z^3 for = pow(z,3.0) */
sine = sine + pow(z,5.0)/120.0;

The trigonometric and other tables in the appendix section of this book were created with the aid of a computer program. Before mechanical or electro-mechanical and-or electronic computers with fast and easy calculations, the only method to create theses tables was manually by "hand" (pen, hand written [scribed], paper, and mind-calculation).   Each entry in the table had to be calculated, often by many people working together so as to decrease the effort and time needed, and to check or validate each others results.

sin (x/2) = + - √ (1 - cos x)/2   : sine of half the angle trigonometric identity

sin (¢/2) = + - √ (1 - 0.8)/2 = √ 0.2/2 = √ 0.1 = 0.316227766

arcsin 0.316227766 = 18.43494882°   : multiplying this half-angle by 2 to find the actual angle

(18.43494882°)(2) = 36.86989765°

For the given SINE() function below, which can be improved upon, for increased accuracy, as mentioned above:

   At sin 0.1, the returned value is: 5.739170477 degrees
   A calculator returns: 5.739170477 degrees

   At sin 0.5, the returned value is: 30 degrees .   At higher values (>0.5) and higher, the ARCSINE() should be
   A calculator returns: 30 degrees.   improved upon, as indicated above.

   At sin 0.6, the returned value is: 36.869893 degrees
   A calculator returns: 36.86989765

   At sin 0.7, the returned value is: 44.426808 degrees
   A calculator returns: 44.427004 degrees

   (c) J.A. 2018
---*/
long double ARCSINE(double sine)
{
long double x=sine;
long double angle;   /* the formula gives the corresponding radian angle value, this will
                             then be converted to its corresponding degrees angle value */

 /* ******** Here is the arcsin x series, and further ahead is a simplified version:

ARCSIN X = X + _(1)X^3_ + _(1)(3)X^5_ + _(1)(3)(5)X^7_ +
                    (2)(3)   (2)(4)(5)   (2)(4)(6)(7)

         + _(1)(3)(5)(7)X^9_ + _(1)(3)(5)(7)(9)X^11_ + _(1)(3)(5)(7)(9)(11)X^13_ +
             (2)(4)(6)(8)(9)   (2)(4)(6)(8)(10)(11)   (2)(4)(6)(8)(10)(12)(13)

         + _(1)(3)(5)(7)(9)(11)(13)X^15_ +   _(1)(3)(5)(7)(9)(11)(13)(15)X^17__
             (2)(4)(6)(8)(10)(12)(14)(15)   (2)(4)(6)(8)(10)(12)(14)(16)(17)

         + _(1)(3)(5)(7)(9)(11)(13)(15)(17)X^19_ + (1)(3)(5)(7)(9)(11)(13)(15)(17)(19)X^21 + . . .
             (2)(4)(6)(8)(10)(12)(14)(16)(18)(19)   (2)(4)(6)(8)(10)(12)(14)(16)(18)(20)(21)

ARCSIN X = X + X^3 + 3X^5 + 15X^7 + 105X^9 + 945X^11 + 10,395X^13 + 135,135 X^15 +
                    6   40   336   3,456   42,240   599,040   9,676,800

      + 2027025X^17 + 34459425X^19 + 654729075X^21 +   654729075(21)X^23   + . . .
          175,472,640   3,530,096,640   (3715891200)(21)   (2)(4)(6)(8)(10)(12)(14)(16)(18)(20)(22)(23)

******** */

Mathization   Edition 1.0   Donate

TRIGONOMETRIC VALUES

```
/*****************************************************************************************************
trigvalues.c

A function to calculate the three main  trigonometric  values (sine, cosine, and tangent)
of a degrees angle value.  The function mainly implements a simplified version of the  TAN X
continued fraction, and the accuracy is very good at about 9 decimal places for angles from 0
through 90 degrees.

(c) J.A. 2018      : If you wrote a program or article, you own the copyright to it, and even if it is never published,
                     however, you may need some other verification that you indeed wrote it at such a date.
*****************************************************************************************************/
#include "stdio.h"
#include "math.h"

double trigvalue(double angle, int function);
/*   The angle argument is in degrees measurement.
     For the argument called function, use 1 for sine, 2 for cosine, 3 for tangent.
     The function's return value is the trigonometric  value of the angle.
     Accuracy is about 9 decimal places from 0 to 90 degrees.  */

/* here is p  ("Pi") to 31 decimal places:  pi = 3.1415926535897932384626433832795    */
/*                                          pi/2 = 1.5707963267948966                  */
/*---------------------------------------------------------------------------------------------------*/
void main(void)
{
double angle=0.0;

for(;;){                      /* main loop */
        fflush(stdin);        /* clear out any potential garbage from the keyboard (the standard input)  memory
                                buffer input line */
        angle=0.0;

        printf("\n\nEnter the degrees angle to find the trigonometric values of, or 0 to exit:  ");
        scanf("\n%lf",&angle);
        if((angle<=0.0) || (angle>90.0) ){ printf("\aUse 0 to 90. Press A Key");  /* a beep sound */ getch(); exit(0);  };

        printf("\nThe sine value of %.6g degrees is %.9g",angle , trigvalue(angle, 1));
        printf("\nThe cosine value of %.6g degrees is %.9g",angle , trigvalue(angle, 2));
        printf("\nThe tangent value of %.6g degrees is %.9g",angle , trigvalue(angle, 3));

     };    /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*---------------------------------------------------------------------------------------------------*/
```

```
/*-----------------------------------------------------------------------------------------------------------------------*/
double trigvalue(double angle, int function)
/*  Use angles 0 to 90 degrees.  set function=1 for sine, 2 for cosine, 3 for tangent */
{
double z=(angle * 3.141592653589793238)/180.0;    /* convert input angle to its corresponding radian angle */

double tangent=0.0;
double sine=0.0;
double cosine=0.0;
int a=0;
double numerator;
double denominator;

if(angle>45.0){ a=1;  z=(3.14159265358979323846 / 2.0) - z;  };      /* used for improved accuracy using the limited
                                                                         number of  terms with the higher angle value */

/* Calculate the tangent value of the angle:  */

numerator  =  (10395*z) - (1260*z*z*z) + (21*z*z*z*z*z);
denominator  =  10395 - (4725*z*z) + 210 *(z*z*z*z) - (z*z*z*z*z*z);
tangent = numerator / denominator;

/* Some basic value checking:  */
if(tangent <= 0.000000001){ tangent = 0.000000001;  };
if(tangent >= 5.6e+14){ tangent = 5.6e+14; };

if(a==1){ tangent = 1.0/tangent; };

if(function==1){    /* return sine value */
                   sine = tangent / sqrt(1+tangent*tangent);   /* : here,  using C's own square root math function */
                    return sine;
               };

 if(function==2){    /* return cosine value */
                   cosine = 1.0 / sqrt(1 + tangent*tangent);
                    return cosine;
               };

if(function==3){    /* return tangent value */
                   return tangent;
               };

};
/*-----------------------------------------------------------------------------------------------------------------------*/
```

A TANGENT FUNCTION

This homemade C-language function uses this tan x continued fraction to calculate tan x.
x is the radian angle equivalent of a degrees angle.

This continued fraction is credited to Johann Lambert:

$$\cfrac{X}{1 - \cfrac{X^2}{3 - \cfrac{X^2}{5 - \cfrac{X^2}{7 - \cfrac{X^2}{9 - \cfrac{X^2}{11}}}}}}$$

. . . (and so on)

-------NOTES */

```
double z=(angle * 3.141592653589793238)/180.0;   /* Convert  the input angle to its corresponding radian angle. */
double N;                                          /* This program uses variable z in place of X in the continued fraction shown. */
double tangent=0.0;

N=17.0;          /* set N to an odd value that is the "lowest denominator" to evaluate in the continued fraction.
                    When N=11, expect 3 decimal places of accuracy at 89 degrees, and 9 decimal places
                    of accuracy for 1 degree.
                    When N=17, expect 9 decimal places of accuracy at 89 degrees.  */

tangent=N;       /* setting tangent to an initial value, here N, the "lowest denominator"  */

for(;;){     /* a loop to process the continued fraction from the bottom (the "lowest denominator") to the top (numerator) */
         tangent=(double)((z*z) / tangent);
         N=N-2.0;
         tangent = N - tangent;
         if(N==1){  tangent = z/tangent; break; };
    };

return tangent;
};
/*----------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM TO CALCULATE A ROOT (OR POWER) OF A NUMBER

```
/* ---------------------------------------------------------------------------------------------------------------
     rootN.c   Calculates the nth root of a radicand value using a successive approximation.
               This program can also calculate a power of a number by using an indicated root (x) of
               of (1/x).  The same function was modified to create a power function.
               The functions created in this program use the standard C language pow() function,
               however a homemade pow() can be created and used instead.  A computer program
               called power.c, that is shown further ahead, will also calculate a power a number, and
               it uses the series sum method, and without using any of the standard C-programming
               language math functions to do so.

               Successive approximation will "zero-in" to the correct result, that is, each new or
               successive, intermediate or temporary result will have a greater numerical precision,
               hence more correct (decimal) digits to the right of the decimal point.

      (c) 2018 JPA
--------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include "math.h"   /* good to include when working with floats and doubles, etc.  It is
                        also needed for this program to work correctly for the C's standard
                        pow() that is used in this program. */

double  rootN(double index, double radicand);
 /* returns the index root value of the radicand
    Ex.   10 =  rootN(2,100);

This function, and specifically its arguments, can also be modified  to find powers:

  To find the second power of 10 = 10^x  = 10^2, the index of the radicand will be
  the reciprocal of this indicated power, here (1/x)=(1/2)=0.5.   100 = rootN(10,0.5);

  To calculate powers of (e), use 2.718281828459045235... as the radicand.  You
  can also make a specific  function(s) to solve for powers of (e).

The rootN() modified to more easily calculate powers is powerN();    */

double  powerN(double base, double exponent);

/* For both of these functions, generally, about 9 loops will usually give 9 or sometimes more
    decimal places of accuracy.  Both of these functions use the C-language pow(), but you can
    create or write your own pow() using the series for (e) and other mathematical steps.  */
/*---------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double  radicandvalue=0.0;
double  indexvalue=0.0;
double  rootvalue=0.0;

double Base1=0.0;
double Exponent1=0.0;
double powervalue=0.0;
```

```
for(;;){

    Rn1 =  (  ((index-1) * Rn)  +  radicand/pow(Rn,(index-1.0))   ) / index;

      temp1=Rn;
      temp2=Rn1;
      if(temp1<0.0){  temp1=-temp1;  };
      if(temp2<0.0){  temp2=-temp2;  };
      diff=temp1-temp2;     if(diff<0.0){  diff=-diff;  };
      if(diff<=0.000000000000001){  break;  };

     Rn=Rn1;       /* for the next iteration of the loop (cycle, recycle) */
     /* printf("\n%g %g",Rn1,Rn);   : optional, to display intermediate results */
     };

return  Rn1;
}
/*----------------------------------------------------------------------------------------------------------------*/
double  powerN(double base, double exponent)
{
/* This function returns the x indicated power value of the base */
/* This function is based on the facts that:

        root^x  =  radicand  = number  =  base^x  = power

      , and that a root of a number can be expressed and calculated as a
        power of a number:
          _____
      x√ radicand  =  radicand^(1/x)

   This function implements a loop used to find a root of a number,
   and was modified to find the power of a number, mainly by taking
   the reciprocal of the indicated power    */

double  root=0.0;
double  Rn=1.0;        /* initial guess, ex. for the 30th root of 10 to calculate faster, fewer loops */
double  Rn1=0.0;
double  temp1=0.0;
double  temp2=0.0;
double  diff=0.0;      /* difference */

double index=0.0;
double radicand=0.0;

int t=0;    /* a toggle, set to 1 if the exponent is negative */

int loops=1;    /* this is generally not needed due to checking the difference between successive values,
                but this is used in case there are certain numerical values that produce an infinite loop,
                as was noticed with a base of 5.6 and a exponent of 3.4                          */

if(exponent==0){ exponent=1; base=1;  };   /* this is for some error checking, prevention and correction, */
                                           /* in particular, if the user enters an exponent of 0.          */
```

A COMPUTER PROGRAM TO CALCULATE A POWER OF (e)

```
/* ---------------------------------------------------------------------------------------------------------

    epower.c

    A homemade function to calculate any indicated power value of (e).

    (c) JA 2018
-----------------------------------------------------------------------------------------------------------*/

#include "stdio.h"

double epower(double exponent);

/*-------------------------------------------------------------------------------------------------------*/
void main(void)
{                                    /*  a sample test program to use the epower()   */
double exponent=0.0;
double power=0.0;

for(;;){                      /* a loop to reuse or redo the following program code over and over  */
        fflush(stdin);       /* clear out any potential garbage from the keyboard (the standard input)  memory
                               buffer input line */

        exponent=0.0;
        printf("\n\nEnter the exponent of (e):  ");
        scanf("\n%lf",&exponent);

        power=epower(exponent);

        printf("\n(e) to the %.16g power is %.16g",exponent,power);

    };   /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*-------------------------------------------------------------------------------------------------------*/
double epower(double exponent)
{
/* Notes---    Here is the power of (e) series:
```

$$e^x = \sum_{n=0}^{n=\infty} \frac{x^n}{n!} \qquad : 0! = 1 \text{ and } x^0 = 1 \qquad\qquad : \text{(credited to Euler)}$$

Expanding the terms of this series:

$$e^x = 1 + x^1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$$

: a derivation of this is shown in the appendix section.

Simplifying the factorials:

```
/*  printf("\n%.16g",sum); getch();  test ok */

while(wholepart>0){      /* calculate:  e^(wholepart)  */
                    power=power *e;
                    wholepart=wholepart-1;
              };

power=power*sum;      /* e^(wholepart) e^(fractionalpart) = (power)(sum) */

if(t==1){ power = 1/power; };   /* if the exponent was negative */

return power;
};
/*--------------------------------------------------------------------------------------------------------------------------*/
```

```
/*-------------------------------------------------------------------------------------------------------------------------*/
double LN(double number)
{
/* This function calculates the natural logarithm of a number less than 2.  It can be used to construct a program
   to calculate the natural logarithm of any value. */

double logarithm=0.0;

double numerator;
double denominator=1;

int termnumber=1;
int maxterms=45;     /* Set this to the number of terms you want to consider and evaluate in the series.  */

double term=0.0;
number=number-1;
numerator=number;

int s=0;         /* a toggle, for the positive and negative sign of the term */

for(;;){         /* a loop to calculate each term of the series and add it to the sum */

        term = numerator / denominator;

        if((s%2)==0){  logarithm=logarithm + term;  };    /* if s, or the term number is even */
        if((s%2)==1){  logarithm=logarithm - term;  };   /* if s, or the term number  is odd */
        s=s+1;

        /*  termnumber=termnumber+1;  if(termnumber==maxterms){  break; };                         */
         /* can also program the loop to terminate when the difference between two successive
            terms is a low value, for example: 0.000001, and this is shown and used right here:          */
          if(term<=0){ term = -term; };  if(term<=0.0000000000000001){    break; };

        numerator=numerator*number;
        denominator = denominator+1;
    };

return logarithm;
};
/*-------------------------------------------------------------------------------------------------------------------------*/
 double LN2(double number)
{
/* ******This function calculates the natural logarithm of any positive valued  number.

   Here is an example of what takes place in this function,and it is described more in this books text.

    Ex.  Find or calculate:  ln 5

     ln 5  =  ln ((e^2) (0.676676416))
     ln 5  =  ln e^2  +  ln 0.676676416
     ln 5  =  2  +  (-0.390562087)
     ln 5  =  1.609437913
****** */
```

A COMPUTER METHOD TO FIND A POWER OF A NUMBER

This method to find a power of a number utilizes the series of terms sum method, rather than successive approximation. This program and method does not rely upon any of the C-language pre-written standard math functions, and is completely homemade and may be modified if necessary. This program utilizes some of the homemade math functions shown previously in this book so as to create a power of a number function.

```c
/* ---------------------------------------------------------------------------------------------------------------------
    power.c

    A homemade function to calculate any power value.

    (c) JPA FEB 2018
-------------------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include"math.h"

double power(double base, double exponent);
/* required for this function  to calculate any power value, are these two function: */
double LN2(double number);
double epower(double exponent);

/*-----------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double base=0.0;
double exponent=0.0;
double logarithm=0.0;
double x=0.0;
double N=0.0;

/*  printf("\n%.16g",pow(0.32,0.32)); getch();  just test to compare to, result is 0.6944612105965807, requires math.h */

for(;;){                        /* a loop to reuse or redo the following program code over and over  */
        fflush(stdin);      /* clear out any potential garbage from the keyboard (the standard input)  memory
                                    buffer input line */

        base=0.0; exponent=0.0;
        printf("\n\nEnter the base of the indicated power:  ");
        scanf("\n%lf",&base);
        if( base<=0.0 ){ printf("\a");  /* a beep sound */ exit(0);  };
        printf("\nEnter the exponent of the base: ");
        scanf("\n%lf",&exponent);

        N=power(base,exponent);
        printf("\n%.16g",N);

    };   /* end of the loop, now go back to the start of the loop and let the user enter another value */

return;
};
/*-----------------------------------------------------------------------------------------------------------------------*/
```

```
        numerator=numerator*N;
        denominator = denominator+1;
    };    /* end of the for loop */

     logarithm=x+logarithm;

return  logarithm;
};
```
/*---*/
```
double epower(double exponent)
{
/* Notes---    Here is the power of (e) series:
```

$$e^{x} \; = \; \sum_{n=0}^{n=8} \; \frac{x^{n}}{n!} \qquad : \; 0! \; = \; 1 \; \text{ and } \; x^{0} = 1$$

Expanding the terms of this series:

$$e^{x} \; = \; 1 \; + \; x^{1} \; + \; \frac{x^{2}}{2!} \; + \; \frac{x^{3}}{3!} \; + \; \frac{x^{4}}{4!} \; + \; \frac{x^{5}}{5!} \; + \; \frac{x^{6}}{6!} \; + \; \frac{x^{7}}{7!} \; + \; . \; . \; .$$

: a derivation of this is shown in the appendix section

Simplifying the factorials:

$$e^{x} \; = \; 1 \; + \; x^{1} \; + \; \frac{x^{2}}{2} \; + \; \frac{x^{3}}{6} \; + \; \frac{x^{4}}{24} \; + \; \frac{x^{5}}{120} \; + \; \frac{x^{6}}{720} \; + \; \frac{x^{7}}{5,040} \; + \; \frac{x^{8}}{40,320} \; +$$

$$+ \; \frac{x^{9}}{362,880} \; + \; \frac{x^{10}}{3,628,880} \; . \; . \; .$$

To use an exponent less than or equal to 1 for the series for (e),   e^x can be factored to smaller powers of (e):

Ex. e^5.2 = e^5 e^0.2   = e^(wholepart) e^(fractionalpart)

The integer power of (e) can be found by repeated multiplication of e=2.718281829...

Here is another method:

Ex. e^5.2 = e^10 e^0.52

The calculation below is set to return 16 decimal places of accuracy.

---------Notes */

double power=1.0;

```
/*----------------------------------------------------------------------------------------------------------------------------*/
double power(double base, double exponent)
{
double logarithm=0.0;
double x=0.0;
double N=0.0;

     logarithm=LN2(base);
      /* printf("\nThe logarithm of %lf is %.10lf",base,logarithm);  printf("\nPress A Key ");  getch();    test ok */

      x=exponent * logarithm;       /*  this calculates the indicated power = x =  exponent of (e)^x  */
     /*  printf("\nx = %.16g",x); getch();   test ok */

      N=epower(x);

return N;
};
/*----------------------------------------------------------------------------------------------------------------------------*/
```

```c
#include "stdio.h"


double mysqrt(double X);        /* A homemade square-root math function.  */
                                /* Here, we let X represent the radicand.      */

/*-----------------------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double radicand;                /* doubles are 8 byte "floating" point data types in C */
double root;

radicand=2.0;
root = mysqrt(radicand);     /* call (transfer computer control or sends the program to) the
                                mysqrt() function with the argument or variable identified as:
                                radicand, and receive the "return(ed) value" (the numerical
                                root) of the function and assign it to the variable identified
                                as:  root.  The argument to the function can also be an
                                explicit constant.  For ex.,   root  =  mysqrt(2.0)  */

printf("\nThe square-root of %.16g = %.16g",radicand,root);        /* display the result on the computer screen */

/*
   The output displayed on the screen will look something like:

   The square-root of 2.0 = 1.414213562...
*/

radicand=10.0;              /*  :Let's also find the square-root of 10.0, hence reusing the function already written. */
root = mysqrt(radicand);
printf("\nThe square-root of %.16g = %.16g",radicand,root);

return;
};
/*-----------------------------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER METHOD TO SOLVE DIFFICULT EQUATIONS

```
/*---------------------------------------------------------------------------------------------------------------
EQSOLVE2.C

This program demonstrates a way to solve a single variable equation (ie., an equation with just one
variable within it, however there may be several instances of it in the equation), and it is very useful
to solve difficult equations when there is no known or simple algebraic solution (ie., an equation to do so).
This program contains an algorithm I made called a Fast Algorithm To Solve Equations.  There is a
alternative version of this algorithm placed in the "late entries" section at the end of this book.

The relatively short C-language program given below solves for (x) in an example equation that is a
multiple term exponential equation of the form:

   N1^x  +  N2^x  =  N3     or even with more terms such as:   RESULT = N1^x  +  N2^x  +  N3^x  +  . . .

, and gives about 10 decimal places of accuracy of x.  It is a difficult equation without an algebraic solution.

The example below solves the hard coded (set, but can be changed) and specific equation of:

   5^x  +  6^x  =  61           : the solution is x = 2.0 ,  checking:  5^2 + 6^2 = 25 + 36 = 61

This method below does not use recursion.  Its more of a fast "trial and error,
precision method" so as to "zero in" (ie., find) the result quicker than simply
incrementing it by only a very small amount after each loop.   Requires only
ten loops max. per correct digit position.  Ex. For 9 places, instead of
1,000,000,000 loops for a net increment of 1 in (x) when incrementing by just
0.000,000,001, my process in the worst case requires only 100 loops, and on
average (ie. digit 5 of 10 possible digits) of only 50 loops will be required to
solve for (x). The program assumes that (x), the variable being solved for is
positive.  (x) is initially set to 0.0.  Following the program are other code
samples for other equations.
(c) J.A. 2014
---------------------------------------------------------------------------------------------------------------*/
#include "stdio.h"
#include "stdlib.h"
#include "math.h"
#include "float.h"
/*---------------------------------------------------------------------------------------------------------------*/
void main(void)
{
double N1=5.0;          /* a value used in the given equation.  Can make this and other values a user input value. */
double N2=6.0;          /* a value used in the given equation  */
double RESULT=100.0;   /* right side of the given equation      */

double x=0.0;           /* to be found, this can set to a low positive value such as
                           0.0000000001 if division by 0 error is to be avoided in the algorithm */
double TEMP=0.0;        /* ie. left side of equation */

int n2=0;               /* outer loop counter variable */
int n=0;                /* inner loop counter variable */

double a=1.0;           /* (variable) increment to be added to current value of the variable (x) */
```

```
  a=a/10.0;
 };
```
/*---*/
This code below will work for cube roots, even when written in a form that
cannot be solved with standard iteration:

For the cube root of 10 (or set 10 equal to a variable such as N):

```
for(n2=0;n2<=14;n2++){

  for(n=0;n<=9;n++){
    if(x>(10.0/(x*x))){ x=x-a; break; };          /* : Here N is "hard-coded" to 10.0,  */
    x=x+a;                                         /*   only for this example code.      */
  };

 if(x==(10.0/(x*x))){  break; };
 a=a/10.0;
 };
```
/*---*/

Here is the loop code for solving the Cubic equation of: x^3 + x^2 + x^1 + C = 14

For this example here, C is arbitrarily set to 0, but should be coded into the program.

```
  x=2.0

  for(n2=0;n2<=14;n2++){

    for(n=0;n<=9;n++){
      if( (x*x*x + x*x + x)>14.0){ x=x-a; break; };      /* : 14 was "hard-coded" as a constant; best to use a variable */
      x=x+a;                                             /*   so as to reuse the function for other values.            */
    };

    if((x*x*x + x*x + x )==14.0){  break; };
    a=a/10.0;
  };
```
/*---*/

Extra notes on the above fast algorithm for solving a single variable equation:

If solving something such as: $100 = 10^x$ and using the above method to solve for (x), the indicated power, it will actually be solving for a logarithm value: x = log 100.   You can use any base and number as needed, perhaps (e)=2.718281828 as the base if you need to find a natural logarithms: logarithm = ln Number

If solving something such as: 0.5 = sin(x), it will actually be solving for: x = arcsin 0.5 , and note the value found will be a radian angle because it is natural, rather than the man-made values of degrees. The standard C language function for sin(x) considers the argument, here (x), as a radian angle.

For a dynamic equation or user input equation, such as a text string, after parsing it, perhaps the hard coded equation in this algorithm can be replaced by a hard coded equation such as:: result = term1 + term2 + ... term20 , and where each term is initially set to 0.0

```c
for(;;){                                           /* main loop, let user enter each expression to calculate */
        operand1=0.0;        /* clear out old values, set them to 0 */
        operand2=0.0;  command=0;  result=0.0;  validcommand=0;

        printf("\n\n> ");
        fflush(stdin);
        scanf("\n%lf %c %lf",&operand1,&command,&operand2);

     if(  (command == '+') || (command == '-') || (command == '*') || (command == '/') || (command == 's') ||
        (command == 'c') || (command == 'p') || (command == 'e') || (command == 'l') || (command == 'L') ||
        (command == 'S') || (command == 'C') || (command == 'T') || (command == 'a') || (command == 'H') ||
        (command == 'h') || (command == 'q') || (command=='x') || (command=='r')  || (command=='Z')  )
        {   validcommand=1;  };

     if(validcommand==0){  printf("\n\aERROR, UNKNOWN OPERATION OR COMMAND, ENTER> 0 h 0 for help");
                           continue;  };  /* the continue command will continue from the start of the loop */

      if(command=='+'){         /* add */
                      result = operand1 + operand2;
                   };

      if(command=='-'){          /* subtract */
                      result = operand1 - operand2;
                   };

      if( (command=='*') || (command=='x')){      /* multiply  */
                      result = operand1 * operand2;
                   };

      if(command=='/'){          /*divide */
                      result = operand1 / operand2;
                   };

      if(command=='s'){          /* square-root */
                      if(operand1<0.0){  printf("\n\aERROR, USE POSITIVE VALUES"); continue;  };
                      result =sqrt(operand1);    /* include math.h for this */
                   };

      if(command=='c'){          /* cube-roots  */
                      if(operand1<0.0){  printf("\n\aERROR, USE POSITIVE VALUES"); continue;  };
                      result = pow(operand1,0.33333333333333333333);
                   };

      if(command=='p'){          /* power */
                      result = pow(operand1, operand2);
                   };

      if(command=='p'){          /* root */
                      result = pow(operand1, 1.0/operand2);
                   };

      if(command=='e'){          /* power of (e) */
                      result = exp(operand1);   };
```

```c
                    /* convert the degrees angle to its  corresponding radian angle for  c-language sine function  */
                    result = cos(operand1);   /* include math.h for this */      };

    if(command=='T'){          /* Tangent  */

                    if(operand2==1.0){                 /* for arctangent */
                                result=atan(operand1);   /* convert this radian angle to its
                                                          equivalent degree angle */
                                result=result * (180.0/3.14159265358979323);
                                goto displayresult;
                                };

                    if( (operand1 <0.0) || (operand1 >180.0)){
                            printf("\n\aERROR, USE AN ANGLE BETWEEN 0 AND 180 DEGREES");
                            continue; };

                    operand1 = (operand1 * 3.141592653589793238)/180.0;
                                               /* convert the degrees angle to its
                                                  corresponding radian angle for
                                                  c-language sine function          */

                    result = tan(operand1);   /* include math.h for this */
                    };

    if(command=='a'){          /* angle conversion  */

            if(operand2==1.0){   /* converting a degrees angle value to its corresponding radian angle value */
                            result  = (operand1 * 3.141592653589793238)/180.0;
                        };

            if(operand2==0.0){   /* converting radian angle  value to its corresponding degrees  angle value */
                            result= operand1  * (180.0/3.14159265358979323);
                        };
                    };


    if(command=='H'){          /* Pythagorean, Right Triangle Theorem , can use letter P also */
                                               /* Solves for the hypotenuse side of a right triangle */

                    result =sqrt( (operand1 * operand1) + (operand2 * operand2) );   /* ex. c = √ a^2 + b^2  */
                    };

    if(command=='Z'){          /* Pythagorean, Right Triangle Theorem  solve for the side of right triangle */
                                               /*input format:  hypotenuse H side  */

                    if( (operand1 < operand2 )){
                        printf("\n\aERROR, HYPOTENUSE SIDE MUST BE GREATER THAN %.16lf",operand2);
                        continue; };

                    result =sqrt((operand1*operand1) - (operand2*operand2)); /* ex. b = √ c^2 - a^2  */
                        };

    if(command=='h'){   goto start;  };       /* program help  */
```

SUMMING CONSECUTIVE TERMS OF AN ARITHMETIC SERIES

Given (N) consecutive terms of an arithmetic series, here is how to find the sum of those terms:

Algebraically, the terms of an arithmetic series can be expressed or noted as:

Term (N):	1	2	3	4	5	6	. . .
	F	F+d	F+2d	F+3d	F+4d	F+5d	. . .

Where (F) is the first considered term, term 1, and (d) is the constant difference between each terms value. Consider for example that the third term is the sum of the second term and the difference (d):

(term 3) = (previous term) + d = (term 2) + d = (term 1 + d) + d = (term 1) + 2d = F + 2d

Algebraically summing up the terms:

Term	Term Value	Total Sum Of Terms (A Running Or Continuous Sum Of This Term and All Previous Terms)
1	F	1F
2	F + 1d	2F + 1d : since: (this term) + (previous sum) = (F + 1d) + 1F = 2F + 1d
3	F + 2d	3F + 3d : since: (this term) + (previous sum) = (F + 2d) + (2F + 1d) = 3F + 3d
4	F + 3d	4F + 6d
5	F + 4d	5F + 10d
6	F + 5d	6F + 15d

Notice in the sum expressions that the first term of each expression can be easily predicted to be NF, however the next term is awkward since the numerical coefficients (C) of (d) appear to be that of some unknown series. There is one "observable formula" for the value of C that will be shown ahead, and can you possibly guess it beforehand? Below is a method of how these constants can be calculated so that the sum of the first (N) terms of an arithmetic series can be evaluated as:

SUM = NF + Cd   : SUM OF N CONSECUTIVE TERMS OF AN ARITHMETIC SERIES (*)
                                    F = value of first term, N = number of terms, d = difference, C = numerical coefficient of d

For this analysis, let's say an arithmetic series has a difference (d) value of 2.0, and the first term is 3. We can express the sum of the first 3 terms as:

$$\begin{array}{rl} S &= 3 + 5 + 7 \\ + S &= 7 + 5 + 3 \\ \hline 2S &= 10 + 10 + 10 \end{array}$$
Algebraically adding the sum to itself, using the commutative law:
(Or by multiplying each side by 2, and then distributing it.)

Notice that each term on the right hand side is algebraically equivalent to (F1 + FN), where F1 is considered as the first term, and FN is considered the "last" term considered. Since there are (N) terms in question being added, and each is the same value, we can express this repeated addition with multiplication:

2S = (F1 + FN) N   solving for S:

$S = \dfrac{(F1 + FN) N}{2}$: **SUM OF N CONSECUTIVE TERMS OF AN ARITHMETIC SERIES**
F1 is the first term value , FN is the last term value, and the common difference (d) between terms is not needed and-or known.

SUMMING CONSECUTIVE TERMS OF A GEOMETRIC SERIES

Given (N) consecutive terms, which can and will be considered as the first (N) terms of a geometric series below, what is the sum (S) of those terms?

Algebraically, the terms of a geometric series can be algebraically (mathematically, symbolically, generally) expressed as:

Term: 1   2   3   4   5   6   . . .   n
         F   Fr   Fr^2   Fr^3   Fr^4   Fr^5   . . .   Fr^(n-1)

Where (F) is the considered as the first term, and (r) is the constant ratio between consecutive terms. Each next term is found from: (term)(r) = next term.   Mathematically ,   previous term = term = (next term) / r

An expression of algebraically summing up a total of (n) terms:

S = F + F^1r + Fr^2 + Fr^3 + Fr^4 + Fr^5 + . . . + Fr^(n-1)

Now, we can possibly derive a simple formula that is equivalent to the right hand side of this equation. Since each term has a factor of F in it, we can factor it from each term and get this equivalent expression:

S = F (1 + r + r^2 + r^3 + r^4 + r^5 + . . . + r^(n-1))

From the indicated factor to F as shown above within the grouping symbols, and considering just the first 4 terms of it, we can get:

  1 + r + r^2 + r^3   factoring (r) out of the terms that contain it:
  1 + r (1 + r + r^2)
  1 + r (1 + r (1 + r))

The only observable possible formula is from the second term: For a total of (n-2) times, starting with (r) and add 1 to it, and then multiply this growing quantity by (r). This total value will be added to 1.   The problem is that this is not a simple formula for the factor F as shown above, since it requires summing and multiplying almost the same number of terms as the series itself.

How about adding the series to itself, just like it was utilized during the derivation of the sum of an arithmetic series? This will not work here since the formula for the sum will simplify to the exact same series. However, rather than effectively multiply each side by 2 as in the sum of an arithmetic series, let's multiply each side of the equation by another value that is directly related to each term of the geometric series, specifically (r):

Let's try a geometric series where the first term is 4, r=2, and 3 terms are to be summed:

Eq. 1:   S   = 4 + 8 + 16   : r = 2, multiplying each side by (r) we get:

          rS = 2(4 + 8 + 16)   distributing:
          rS = 8 + 16 + 32   which can be written or expressed as:

Eq. 2:   rS = 32 + 8 + 16

Subtracting the first equation from the second we get:

          rS - S = 32 + 8 + 16 - (4 + 8 + 16) = 32 + 8 + 16 -4 -8 -16 = 32 - 4 = 28

Expanding the indicated series:

1 + r^1 + r^2 + r^3 + r^4 + r^5 + . . . + r^(n-1)

Hence when the first term is F=1:

$$S = \frac{1}{1-r} = 1 + r^1 + r^2 + r^3 + r^4 + r^5 + . . .$$:for this to converge   -1 < r < +1

Or as:

$$1 + \sum_{x=1}^{x=(n-1)} r^x$$

Expanding the indicated series:

1 + (r + r^2 + r^3 + r^4 + r^5 + . . . + r^n-1)

Equating the two sum formulas:

$$\frac{F((r^n) - 1)}{(r-1)} = F(1 + r + r^2 + r^3 + r^4 + r^5 + . . . + r^{(n-1)})$$

By dividing both sides of this equation by F, or simply equating both of the factors to F, we get:

$$\frac{(r^n) - 1}{(r-1)} = 1 + r^1 + r^2 + r^3 + r^4 + r^5 + . . . + r^{(n-1)}$$

Transposing the (+1) term, and switching sides:

$$r^1 + r^2 + r^3 + r^4 + r^5 + . . . + r^{(n-1)} = \frac{(r^n) - 1}{(r-1)} - 1$$

If we add (r^n) to each side, we can get a formula (shown on the right side) for the sum of the first (n) integer powers of any value (such as r shown on the left side):

$$r^1 + r^2 + r^3 + r^4 + r^5 + . . . + r^{(n-1)} + r^n = \frac{(r^n) - 1}{(r-1)} - 1 + r^n$$

Combing terms (ie., the fractions) on the right hand side:

$$r^1 + r^2 + r^3 + r^4 + r^5 + . . . + r^n = \frac{r((r^n) - 1)}{(r-1)}$$: This is essentially the same expression previously derived, but here, r=F.

Another observation here is to note how a function such as on the right hand side of the equation above can be expressed as a sum of a specific series of terms.

Expressing this with summation notation, and letting the base = b = r :

Note that the above could be expressed as: $x^1 + x^2 + x^3 + ... + x^n = 1$: where x=0.5 and n -> approaches infinity.
And for just the first two terms: $x^1 + x^2 = x^2 + x = 1$: x = GR = ~0.618 , Extra: $x^2 + x - 1 = 0$

By repeatedly dividing an object (ie., whole=100%) by two and making two pieces that are 50% of the original, it is the fastest way to create very small pieces of a certain size.

/*--*/

Here is an example of the output of the above computer program:

Input the numerator, ex. 1.0: 1

Input the denominator, ex. 2.0 , (r=1/denominator): 2

term 1 = 0.5000000000000000 , sum = 0.5000000000000000
term 2 = 0.2500000000000000 , sum = 0.7500000000000000
term 3 = 0.1250000000000000 , sum = 0.8750000000000000
term 4 = 0.0625000000000000 , sum = 0.9375000000000000
term 5 = 0.0312500000000000 , sum = 0.9687500000000000
term 6 = 0.0156250000000000 , sum = 0.9843750000000000
term 7 = 0.0078125000000000 , sum = 0.9921875000000000
term 8 = 0.0039062500000000 , sum = 0.9960937500000000
term 9 = 0.0019531250000000 , sum = 0.9980468750000000
term 10 = 0.0009765625000000 , sum = 0.9990234375000000   : the sum is practically equal to 1.0 about here
term 11 = 0.0004882812500000 , sum = 0.9995117187500000
term 12 = 0.0002441406250000 , sum = 0.9997558593750000
term 13 = 0.0001220703125000 , sum = 0.9998779296875000
term 14 = 0.0000610351562500 , sum = 0.9999389648437500
term 15 = 0.0000305175781250 , sum = 0.9999694824218750
term 16 = 0.0000152587890625 , sum = 0.9999847412109375
term 17 = 0.0000076293945313 , sum = 0.9999923706054688
term 18 = 0.0000038146972656 , sum = 0.9999961853027344
term 19 = 0.0000019073486328 , sum = 0.9999980926513672
term 20 = 0.0000009536743164 , sum = 0.9999990463256836
term 21 = 0.0000004768371582 , sum = 0.9999995231628418
term 22 = 0.0000002384185791 , sum = 0.9999997615814209
term 23 = 0.0000001192092896 , sum = 0.9999998807907105
term 24 = 0.0000000596046448 , sum = 0.9999999403953552
term 25 = 0.0000000298023224 , sum = 0.9999999701976776
term 26 = 0.0000000149011612 , sum = 0.9999999850988388
term 27 = 0.0000000074505806 , sum = 0.9999999925494194
term 28 = 0.0000000037252903 , sum = 0.9999999962747097
term 29 = 0.0000000018626451 , sum = 0.9999999981373549
term 30 = 0.0000000009313226 , sum = 0.9999999990686774
term 31 = 0.0000000004656613 , sum = 0.9999999995343387
term 32 = 0.0000000002328306 , sum = 0.9999999997671694
term 33 = 0.0000000001164153 , sum = 0.9999999998835847
term 34 = 0.0000000000582077 , sum = 0.9999999999417923
term 35 = 0.0000000000291038 , sum = 0.9999999999708962
term 36 = 0.0000000000145519 , sum = 0.9999999999854481
term 37 = 0.0000000000072760 , sum = 0.9999999999927240
term 38 = 0.0000000000036380 , sum = 0.9999999999963620
term 39 = 0.0000000000018190 , sum = 0.9999999999981810
term 40 = 0.0000000000009095 , sum = 0.9999999999990905
term 41 = 0.0000000000004547 , sum = 0.9999999999995453
term 42 = 0.0000000000002274 , sum = 0.9999999999997726
term 43 = 0.0000000000001137 , sum = 0.9999999999998863
term 44 = 0.0000000000000568 , sum = 0.9999999999999432
term 45 = 0.0000000000000284 , sum = 0.9999999999999716
term 46 = 0.0000000000000142 , sum = 0.9999999999999858

```
   printf("\nPress A Key, Use 0 to exit: "); ch=(unsigned char) getch();  if(ch=='0'){ exit(0); };
};

return;
};
/*-----------------------------------------------------------------------------------------------------*/
double infinitesum(double number, double divisor)
{
double sum=number;
int loop=0;  /* not needed, but can be used for testing */
int term=1;

printf("\n%d   %lf",term,number);  term=term+1;

for(;;){
        number=number / divisor;
        if(number<=0.000001){ break; };
        printf("\n%d  %lf",term,number);
        sum=sum+number;

        /*  loop = loop + 1;  if(loop ==10){ break; };   for testing */
        term = term+1;
     };

return sum;
};
/*-----------------------------------------------------------------------------------------------------*/
```

Here is an example of running the above program:

Input Starting Number: 100
Input Divisor: 4

1 100.000000
2 25.000000
3 6.250000
4 1.562500
5 0.390625
6 0.097656
7 0.024414
8 0.006104
9 0.001526
10 0.000381
11 0.000095
12 0.000024
13 0.000006
14 0.000001
Result = 133.333333
The change from 100.000000 is 33.333333
Press A Key, Use 0 to exit:

In general, if you split the interest total time period into N intervals, the (simple, non-compounded) interest you should receive per period is:

$\dfrac{100\% \text{ of the total interest}}{n}$ = $\dfrac{1 \text{ (interest)}}{n}$: in decimal form

If n was 4 intervals:

time of each interval = (total time period) / (n time periods) = 1year/4 = 12months/4 = 3 months = 0.25 years

Since if n = 4 intervals, the (simple) interest to expect per interval period of 0.25 years or 3 months is also 0.25 of the total interest: (I):

$\dfrac{\text{Total Interest}}{n}$ = $\dfrac{100\% I}{n}$ = $\dfrac{100\% I}{4}$ = 25% of the Interest, and expressing this mathematically:

Mt = Mo + 0.25 (Total Interest). Since: (Total interest) = 100% (Total interest) = 100% (1) = 1 = Mo :

Mt = Mo + (25% of Mo) = Mo + 0.25Mo = $\dfrac{Mo}{1}$ + $\dfrac{(1)(Mo)}{(4)\ (1)}$ = $\dfrac{Mo}{1}$ + $\dfrac{Mo}{4}$

Notice the correlation to the number of time intervals chosen (n) and the denominator of the fractional value of the interest the bank will give you. If a time interval is a certain percentage of the total time period, the partial interest will be the same percentage of the total interest. Rewriting the formula, we get:

Mt = Mo + IMo : Here, initially considering the interest as: I = 100% = 1 , that is, as undivided. For each and every interval of interest payment, I is now calculated to be: I = 100%/n = 1/n :

Mt = Mo + $\dfrac{(1) Mo}{(n)}$ Factoring out Mo from each term:

Mt = Mo + $\dfrac{Mo}{n}$ or * Mt = $\dfrac{Mo}{1}$ $\left(\dfrac{1}{1} + \dfrac{1}{n}\right)$: * (more will be said about this further ahead)

For this example where n=2, Mo=$1, and It = $1, at the end of the first interval of the 2 total intervals :

Mt = $1 + (0.50) $1.0 = $1 + $0.50 = $1.50

If you then withdrawal this total money value of $1.50, which can be expressed as just 1.50, at the end of the first interval (of a year divided into two intervals) and deposit it at a similar bank called: Another Simple Growth Bank, they will recognize your initial savings deposit as Mo = 1.50 instead of 1.00 as the First Simple Growth Bank did. Then at the end of the first interval (of a two intervals per year period, hence after 6 months) of one total time period (here, it's a year also), you should again receive 50% or half of the interest of $1.50. Since I = 100%Mo, It =100%(1.5) = 1.5.

In = $\dfrac{It}{n}$ = $\dfrac{It}{2}$ = $\dfrac{1.5}{2}$ = (1.5)(0.5) = 0.75

Mt = $\dfrac{Mo}{1}$ + $\dfrac{Mo}{2}$ = $\dfrac{1.5}{1}$ + $\dfrac{1.5}{2}$ = 1.5 + 0.75 = 2.25 : By Dec. 31. Here at this bank, Mo = 1.50

This is certainly larger than the 2.00 you would of received if you kept your money in the First Simple Growth Bank after 1 year. The above "process" you have done is essentially known as compounded growth (due to interest being added to

At the end of a third interval (n=3), the total value of the growth would again be $(1 + \frac{1}{n})^1$ times larger:

$$Mt = \frac{Mo}{1} \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{n}\right)^1 = Mo\left(1 + \frac{1}{n}\right)^3 \qquad \text{: after 3 intervals of the total time or period}$$

Clearly, an obvious pattern or correlation is that the value of the exponent of the expression for Mt is equal to the total number of intervals (n) in that total time period.   Reflecting this correlation into the formula:

$$Mt = Mo\left(1 + \frac{1}{n}\right)^n \qquad \text{:Total Compounded Growth Formula of a (total time) period divided into n}$$
(compound growth) intervals.

We already know that for two (2) intervals of compounded growth that Mt = would be 2.5 times greater than the original or starting value of Mo. That is, mathematically:

$$\frac{Mt}{\text{The initial starting value of Mo}} = \frac{2.5}{1} = 2.5$$

$$Mt = Mo\,(2.5)$$

Increasing the intervals (n) of this compound growth process to see how many more times greater the total value (Mt) can be due to various compounded growths with an increasing number of intervals (n) :

$$\left(1 + \frac{1}{100}\right)^{100} = 1.01^{1,000} = 2.70\underline{5} \qquad \text{: after 100 intervals of compounded growth in the period}$$

$$\left(1 + \frac{1}{1000}\right)^{1,000} = 1.001^{1,000} = 2.716923932 \qquad \text{: after 1,000 intervals of compounded growth in the period}$$

$$\left(1 + \frac{1}{1,000,000}\right)^{1,000,000} = 1.000001^{1,000,000} = 2.718280469 \qquad \text{This can also be expressed as:}$$

$(1 + 0.000,000,1)^{1,000,000} = (1.000,000,1)^{1,000,000}$: This clearly shows the base of the indicated power is only slightly greater than 1. A value of 1 raised to any indicated power greater than 1, including very high values, is only, and equal to 1. If 1 is increased by a very small , seemingly negligible, value, the result of that value raised to a very high value is significant. 1 will increase or grow by a factor slightly greater than 2.7

$$\left(1 + \frac{1}{1,000,000,000}\right)^{1,000,000,000} = 1.000000001^{1,000,000,000} = 2.718281827$$

Rather than being some vary large value (as might be intuitively expected due to the large exponent) we see that this value is actually only approaching, converging to, or reaching a maximum limit (final value) of about 2.71828 as n approaches a (endlessly, infinitely) very high value. Here is why the value is not actually larger:

Given the power of: $y = \left(1 + \frac{1}{n}\right)^x$

Note that when n approaches a value of infinity that the fraction (1/n) gets less and less, approaching a value of 0, and therefore, the base of this power actually approaches a value of 1. When n is approaching infinity and the indicated exponent x=n, the rate of change of this power value with respect to n is approaching a value of 0. That is, the slope of

Ex.   Let: ct = n = e^x   hence:

    n = 1e^x   : The constant of 1 will undergo continuous compounded growth for x periods.
                                    Solving for x by taking the natural logarithm (ln) of both sides of the equation:

   ln n = ln 1e^x
   ln n = x ln e   since ln e = 1 and switching sides:
   x   = ln n   hence also, the expression n = e^x could be written as: n = e^x = e^(ln n)

   Ex.   5 = 1e^x = 1e^(ln 5) = e^1.60948   : for 1 to grow to a value of 5, 1.60948 periods are needed

    Ex. 10 = 2e^x   Here 2 will undergoing continuous compounded growth for x periods.   Solve for x.
                        After dividing both sides by 2 for some simplification:

     5 = 1e^x = e^(ln 5) = e^1.60948   : x = ln 5 = 1.60948

In general, given: n = ce^x   dividing both sides by (c):

                  n/c = e^x   taking the logarithm of both sides:

            ln (n/c) = ln e^x
            ln (n/c) = x ln e   : ln e = 1

               x = ln (n/c)   also, according to the basic definition of a logarithm:

            e^x = ln (n/c) = x

Ex. Find x in: 37.6 = 1.6e^x

    x = ln (37.6/1.6) = ln 23.5 = 3.157

Ex. Find c in: 37.6 = ce^3.157

    37.6 = ce^3.157 = c 24.5   ,   c = 37.6/24.5 = 1.5347

When (c) is not multiplied by a power of (e), but divided by a power of (e), the value of (c) will undergo natural decay:

ct = _c_ = ce^-x   : a common natural decay expression. Remember that e^0 = 1 , ie.x=0, and x<1.
     e^x   With ce^+x , there is a natural growth, even if c<1and is a fractional value.
                       With ce^-x , there is a natural decay

For example, when a heated object naturally cools down to any temperature, it's temperature will undergo natural decay and these facts will be included in a given formula relating the objects temperature with respect to time.

A derivation of the reciprocal of (e) is nearly identical to that for (e), except that instead of adding a value, a value is subtracted per interval of the period:

e^-1 = (_1_ - _1_)^n = (_n_-_1_)^n = 0.367879441 = _1_   : as n approaches infinity
       (1   n)   (n   )   e

complete that cycle or wave. Frequency (f) and corresponding time (t) value are mathematical reciprocals of each other.   (f)(t) = 1

PERT , The PERT formula

     "Pert" or Pe^(rt) is a common formula for a bank that offers continuously compounded interest. It will be shown below how Euler's Number (e) comes into use at the bank and in other formulas elsewhere. Below, P = Principle (similar to Mo=initial money amount) is the starting amount that will be grown by the growth formula applied to it. Variable (I) = the interest rate.

  Total Value = Starting Value + Interest

  Pt = P + PI   Pt = total principle after interest is applied to it. Factoring P from each term:
  Pt = P (1 + I)   If interest is to be calculated again after another period:
  Pt = Pt + (Pt)(I) = Pt (1 + I) = P(1 + I) (1 + I)   Which can be expressed mathematically as:
  Pt = P (1 + I)^2

If there are several intervals (n) in the period, the interest for each interval will be the total interest for that period divided by that number of intervals, and the exponent will also be equal to that number of intervals (of interest calculation) in the period:

  Pt = P (1 + $\frac{I}{n}$)^n

Now in relation to Euler's Number (e) where the number of intervals approaches infinity for naturally compounded growth:

Letting interest rate = I = 100% = 1 = r , and n=x to express this as the classic power of a binomial expression for (e) which can be expressed with some grouping:

  Pt = P (1 + $\frac{1}{x}$)^x   If the interest rate was not 1, but another value, this can be expressed as:

  Pt = P (1 + $\frac{r}{x}$)^x   And this is equal to the following expression.

  Pt = P (1 + $\frac{1}{x}$)^{xr}   We see that the rate (r) essentially becomes a power to (e). This is verified below at
                        (*) and (**). This can be expressed as a power to a power:

  Pt = P $\left[(1 + \frac{1}{x})^x \right]$ r   As the number of intervals (x) in the period approaches infinity, this is equivalent to:

  Pt = P e^r   If there are several periods (t) of time of this growth calculated, then the formula will include
                     a power of (t) due to repeated multiplication, and which (t) could even be a fractional value.

  Pt = (P e^r)^t   which can be expressed as:

  Pt = P (e^r)^t   :this is a power (of e) raised to another power, and this can be mathematically expressed as:

$$(e^x)^2 \quad = \quad e^{(2x)}$$

$$\left[\left[\left(\frac{1}{1} + \frac{1}{n}\right)^n\right]^x\right]^2 \; = \; \left[\left(\frac{1}{1} + \frac{1}{n}\right)^n\right]^{2x} \; = \; \left(\frac{1}{1} + \frac{1}{n}\right)^{2nx} \; = \; \left(\frac{1}{1} + \frac{2}{n}\right)^{nx}$$

as shown below:
(x can be 1 also)
When n--> infinity, that power of the binomial is defined as e=2.718 **…**

$e^2 = e^1 e^1 = (1/1 + 1/n)^n \; (1/1 + 1/n)^n \; = \; ((1/1 + 1/n)^n)^2 = (1/1 + 1/n)^{2n} = e^2$: when n is a very high value

If interest rate (I = r) doubles, say from 1 to 2 we would find this expression for the value of growth after (n) periods of time.

$(1/1 + 2/n)^n$ when n is very high, this is equal to:

$(1/1 + 1/n)^n \; (1/1 + 1/n)^n \; = \; e^2 \; = \; (1/1 + 1/n)^{(n+n)} \; = \; (1/1 + 1/n)^{2n} \; = \; ((1/1 + 1/n)^n)^2 \; = \; e^2$

(**) We see a pattern that the indicated power; the exponent of e is equivalent in value to the interest rate. Also consider when n is a very high value:

$e^2 \; = \; ((1/1 + 1/n)^n)^2 \; = \; ((1/1 + 1/n)^2)^n \; = \; [\, (1/1 + 1/n) \, (1/1 + 1/n) \,]^n$

$[\, (1/1 + 1/n) \, (1/1 + 1/n) \,]^n \; = \; [\, 1^2 + 1(1/n) + 1(1/n) + (1/n)^2 \,]^n$

$[\, 1^2 + 1(1/n) + 1(1/n) + (1/n)^2 \,]^n \; = \; [\, 1 + 2(1/n) + (1/n^2) \,]^n$: as n--> infinity, this last term approaches 0 very quickly, and is practically of no effect

$e^2 \; = \; [1/1 + 2/n]^n$, and if this is multiplied by $[1/1 + 1/n]^n = e$, the result is:

(e^2) $\qquad$ (e^1) $\qquad$ which can be expressed as:

$[1/1 + 2/n]^n \quad [1/1 + 1/n]^n \; = \; (\, [1/1 + 2/n] \, [1/1 + 1/n] \,)^n \; = \; (\, [\, (1/1) + (1/n) + (2/n) + (2/n^2) \,] \,)^n \; =$

$(\, [\, (1/1) + (3/n) \,]^1 \,)^n \; = \; [\, (1/1) + (3/n) \,]^n \; = \; e^3$ as n---> infinity

The first term is: $a^n b^0 = a^n(1) = a^n$. For each next term, the exponent of (a) will be one less, and the exponent of (b) will increase by one. The last term is: $a^0 b^n = (1)b^n = b^n$.

The (numerical) coefficients of the (a) and (b) variables are symmetrical (ie. the same value) on each side of the middle of the sum of all the terms.

For the example above, the coefficients of the variables in each consecutive term are: 1, 4, 6, 4, 1, and note the similar value (here, 1 and 4) at both ends. Here is a formula for the numerical coefficients (c1, c2, c3, . . .cn+1) of those variables:

1. The first and last coefficient is 1.

2. The second coefficient is (n)

3. The next term's coefficient (ct+1) is found from it's previous term's coefficient (ct):

ct+1 = _ct (exponent of variable (a))_   : t = term number.   All values on the right side correspond
                    t   to term (t) which is essentially the previous term used to find
                                                                 the constant of the next term (t+1).

Any constants past the (n+1) term are 0, and therefore all those terms will have a value of 0. Here is a (non-standard) formula for these constants that does not rely upon the previous constant. This may be skipped over (to: *) if its excessive to the reader, and since its non-standard:

Notice that for the first term of each expansion, the exponent of (a) is equal to (n). For the next term, the second term, the exponent of (a) is one less or (n - 1) and for the next term, the exponent of (a) is two less than (n) or (n-2), and so on:

$$c3 = \frac{c2 (n - 1)}{2} = \frac{\frac{n (n - 1)}{1!}}{\frac{2}{1}} = \frac{n (n - 1)}{(2)(1!)} = \frac{n (n - 1)}{2!} \qquad : 2! = (t-1)! = (3-1)!$$

$$c4 = \frac{c3 (n - 2)}{3} = \frac{\frac{n(n - 1) (n-2)}{2!}}{\frac{3}{1}} = \frac{n (n - 1) (n-2)}{3(2!)} = \frac{n (n-1) (n-2)}{3!} \qquad : 3! = (t-1)! = (4-1)!$$

Summarizing the results:

$$c1 = \frac{1}{0!} = 1 \quad , \quad c2 = \frac{n}{1!} \quad , \quad c3 = \frac{n(n-1)}{2!} \quad , \quad c4 = \frac{n (n-1)(n-2)}{3!}$$

To be consistent with (t - 1)! , 0! is shown above and it is formally defined as: 0! = 1.

The formula resulting from these expressions for the constants is:

1. Starting from the second term, each numerator has (t - 1) factors.

The first factor is (n).
Each next factor is 1 less:
        The second factor is: n - 1 = (n - 1)
            The third factor is: (n - 1) - 1 = n - 1 - 1 = (n - 2)   , and so on.

When one of the variables, or terms, of the binomial is equal to 1, the Binomial Formula can be simplified or reduced to what is commonly called the **Binomial Series**. If (a) = 1, all of the powers of it are equal to one, and it can essentially be eliminated or removed as a factor in each term. Expressing this:

$$(1 + b)^n = c1 + c2b^1 + c3b^2 + . . . + cn+1b^n \qquad : \textbf{Binomial Series}$$

Even though the (a) variables were eliminated from the expansion, the same concepts and expressions for the constants are the same.

The results of using the Binomial Series are somewhat, but not exactly similar to that of the Binomial Formula:

Expect (n + 1) terms if (n) is an integer.

When (n) is not an integer, the expansion of the binomial becomes an infinite series of terms. The value of variable (b) must also be less than 1, otherwise, the terms will become larger and larger (since its' exponent is becoming larger and larger), and therefore, the series will not converge to a specific value. The lower (b) is, the fewer terms required for a specific number of (most significant, higher numeric weight, leading) accurate digits in the result.

The binomial series can be used for general mathematical calculations such as for taking powers or roots of values. Below, the binomial series is used to find a series for powers of (e):

The core or fundamental expression for (e) is a binomial raised to an indicated power (ie., indicated with an exponent). Since the first term (a) in the binomial is equal to 1, the binomial series can be utilized to "expand" it so as to find its value.

$$e = \left(\frac{1}{1} + \frac{1}{n}\right)^n \qquad \text{or} = (1 + 1/n)^n \qquad : \text{(e) is defined where (n) is very high (infinite) in value.}$$
Raising each side to the (x) power:

$$(e^1)^x = \left[\left(\frac{1}{1} + \frac{1}{n}\right)^n\right]^x \qquad \text{using the power to a power rule on the right hand side:}$$

$$e^x = 2.718281828^x = \left(\frac{1}{1} + \frac{1}{n}\right)^{(nx)}$$

To find any power of (e), this binomial can be expanded into a (power) series of terms by using the Binomial Series, where the exponent will be (nx), and (b) will be (1/n). Since (a)=1, it will essentially not be needed in this expansion since all powers of 1 have a value of 1, and 1 as a factor does not change any value of a term. Showing this algebraically, we will arrive at a simplified formula for finding powers of (e):

$$e^x = c1 + c2b^1 + c3b^2 + c4b^2 + . . . \qquad : e^x \text{ expressed as a power (of b) series}$$

$$e^x = c1 + c2\left(\frac{1}{n}\right)^1 + c3\left(\frac{1}{n}\right)^2 + c4\left(\frac{1}{n}\right)^3 + . . . \quad \text{distributing the power to the num. and den.:}$$

$$e^x = c1 + c2\left(\frac{1^1}{n^1}\right) + c3\left(\frac{1^2}{n^2}\right) + c4\left(\frac{1^3}{n^3}\right) + . . . \quad \text{simplifying:}$$

$$e^x = c1 + \frac{c2}{n^1} + \frac{c3}{n^2} + \frac{c4}{n^3} + . . .$$

$$c1 = 1$$

DERIVATION OF THE DERIVATIVE OF e^x

We can find the derivative of the function: e^x by taking the derivative of the expansion of e^x. To do this, take the derivative of each term and sum the derivatives. The result is amazing since the derivative of this function is actually the same function. That is, the derivative of e^x is e^x. Given (x), the derivative (dy/dx = de^x/dx) or slope of the curve at that point is equal to e^x = f(x) = (y) which is the corresponding coordinate of (x) on the curve. Obviously, the greater (x) is, the greater the value of e^x = slope at that point on the curve.

Since the first term is a constant, its derivative is 0.

The derivative of the second term is: $\dfrac{1\,x^{(1-1)}}{1!} = \dfrac{1x^0}{1!} = 1$

The derivative of the third term is: $\dfrac{2\,x^{(2-1)}}{2!} = \dfrac{2x^1}{(2)(1!)} = \dfrac{x^1}{1!}$

The derivative of the fourth term is: $\dfrac{3x^{(3-1)}}{3!} = \dfrac{3x^2}{(3)(2!)} = \dfrac{x^2}{2!}$ therefore:

$\dfrac{dy}{dx} = \dfrac{d(e^x)}{x} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \;.\;.\;.\; = e^x$: **Derivative of e^x , series expression**
The derivative of e^x is e^x.

By knowing that the derivative of e^x is e^x, and that e^0 = 1, you can also write a Maclaurin series for e^x which will produce the same results as shown above for the series for powers of (e).

As another advanced verification to the derivative of e^x:

   From: y = a^x   : an exponential equation, and it's derivative is:

      $\dfrac{dy}{dx} = \dfrac{d(a^x)}{dx} = a^x \ln a$,   if a=2.718281828... = (e),   we have:

      $\dfrac{dy}{dx} = 2.718281828^x\;\ln 2.718281828$

      $\dfrac{dy}{dx} = e^x \ln e = e^x (1) = e^x$

      We see that the derivative equals the original function. For other values of (a), lower or higher than 2.718281828 = (e), the derivative of a^x will not equal just the same function of a^x because it will also include the factor of (ln a). Since (e) is a valid value for (a), it gives a verification to the formula for the derivative of an exponential equation of the form: (a^x).

A similar formula for (e) is: e = lim n-->0   of: (1 + n)^(1/n)

Euler's Formula or Equation is: e^(ix) = cos x + i sin x   : i is the imaginary unit or number of: sqrt(-1)

For example $\sqrt{-7} = \sqrt{(-1)(7)} = \sqrt{-1}\,\sqrt{7} = i\,\sqrt{7}$

In the series for (e), if you substitute (ix) for (x), you can get this expression for Euler's Formula.

and the equation are always positive in sign, regardless if (x) is positive or negative in value, and due to the rules for multiplying signed values . Here, the squaring of the variable creates the curve ("mirror") symmetry about the (y) axis.

Below are some examples of typical curves which involve (e) as a base of a power in an exponential equation, and the results are similar for values other than (e). In general, the greater (e) and-or (x) is, the steeper the slope of the curve. [FIG 210]
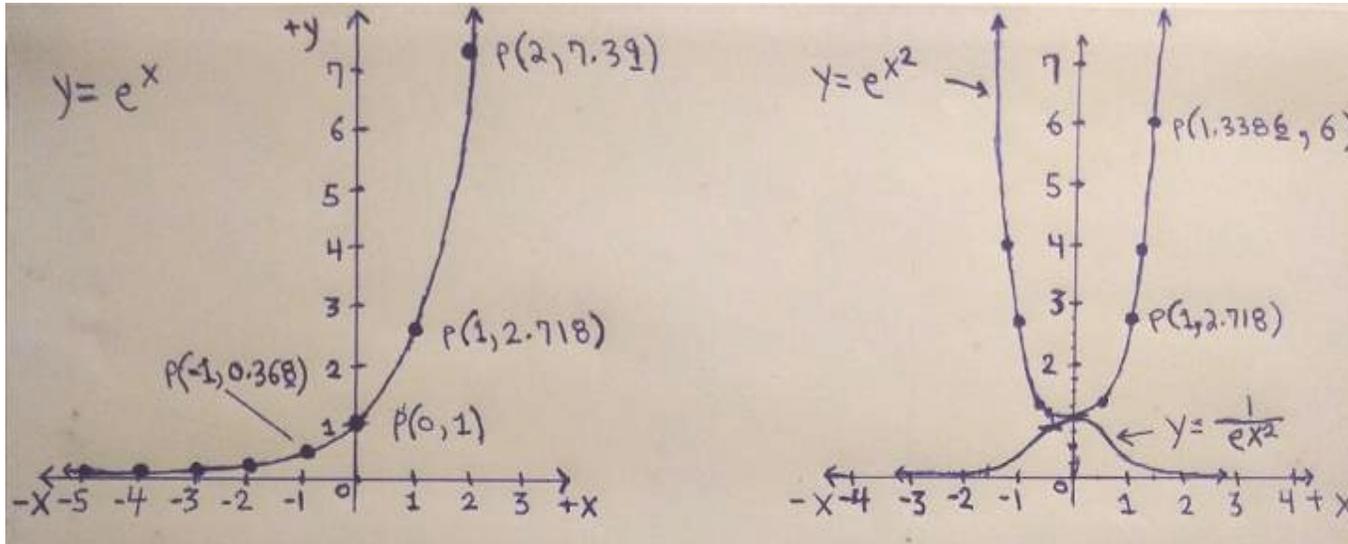
For curve above for y=e^x, notice that the values of (y) when x<0 (ie., negative in value) are reciprocal in value to the (y) values when (x) is positive, and that their product is therefore 1. For example: when x = 1, y = 2.718, and when x = -1, y = 0.368. The product of (2.718) and (0.368) = 1.   (e^x) (1 / e^x) = 1 = (e^x) (e^-x) = e^(x-x) = e^0 = 1 .   For this curve, when (x) is negative in value: y = e^(-x) = 1/e^x, and this curve is shown further below below.

On the right side is the bell curve of: y = 1/e^(x^2) , and the reciprocal of this equation and curve is: 1/y = 1 / e^(x^2) = e^(x^2), and this curve is also shown. Here, the corresponding values of (y) for the same value of (x) on each curve, are reciprocal in value.

The reciprocal curve, equation and values of: y = e^x is 1/e^x = e^-x , and this curve is shown below:

[FIG 211]
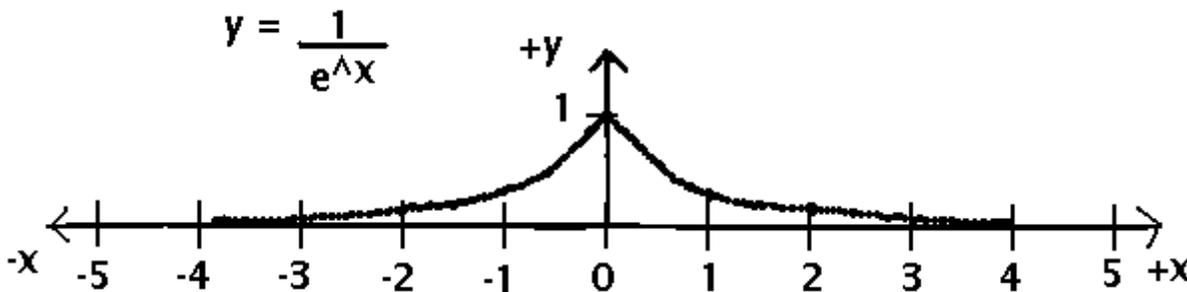
Note that: y = e^x + 1/e^x = e^x + e^-1x Is the catenary curve.   This curve could be described as the natural sag or bend of a suspended (supported at its ends) line that is perpendicular to a field of attraction such as gravitational force.

A DERIVATION OF THE NATURAL LOGARITHM SERIES

This derivation for the natural logarithm series stems from the facts that the derivative of e^x is itself, and from the derivative of the natural logarithm of (x) as will be explained here:

Given: e^y = x   : taking the logarithm of both sides of this equation

      ln e^y = ln x   by the log rules, this can be expressed as:

      y ln e = ln x   : since ln e = 1, this simplifies to:

        y = ln x   according to the basic logarithm definitions:

      e^y = x   taking the derivative (with respect to y) of both sides of this equation:

$$\frac{d(e^y)}{dy} = \frac{dx}{dy}$$: since the derivative of a power of (e) is equal to that same power of (e), for example, given: e^x , d(e^x)/dx = e^x . We now have:

$$e^y = \frac{dx}{dy}$$ since e^y = x, and y = ln x as shown above, and using substitution, we have:

$$x = \frac{dx}{d(\ln x)}$$ taking the reciprocal of both sides of this equation, and switching sides:

$$\frac{d(\ln x)}{dx} = \frac{1}{x}$$: **DERIVATIVE OF THE NATURAL LOGARITHM OF A NUMBER**

Note that the derivative of the natural logarithm of a number is equal to the reciprocal of that number. The higher the number (x), then the slope at that point on that log curve or graph approaches 0 which is the slope of a horizontal line. This is since given y=ln x, and when (x) is high in value, changes in (x) will cause almost no effect or changes in the value of (y). Note also that while (1/x) is algebraic , (ln x) is not algebraic, but is transcendental.

Now that we have the derivative of the natural logarithm of (x), we obviously cannot simply take the inverse (anti-) derivative of (1/x), which results to: (ln x), so as to find a usable expression (such as a series) for the natural logarithm of (x). Even if we tried taking the inverse derivative of (1/x) = (x^-1) using the method shown previously, it will also not produce a usable result since according to that method: increasing (adding) the exponent of (x), here -1, by 1 is 0, but then the division by this value of 0 is not mathematically allowed:

$$\frac{1}{x} = x^{-1}$$ taking the anti-derivative: $$\frac{x^{(-1+1)}}{0} = \frac{x^0}{0} = \frac{1}{0}$$: The quotient is undefined; has no specific value.

In order to find a usable expression for (ln x), we need more to work with, and this is an equivalent expression for (1/x). Perhaps there is a reciprocal series that is equivalent to (1/x) that we can use to find the inverse derivative of, and therefore have an expression for (ln x). As it turns out, the general reciprocal series is a bit more complex to work with than this "modified" reciprocal series we can use that will (only) evaluate the reciprocals of numbers (N) between 0 and 2:

$c1 = \dfrac{1}{0!} = 1$, $c2 = \dfrac{n}{1!}$, $c3 = \dfrac{n(n-1)}{2!}$, $c4 = \dfrac{n(n-1)(n-2)}{3!}$

$c1 = 1$, $c2 = \dfrac{-1}{1}$, $c3 = \dfrac{-1(-1-1)}{2}$, $c4 = \dfrac{-1(-1-1)(-1-2)}{6}$

$c1 = 1$, $c2 = -1$, $c3 = 1$, $c4 = -1$

Substituting the constants:

$(1 + x)^{-1} = \dfrac{1}{x+1} = 1 + (-1)(x^1) + (1)(x^2) + (-1)(x^3) + . . .$ using distribution:

$(1 + x)^{-1} = \dfrac{1}{x+1} = 1 - x^1 + x^2 - x^3 + . . .$: checks , (modified) Reciprocal Series

Finding the inverse derivative of each term above to have the complete inverse (or anti-) derivative of this series, we have the following logarithm series which was developed by Nicolaus Mercator:

$$\ln(x+1) = \sum_{n=1}^{n=\infty} (-1)^{(n+1)} \dfrac{x^n}{n} = +\dfrac{x^1}{1} - \dfrac{x^2}{2} + \dfrac{x^3}{3} - . . . : x < 1$$

For this series to converge to a specific value, (x) must be less than 1. The terms will get smaller and smaller in absolute value. This series can evaluate the natural logarithm of values less than 2. This series is very important and-or useful, and a method has been developed as shown in this book of how to calculate the natural logarithm of numbers greater than 2.

For a simple formula for ln x , you can use the following, however without a computer or calculator, you will need to compute a power of (x) which for here, is actually a root of (x):

   $\ln x = n(x^{(1/n)} - 1)$: as n-->inf , formula by Leonhard Euler
                        After simplifying, this is equal to:

   $\ln x = nx^{(1/n)} - n$: as n-->infinity

A similar formula is:

   $\ln x = \dfrac{x^n - 1}{n}$: as n --> 0

From the derivative of ln x, as shown previously, we mathematically have:

   From: $y = \ln x$, $dy/dx = d(\ln x)/dx = 1/x$:

   $\dfrac{d(\ln x)}{dx} = \dfrac{1}{x}$ from this equation, we mathematically have:

   $d(\ln x) = \dfrac{1}{x} dx$ All the small parts or pieces of something considered as one whole thing will sum to that whole thing. This is called integrating. Each small part of (ln x) is noted as d(ln x), and all

BASIC DEFINITIONS OF THE TYPES OF NUMBERS

Below are some of the various general types, categories, sub-categories, or classifications of numbers that you may sometimes see and-or read about, and it is would be helpful to at least know a basic or minimal definition of them. As you work more and more with numbers, you will obtain a natural, uncomplicated feel for what these numbers are.

natural number - Natural numbers are those used for basic counting and representing a
quantity of whole (entire) things or units often found nature (plants, rocks,
trees, stars). Ex. 1, 2, 3 , ...

counting number - Our man-made counting numbers are essentially the same as the natural
or "natures numbers".

whole number - An integer value greater than or equal to 0, hence it is also positive in value.
A whole number will also not have, include or indicate a fractional part (of 1).
Ex. 0 , 3 , 7 , 1000   Whole numbers are used to enumerate complete or whole
units, objects or entities. 0 is included since it numerically or symbolically
represents an absence.

integer number - A signed counting number, hence can be positive or negative in sign. An integer
also does not have, include or indicate a fractional part.   Ex. -5 , 0 , +5 . The
word "integer is rooted in the words: "wholeness", "complete", "integrity",
"untouched" and probably; an "instance" or "next step" of.   The word "integer"
has some companion words such as: "increment" which means an "increase".

fractional number - A number that represents a value or quantity of something that is less
than 1 or 1 whole entire unit or entity. A fraction of a thing is smaller piece
or part of that entire or whole thing, hence it is not the whole thing. Its
numerical value or representation of that part or quantity of the whole
(considered as having a "relative" numerical value of 100% = 1), will
therefore be less than 1.

Ex. 0.5 = 1/2 = 5/10, etc. , 0.5<1   ,   Ex. 0.1 = 1/10 = 2/20, etc. , 0.1 < 1

Although a fraction is always numerically less than, and only a part of the
whole thing, if the whole thing is greater than 1, such as a group of things
or units being considered, the actual or physical fractional part may then
be numerically a value greater than 1. For example, if you have 100 bricks,
a tenth (1/10 = 0.1, a fraction of 1 or the entirety) of those 100 bricks is
10 bricks, and this is more than 1 brick. These 10 bricks may also be said
as being "ten percent" (10% = 0.1) of all the bricks. 10 bricks is still a
fraction or part of all the 100 bricks considered as the 1 entirety, 100%,
whole or all.

mixed number - A number containing a whole number part, and a fractional part or portion.
Symbolically: mixed number = whole number part + fractional part
or: mixed number = whole number part . fractional part
Ex. 5.2 ,   5.2 = 5 + 0.2   ,   5 is the whole number part, and 0.2 is the

rational number - A rational number is any number that can be expressed as the quotient
or result of the division of two integer values. The fractional part of the
result will be either 0, a number with fixed or certain number of decimal
places, or a number which contains an "endless" repeating of the same
decimal digits after a certain decimal position. All rational numbers are
said to be "algebraic" (ie. can be produced (ie., a result) by a single
variable polynomial equation).

Ex. $4/2 = 2.0$, $5 / 2 = 2.5$, $1/3 = 0.3333....$, $4/7 = 0.571428571428...$

The word "(to) rationalize" basically means "to make (simple) sense or
reason of".

irrational number - An irrational number cannot be expressed as the quotient or result of
the division of two integer values. That is, it cannot be adequately
expressed as a single fraction. The "expansion" or continuation of further
decimal digits is endless and does not repeat (ie., have a constant pattern).

Ex. PI $= 3.14159....$,   e $= 2.71828 ....$,   square-root 2 $= 1.4142135....$,
and the golden ratio $= 0.618034....$ are irrational.

The word "irrational" basically means: "does not make (simple) sense or
reason", or is "beyond (simple or rational) reason and-or mathematical
representation or expression".

transcendental number - A transcendental number is a number that is "not algebraic", that
is, it is not the root of a single variable polynomial equation with
rational coefficients.   A root is the solution of that variable in the
polynomial equation that is set equal to 0. All transcendental
numbers are also irrational (non-rational) numbers.
Ex. PI and Epsilon (e) are the best known transcendental numbers.
Note for example that squarer-root 2 is an irrational number, but it is
not a transcendental number since the square-root of 2 is the
"algebraic" solution of: $1x^2 - 2 = 0$.
The golden ratio (GR) is also irrational, but is not transcendental
since it is the "algebraic" solution of: $1x^2 - 1x - 1 = 0$, and the
solution here is found to be:
$1/GR = 0.618033988749894 = 1.618004...$
The equation shown can also be expressed as: $x^2 - x = 1$

The word "transcend" basically means "outside of" (such as standard
mathematics such as basic algebra, or beyond "algebraic rationality").
 "A number from out of the blue". "A number from out of nowhere".
Transcendental numbers are real numbers.   The trigonometric
functions are considered as transcendental functions since the results
take an infinite number of steps to complete, and therefore can't be
represented or expressed as a ratio of two numbers or with a simple
algebraic formula such as the angle or trigonometric value divided by
some other value. In general, logarithm and exponential functions are
to be considered as transcendental functions that generally result in a
transcendental value, and especially due to how it may be calculated as
a sum of an infinite series of terms that will converge to a specific value.

Some Common Constants, Formulas, Equivalents And Conversions Within And Between Measuring Systems

SOME COMMON CONSTANTS:

pi = 3.14159265358979323846...   : the ratio value of: (circumference / diameter) of all circles. (pi) = C / D

e = 2.71828182845904523536...   : epsilon, Euler's number, base (of measurement) for natural logarithms

GR = 0.618033988749894...   : Golden Ratio (GR), a ratio of two specially related lengths or amounts.
                                     1/GR = GR+1 = 1.618033988749894... ,   GR^2 = 1 - GR and GR^2 + GR = 1 ,
                                     GR^2 = 0.38196601

Vc = 186282.4... miles/1s = 299,792,458 meters/1s = about 300 million m/s   :speed or velocity of light (c).
                                                                                (in a vacuum with no matter)

        Using (300 million) m/s is only about 0.0692% =~ 0.07 = 7 hundredths of 1 percent greater than the true value of
        speed of light, and is also ~= 186,000 miles/s

Vs = 1125 ft/s = ~ 343 meters/s =~ 767.27 miles/h   : velocity or speed of sound through air, at about room temperature
              0.343 km/s =~ 1235 km/h   ~1100 ft/s or= ~ 350meters/s . Thunder sound called "thunder"
                                                         will travel about 1 mile in 5 seconds.

g = 9.807m/s^2 = 32.1752ft/s^2   : Earth's constant gravitational acceleration of an object due to gravity which is the
                                     attraction of masses (material, substances, matter), and is measured as a force.
                                     This specific value is due to the specific mass of the Earth, and is generally not the
                                     same value on the Moon or other planets since their mass is more or less than
                                     that of the Earth.

                     From: weight = force = (mass)(acceleration) = ma = mg :

                     g = weight / mass = a = f / m   : force = weight is measured with units of newtons (N),
                                                     : mass is measured with units of kilograms (kg)
                                                     : 1gram of mass = 1kg/ 1000 = 0.001kg of mass
                                                      (g) and (weight) are directly proportional to each other
                                                      for a given mass.

                     Ex. Using a known mass of 1kg, and if it was weighed on a force scale to be 9.8N,
                         and on some other planet or moon, it might weigh 1.6N, then (g) can be
                          calculated as:

                     ge = a = _f_ = _9.8N_ = _((9.8kg)/1m) / 1s^2)_ = _9.807m_   : for Earth
                              m   1kg   1kg   1s^2

                     gm = a = _f_ = _1.6N_ = _((1.6kg)/1m) / 1s^2)_ = _1.6m_   : for Moon
                              m   1kg   1kg   1s^2

                     The ratio: gm / ge = 0.16315 =~ (1/6). A mass on the Moon will weigh about
                         16% or one-sixth as that of the weight Earth. Since weight is directly
                         proportional to (g), the ratio of the weight of an object on the Moon to that of
                         the Earth also has the same ratio value:

two masses with direct contact, and when considering forces that are not directly applied, such as gravity, and the fact that its strength is inversely related to the distance between those two masses, (r^2) is used as a divisor, and it is not equal t, but analogous to:
 force = mg / r^2 or= gm / r^2 .

Also, as a reminder, a given force can more easily move a smaller mass, and-or do more work (force x distance) with it due to that the more massive object has more inertia and will require more energy to move it, and-or accelerate it to change its velocity. Also of note is that if one mass is less than 1 kilogram, there will be no effective magnification or amplification in the resulting force. This leads to the question of whether a more massive object has that much more gravity, and due to this equation, it can only be said that a more massive object has more gravity force potential, influence or possibility upon another smaller mass. Note also that:
G = Fn r^2 / (M1 M2)

A basic derivation of the gravitational force equation:

First, the gravitational force Fg between two objects or masses (M1 and M2) is the same value (regardless of their amount of mass) due to the concept of equal and opposite forces, and this could be expressed as:

Fg = F1 = F2   : here the Force(s) is the gravitation force

Fg = (M1 a1) = (M2 a2)

We know that the more massive an object is, the more gravitational force or attraction to another mass it will have, and its value of (a) will depend upon the size of that mass, and is directly related to that value. On Earth, the specific value of (a) is (g) = 9.81 m/s^2, and for another object or planet with a different mass, the local or surface value of (g) will be different. Since Fg = F1 = F2 is the same between both masses, and that F = ma, the resulting kinetic acceleration and velocity will be higher of the smaller mass than that of the larger mass.

We know that M1 is generally not equal to M2. Given a mass having a weight or force, if that mass increased by a factor, its corresponding weight (ie., an effect or force due to gravity [force]) will also increase by that same factor.

Something having twice the weight will make a dent twice as deep in the ground when dropped from the same height, and this is due to that the force upon reaching the ground was also increased by that same factor.
From :   F = weight = ma ,   multiplying both sides by a factor (n): nF = (n) weight = (n)ma

This implies that the Fg between a mass and another will be a multiple or relative (in numerical reference) or factor the mass value, and a multiple can be expressed as a product. Each mass will have its own local gravitational attraction (g) upon another mass. This force (Fg) is like an effective or resultant force due to an effective or resultant mass value so that Fg between them is the same value, hence of an equal and opposite value. The two masses essentially amplify the force of each other and can be expressed as a multiple or product of each other.

In F = ma , both (m) and (a) can amplify a (force=weight) when those values are amplified (increased by a factor of (n), and if (m) is constant, then (a) determines that force and-or weight.   Since (a) can amplify the force or weight, it is as if the given mass rather was amplified instead. The effective result for a two mass gravity system is that each mass is essentially amplifying each other so as to have an amplified, equal and opposite force (Fg). An equivalent force (Fg = Ma) upon each mass implies an effective or apparent amount of mass, and at the same acceleration (a=G) and its value was first mathematically determined by Henry Cavendish using a torsion (ie.,twisting) balance and various masses to cause a gravitational attraction force on the masses, and then a twisting of the balance by an angle

METRIC PREFIXES: [For multiples or fractions of a unit or reference of measurement. Here are some common
[prefixes, and this topic was previously mentioned in this book.]

kilo means: (1000) = "a thousand of"
centi means: (1/100) = "one-hundredth" or "one-hundredth of" the units used, indicated or expressed
milli means: (1/1000) = "a thousandth of"

ABBREVIATIONS:

in=inches, m=meters, cm=centi-meters, km=kilo-meters
ft=feet, g=grams, L=liters, lb=pounds, oz=ounces, fl=fluid (ie., a [liquid] volume)
s= seconds of time, hr=hours, m=minutes

FREQUENCY AND TIME:

Time=1/Frequency = Ts=1/Fhz. hz=hertz = vibrations, cycles or waves per second. Time is the total time of each and
every similar cycle or wave, and it is mathematically inversely related to the frequency of that vibration, cycle or wave.
The higher the frequency of a wave, the lower or shorter (ie., "faster" , "quicker") the time needed to complete that 1 wave.

Wavelength is essentially the physical distance or length between similar or corresponding points (such as the peak
[maximum] amplitude value, "peak to peak") on the wave (of vibration, energy cycle or pulses) during its cycle as it travels
outward from the source. The wavelength of a wave will be different for energy waves that travels faster or slower such
as the speed of light and the speed of sound in air. Wavelength can also be calculated using the frequency of the wave.

Since wavelength is a length or distance, and distance = (speed of movement)(time of movement) = (speed)(time):

electromagnetic wavelength = (speed of light)(Ts) = (Vc)(Ts) = (Vc)(1 / Fhz) = Vc / Fhz

Ex. For typical AM radio wave with a 1 Mega-hertz-cycle , the signals wavelength and basic antenna length is:
(300,000,000 m/s) / (1Mhz 1/s) = (300(10^6) m/s) / (1 (10^6) 1/s) = 300m
This leads to a simple and commonly spoken formula for radio wave wavelength in meters:
(300 / frequency in Mhz) = wavelength in meters

Frequency (F) and Wavelength (W) are inversely related as shown in the formula above. Also consider
these inverse proportions: Vc = (W1)(F1) = (W2)(F2) , Therefore mathematically: (W1/W2) = (F2/F1) ,
and this is an example of what is described as an "inverse ratio (of values and units)".

As a simple example, if a given frequency is multiplied by 2, the resulting wavelength is divided by 2.
For example, for a typical FM radio wave of 100Mhz, the wavelength is: 300m/100 = 3m
For a 1Ghz radio wave, 1Ghz = 1000Mhz, the wavelength is: 300m/1000 = 0.30m = 30cm

Compared to low frequency, long wavelength signals such as for AM radio, a high frequency, short wavelength
FM radio transmitter and-or receiver antennas are much shorter in length, and may even be shorter by using a half
or quarter wavelength antenna which are very practical, but they will reduce the transmission efficiency of the input
electrical energy. You may research a common antenna called a dipole (ie., two pole).

Wavelength And Frequency Of Visible Light: 1Thz = 1 terahertz = 1(10^12)hz=cps = 1 trillion hertz = 1000Ghz
1nm = 1(10^-9)m = 1 billionth of a meter

(**RGB**) =: **R**ed = 700nm , 428Thz , **G**reen = 530nm , 566Thz , **B**lue = 470nm , 638Thz = 638 (10^12)hz

White light appears as the brightest color, and it is due to the primary colors: RGB and where their frequencies
are overlapping or combining and effectively simulating a higher frequency when they are all received and sensed

Sun to Earth = 93 million miles	500s = 8.333m	: 93 million miles = 1 AU of distance
Earth to Moon	~ 1.3s	: 240,000 miles away
1 Million miles	~ 5.38s	: 1 million miles =~ 0.0107527 AU (=~ 1/100 AU = 0.01 AU)
1 Billion miles	~ 1.5h	: 1 billion miles =~ 10.753 AU. (=~ 10 AU)
		Saturn is almost a billion miles away.

In 1 second, light will travel about 186000 miles = 0.002AU per second , and:
In 1 minute, light will travel about: (186000miles/s)(60s) = 11.16 million miles = 0.12 AU per minute
In 1 hour, light will travel about 670 million miles = 7.2AU per hour
In 24 hours = 1 day, light will travel about 16.08 billion miles =~ 16 billion miles = 172.8 AU =~ 173 AU per day
In 1 year = 1 light-year = 365 days, light will travel about: 5.870 trillion miles = (365 days)(173 AU per day) = 63145 AU

TEMPERATURE:

The temperature (ie., heat, thermal energy) of a substance (matter, mass) is a measure of its atom or molecule vibration (movement, motion, moving, or "kinetic") and its corresponding energy level. When this type of motion does not increase in value, it is considered that the substance or mass did not absorb or acquire any more external energy. When a mass with a higher temperature is joined to one with a lower temperature, energy, via the atomic movements or vibrations, the energy of the mass with the higher temperature will be transferred to the mass with a lower temperature until the temperature of the two masses are the same value.   The mass with the higher temperature will loose some energy, and the mass with the lower temperature will gain that same amount of energy.

When all atomic or molecular vibration or motion stops, this temperature is defined as "absolute zero" and is the reference of measurement for the Kelvin (K), or "absolute" [scientifically true or natural] temperature scale, and each degree of temperature increase in this scale is identical to that of the Celsius scale. Water freezes at about 273K = 0°C = 32°F, and boils at about 373K = 100°C = 212°F.

If an object or mass is now twice its initial starting temperature of reference value, it is thought to have gained the same value of its starting atomic or molecular thermal energy. The (real, solid) matter or mass (technically, the measure of matter) is storing that energy.

"Absolute Zero" where all thermal energy is depleted and atomic vibration or motion stops or "freezes" is defined as:
0°K = -459.67°F = -273.15°C. The Kelvin scale is credited to William Thomson who was also the 1st Baron of Kelvin (a river and area) of Britain, and he is more commonly known as "Lord Kelvin". He conceived this absolute or true temperature scale in about 1848.

Fahrenheit temperature = F° = 1.8C° + 32   : a derivation of this formula is shown in this book
    Celsius temperature = C° = (F° - 32) / 1.8

Some selected temperatures and their converted values:

	-28.88°C = -20°F	
	-23.3°C = -10° F	: 10 below zero, or 10 below or less than a 0°F reference temperature
	-17.8°C = 0° F	: 0°F was defined as the freezing point (value) temp. of a special "brine" fluid
water freezes: 0°C	**= 32°F**	: water freezing point at standard atmospheric pressure of 14.7psi
"a chilly day": 10°C	= 50°F	: a chilly (cool, cold) day can be perhaps from ~ 40°F to ~60°F
"room temperature": 22.2°C	= 72°F	: this is a human subjective, average comfortable living or "room" temperature
"a hot day" : 30°C	= 86°F	: The air temperature measured in the shade. ~87° or greater.
		A high humidity level can often make the air temperature feel much warmer,
		especially when the temperature is above 72°F.
water boils: 100 C°	**= 212°F**	

For temperatures above 200°F, the equivalent Celsius scale temperature is approximately half. Likewise, for

Power (P) is the (average) rate of using energy during a time interval, and has units of Watts. Power is therefore not the total amount of energy transferred and-or used during a length of time. Power is how much (average) of total amount of energy was needed, transferred, and-or used during a period or amount of time.

Power = (total energy / time) = (joules / time) watts.   : rate of using energy or= doing work   ,   Also: 1W = (1V)(1A).

   1 Joule (J) of energy used for 1 second of time can be defined as 1 watt of energy:

   1W = $\dfrac{1J}{1s}$, mathematically: **1J = 1W-s = 1 watt-second**

   If 1 watt of power is applied or used for 1 hour = 3600 seconds of time:

   Total energy used = (power)(time)   : derived from the power equation above

   (1W)(1hour) = 1 w-h = $\left(\dfrac{1J}{s}\right)$ (3600 s) = 3600J of energy   : 3600Joules = 1W-h = 1 Watt-hour

   Note the difference of the meaning of the units mentioned above:

   watts is the energy used per an amount of time, hence watts is a rate of using energy ,   and

   watt-hours is the total amount of energy used   : Wh = (watt-hours) = (watts)(hours) = (J/s)(3600s) = J / 3600

If 1 watt is applied , used or converted for 1 hour, this can be expressed as: 1 watt-hours = 1w-h = 3.412 **BTU**/hour = (1056 joules/BTU) (3.412 BTU) = 3603 joules/hour ~= 3600J/hour. Consider that if 1watt-second = 1w-s = 1J/s, and if the time length or duration of using this energy is 1 hour = 3600 seconds, then the amount of energy would be 3600 w-s= 3600 J/hour = 3.412 BTU/hour.   Note that 3600 w-s is not equal to 3600 watts per second, but is rather the total amount of watts used during the 3600 seconds = 1 hour of time, hence a total of 3600 watts (total) per hour. If 3.142 BTU/hour = 1 watt-hours, then after multiplying both sides of this equation by 3600, we have: (3600)(1) watts-hours = (3600)(3.142 BTU/h) = 12283 BTU/hour. Also from the above information, 1BTU/hour = 0.293 watt-hours =~ 0.3 Wh.

For electricity: 1J = 1 w-s = for example =: (1V)(1A) (1s) or for example =: (10V)(0.1A)(1s) :Joules = watt-seconds

A **calorie** (thermal) energy unit is defined as the thermal energy gained within a gram = 1g of water when its temperature is increased by 1 degree Celsius. **1 calorie = 4.185J (joules) of energy**, and it is roughly equal to 4.2J   For a given temperature, it is not too difficult to conceive that when the mass of an object is (n) times more than that of a lesser mass object, then it can hold that many (ie., n) times more total thermal (heat) energy. This book discusses calories, specifically for food, in more detail in this book.

Freezing Sea Water To Remove The Salt, and other related topics:

Because of the salt (sodium chloride molecule, table salt) in ocean or sea water, salt water is more difficult to freeze, that is, it takes a slightly lower temperature for it to freeze at about 28.4°F =~ -2°C. Pure water takes 32°F = 0°C to freeze. This difference in the freezing point is about 3.6°F or 2°C less. This is due to the interference of the salt molecule with the water (H20 molecule) ice crystal being formed.   When salt water freezes, the ice will not have the salt in it, and this is one method, called freeze desalination,so as to **desalinate** salt water. It is generally incorrect to think that freezing water is the best way to remove impurities and-or make it safe to drink. If very cold ocean water can be placed in a pond in a cold climate, it will be able to freeze naturally and more rapidly since there is (hopefully) less wind and no ocean wave movement giving the water some kinetic energy and preventing it from freezing faster into a nice crystal form. Sea or ocean water is about 3.5% salt by weight, hence 1kg = 1000g of salt water will have about 3.5g of sodium chloride which is table salt. Ice (ie., frozen water) can be melted to a liquid water form by using various solar heating methods such as a

1/64in = 1 sixty-fourth of an inch = 1in/64 = 0.015625 in = 0.396875mm = about 0.4mm = slightly less than half of 1mm

1/10 in = 0.1 in = (2.54cm / 1 in)(0.1in) = 0.254cm = (10mm/1cm)(0.254cm) = 2.54mm
1/12 in = 0.0833 in = 0.2116cm =~ 2.12mm

3 in = 12in/4 = 1ft/4 = 0.25ft , 3 in = one quarter of 1 foot
4 in = 12in/3 = 1ft/3 = 0.333...ft = 10.16 cm = about 10 cm = 100mm   , Note: 10cm length x 10cm width x 10cm height
            = 1000cm^3 = a volume called a Liter (L).   4 in = one third of a foot
6 in = 12in/2 = 1ft/2 = 0.5ft   , 6 in = one half of a foot
8 in = (4in)(2) = (0.333...ft)(2) = 0.6ft

1ft = 12in. , 12in (2.54 cm/in) = 30.48 cm =~ 30.5cm ~= 30 cm = 30.48cm x 10mm/cm = 304.8mm (roughly 300mm)
1ft = 0.3048 m = 30.48 cm = 304.8 mm =~ 305 mm
10 centimeters = 3.93701 inches , = ~ 4 inches
12 in =1 foot = 1ft = 0.3048 meters (m) ,   : average length of an adult male human foot, but later: 1/5280 of a mile (*).
1ft = 0.3048m , 5ft = 1.524m =~ 1.5m , 10ft = 3.048m =~ 3m , 100ft = 30.48m =~30m , 1000ft = 304.8m =~ 300m

1 mile = 1mi = 5280ft = 1609.34 meters = 1.60934 km   : (*) a Roman mile was originally and arbitrarily just 5000ft
   which was the length of 1000 (ie., millennium or mil, for mile) complete cycles of average sized both (2) left and right
   steps or "paces" of an adult man out walking a distance for traveling. 1 stride = 2 paces. 1 step or pace = about 2.5
   feet. At an average of 5 feet per stride, the length of 1000 strides is 5000 feet. The English mile is based on the
   Roman mile. The English or British mile is the multiple of 660 and 8 = 5280. 660ft is the length of 1 furlong
   commonly used for farming and property such as for an acre (an area of land). 1 furlong = (1/8) mile.   A speed or
   velocity of: 1mi/1hr = 5280ft / hr = 5280ft / 60min = 88ft / min = 88ft/60s = 1.46<u>7</u>ft/s =~ 1.5 feet / s. Or:
   5280ft / 1hr = 5280ft / 3600s = 1.46<u>7</u> ft / 1s =~ 1.5ft / 1s = 18in / 1s

  An average walking speed of an adult is about 2mi/hr =~ 3ft/s. A casual bicycle speed is about 6mi/hr on flat ground.
   0.625 mi =~ 1 km

    1 mile = 1.609 km =~ 1.6 km = 1609 meters = roughly one and a half kilometers = 1.5km
    5 miles = 8.04672 km =~ 8 km
   10 miles = 16.0934 km =~ 16 km   : 10mi/hr = 14.66 ft/s =~ 15 ft/s
   15 mi = ~ 24 km , 20 mi =~ 32 km , 25 mi =~ 40km , 30 mi =~ 48 km , 35 km =~ 56 km , 40 mi =~ 64 km
   50 miles = 80.4671 km =~ 80 km , 60 miles = 96.561km =~ 100km
   62.135mi = 100 km
  100 miles = 160.934 km =~ 160 km

1 nautical mile = arc length distance of 1 arcminute on Earth = 1min of arc ¢ from N. pole to S. pole. = 1.852km = about
            = 6076ft = 1.15078 miles.   Nau is an old word prefix for things pertaining to boats and navigation hence.

   360° = 360 arcdegrees = 360 (60) = 21600 arcminutes = 21600 (60) = 1296000 arcseconds
      From the center point of the Earth: A 360° arc = an arc length of 1 circumference of the earth =~ 25000 miles
      1° or 1 arcdegree on the surface of Earth corresponds to: ~ 25000 miles / 360° =~ 69.44 mi = ~ 70 mi
    1 arcdegree = 360°/360 = 1° = 1 degree   ,   1 arcdegree = 60 arcmin = 3600 arcseconds
      From horizon to horizon = 360° / 2 = 180° = 21600 arcminutes / 2 = 10800 arcminutes = 648000 arcseconds
    1 arcmin = 1arcdegree / 60 = 1°/60 = 0.016666...° = 0.01666<u>7</u> arcdegrees , 1 arcmin = 60 arcseconds
      The Moon appears to us on Earth as being 0.5° wide = 0.5° / (0.01667°/1 arcmin) = 30 arcmin = 1800 arcseconds
      The Moon's diameter is about 2159 miles , 1800 arcseconds / 2159 miles =~ 0.83<u>4</u> arcseconds / mile
         Or: 0.5° / 2159 mi. = 0.0002316° / mile
    1 arcsecond = 1arcmin / 60 = 0.01667° / 60 = 0.00027777...° =~ 0.00028° arcdegrees

   For what each degree on the Earth's circumference length corresponds to, we can set up an equivalent ratio or
   proportion equation: 360°/ 25kmi as = 1° / x mi , x mi = 69.44mi , (1°/60) = 0.01667° = 1 arcminute angle = 1"

1m = 100cm = 39.37 inches = 39in + 1(1"/4)in + 1.92 (1"/8) =~ 39.5in = ~40in = ~= 4 (10in) = 1.093yards
1m = 100cm = 1000mm = 1,000,000um   : 1cm = 10mm and 1mm = 1000um ,   Extra: 1cm = 10000um

1 km = 1000meters = 0.621373 miles =~ 0.6 mi
    1 km = nearly 5 furlongs ~= 5 (660ft) = 3300 ft . 5 furlongs = 1.00584km exactly defined
    1 km = 3280.84 ft =~ 3281ft .   3280.84 ft / (5280 ft/mi) =~ 0.6214m =~ 62% of a mile , 1mi = 5280ft
    1.609km =~ 1.61km =~ 1 mi
    5 km = 3.107mi   =~ 3 mi   ,
    8 km =~ 5 mi
    10 km = 6.2137mi =~ 6 mi , 15km=~9km , 20 km =~12 mi , 25 km =~ 15 mi , 30 km = ~ 18 mi , 40 km =~ 24 mi
    50 km = 31.069mi =~ 31 mi = about 30 miles
    100 km = 62.137mi =~ 62 mi = about 60 miles

1mm = (1/1000) m = 1(10^-3)m = 0.001m = "one-thousandth" of a meter = "one millimeter"
   A poppy seed is about 1mm in length and weighs about 0.0003g, and there are about 3333 poppy seeds per gram.
1mm = (1/10) cm = 0.1cm =   "one-tenth" of a centimeter =~ 0.03937in = ~ 0.04in =~ 0.63 sixteenths of an inch
   Ex. 2.5 cm = 2cm + five-tenths of a (1) cm = 2cm + 5mm : (since a millimeter is a (one) tenth of a centimeter)
   Ex. 1 in = 25.4mm = about 25mm ,   2in = about 50mm   ,   4in = about 100mm
1um = 1 micrometer = (1/1,000,000)m   = 1(10^-6)m = "1 micron" = "one millionth of a meter" = 0.001mm
1um = (1mm/1000) = 0.001mm
1nm = 1 nano-meter = (1/1,000,000,000) = 1(10^-9)m = "one-billionth of a meter" = "1 thousandths of 1 mico-meter"

1A = 1 angstrom = 1(10^-10)m = (1/10)nm = 0.1 nanometer = "one-tenth of a nanometer"

1 barleycorn (average barley seed length, from the center of the seed pod) = (1/3) inch = 0.3333 in
   Barley is one of the ("cereal") grass plants (barley, wheat, oats, corn, rice) that produce edible seeds (or "corns")
   called grains. **Three average size barleycorn lengths from the center of the seed pod were used to defined an
   inch unit**.

   The English shoe scale is based on the barleycorn = (1/3) in. For the length of the shoe, size 12 = 12 inches long,
   and each size lower is 1 barleycorn less in length, hence (1/3) of an inch less. Size 11= (12in - 0.33in) = 11.66 in.
   For some extra comfort, many will wear up to a size bigger (ie., longer) so as to have some extra toe ("wiggle room")
   space, and-or to consider a bigger size that may be usable for wide feet when wide-width shoes are unavailable.

1 cubit = 20.625" = 0.5236m   (about 20" or half a meter)   Some cubit measurements vary from 18" (Bible) to 22", and
 20in is an average = 50.8cm. **20.6" = 52.324cm is the Egyptian Cubit; typically used**. This value is approximately half
a meter = 0.5m. A cubit is an ancient measuring unit for length, and mentioned in the Torah, Old Testament book which is
about 4000 years old at the current time of writing of this book. The Egyptian cubit was further divided into 28 (finger
("digit") sized widths) markings or steps. The cubit was often divided into steps of basic fractions (1/2, 1/3) of a cubit. A
"palm" or "palm width" was "4 fingers wide".   A cubit has 7 of these palm units, hence a palm unit is equal to a 1/7
fraction of a cubit unit. **As an example of expressing the difference and percentage between a measurement and a
reference value:**   If 20in is used as the cubit, then it is slightly less and in error of the 20.6in Egyptian Cubit by only:
error = (value - reference value) = (20in - 20.6in) = -0.6in, and this error value as a percentage of the reference value is:
(error value / reference value) =   (-0.6in / 20.6 in) = -0.02913 = -2.913% =~ -3%. As a ratio of the two given
measurements: (value / reference value) = (20 in / 20.6 in) = 0,97087 =~ 97%.   That is, the average value is about 97%
of the Egyptian reference value, and about: (1 - 0.97) = 0.03 = 100% - 97% = 3%   percent less. Though this is a small
percentage of error, it will become significant if may cubits are involved.

Mils. A mil is 1/1000 of an inch. A mil is sometimes used by a machinist so as to be more precise than say a value
    with units of 32nds or 64ths of an inch. Since metric units are now common, the mil unit is not used much.

1 mil = 1in / 1000 = 0.001 inch =   a thousandth of an inch   : note, 1" = 1in = 1 inch = 2.54cm = 25.4mm
1 mil = 0.001 in = 1 in / 1000   = 2.54cm / 1000 = 0.00254cm   , after dividing this number of cm by 10:

Ex. 52.4 cm = 52cm + 0.4cm = 52cm + 4mm
Ex. 1.542 m = (1.542m)(100cm/m) = 152.4 cm = 100cm + 52.4cm = 1m + 52cm + 0.4cm = 1m + 52cm + 4mm

Here is an similar example of the above, but it is for how to comprehend a mass value with metric units such as kilogram (Kg) units. Note that 1 Kg = 1000g units.

1.352794 kilograms (Kg)

kilograms   grams   milligrams
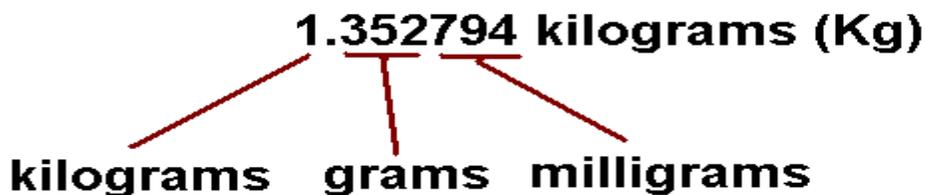
1Kg = 1000g
1g = 1000mg

1 kilogram + 352 grams + 794 milligrams

Ex. 1.524378649 Kg = 1 kilogram + 524 grams + 378 milligrams + 649 micrograms =
            = 1kg + 524g + 378mg + 649ug   : here the value after the decimal point is less than 1 Kg
                                      : micrograms may be expressed as mcg or ug
Ex. 0.003 Kg = 0 kilograms + 3 grams = 3g

Ex. 1.234 g = 1 gram + 234 milligrams = 1g + 234mg   : here the values after the decimal point is less than 1 gram
Ex. 0.5g = 0 grams + 500 milligrams = 500mg
Ex. 0.001g = 0 grams + 1 milligram = 1mg

For a given amount of time, distance and velocity are directly related and proportional in value.

distance = (velocity)(time) = v t

From: d = v t   , $\dfrac{d}{v}$ = t   , and if (v) changes by a factor of (n), then distance will also change by that same
                        factor of (n):

   (n)d = (n)v t   , $\dfrac{d}{v}$ = $\dfrac{d(n)}{v(n)}$ = t = $\dfrac{d1}{v1}$ = $\dfrac{d2}{v2}$, and mathematically: $\dfrac{d1}{d2}$ = $\dfrac{v1}{v2}$

Note also that: d = vt is a linear or line equation of the form: y = mx , and $\dfrac{y}{x}$ = m = v = slope = a constant for a line.

Likewise, in a similar manner, if (v) is a constant value, (d) and (t) will be directly related and proportional in value:

   d = v t   , mathematically:

   $\dfrac{d}{t}$ = v ,   (n)d = v (n)t   , v = $\dfrac{d}{t}$ = $\dfrac{(n)d}{(n)t}$ = v = $\dfrac{d1}{t1}$ = $\dfrac{d2}{t2}$, and mathematically: $\dfrac{d1}{d2}$ = $\dfrac{t1}{t2}$

Surface Areas

A cube has 6 square sides of equal areas. Each side is s^2, therefore, the surface are of a cube is:

Surface Area Of A Cube = 6 (Area of 1 of the cube faces) = 6s^2   **: surface area of a cube**

For a rectangular prism, much like a stretched cube:

Surface Area Of A Rectangular Prism = Sum Of Area Of Its 6 Sides . There will be 3 pairs of 2 equal area sides.
Each area can be found from:
Any rectangular area = (length of side)(width of side) = (L)(W) sq. units

At = (areas of side pair 1) + (area of side pair 2) + (area of side pair 3)
At = 2(L1 W1) + 2(L2 W2) + 2(L3 W3)   : **surface area of a rectangular prism**

For the surface area of a sphere:

As = 4 (pi) r^3   : **surface area of a sphere**

Smaller Objects, In Or Of A Larger Object, Will Have A Larger Total Surface Area

If you decrease the radius of a sphere, you obviously (by observing the formula for surface area of a sphere) will decrease its surface area. If you have a large amount of tiny spheres, say within another known amount of volume, such as that of a sphere or cube, their net or total surface area is greater than that of the largest possible sphere in that volume.   A large value of surface area has some practical applications, particularly when trying to get maximum contact between substances and-or for more efficient processes. Consider a block or cube cut in half, there will now be more surface area.

The ratio of a spheres surface area to its volume is found using division of the two formulas, and is: (3/r). The smaller the radius (or diameter) of a sphere, the resulting value of (3/r) will increase, and this effectively means that the ratio of the surface area to volume, of a smaller sphere, is larger.

One way to help verify this is to imagine a unit cube or volume of 1 (unit^3)=cubic unit. If you divide its dimensions in half, each part, cube or smaller volume, will be only: (1/2)^3 = 0.5^3 = 0.125 unit^3 , but the total volume will remain the same. (1 unit^3 / 0.125 unit^3) = 8. There will be 2^3 = 8 smaller cubes if you divide a cubes dimensions in half. This is somewhat similar to dividing the radius of a sphere in half, but with a sphere, its surface area will decrease by 4, and its volume will decrease by 9. The total volume will be the sum of all the smaller segments of a certain volume. The total surface area will be the sum of all the smaller segments of a certain surface area. Dividing a volume up into smaller volumes will not result in a sum of those volumes being greater than the source volume.

The surface area of the cube with 1 unit, with 6 sides or surfaces, is: 6 (length x width) = 6 (1 x 1) = 6(1) = 6 (units^2).
The surface area of that cube divided up into 8 smaller cubes when its dimension are halved (divided by 2) is:

8 (surface area of one small cube)
8 ((6) (length x width)) = 8 ((6) (0.5 x 0.5)) = 8(6)(0.25) = 48 (0.25) = 12 square-units
Note the total number of (smaller) surfaces is 8 times more, and that the total surface area doubled.

For a thought experiment consider a circle, and all about that circle's circumference on the inside or outside of it are smaller half-circles side by side, and clearly, the (arc) length of each half-circle or semi-circle is longer than the portion of the larger circle's circumference it subtends or is on. This helps verify that the surface area is also larger for the smaller semi-circle(s) and theoretical smaller spheres.   Consider a very large circle where the circumference is line-like for a small segment of it and the shortest distance is directly along its path, clearly an arc of a semi-circle placed on that segment is longer, and it has a larger surface area.

Weight and mass are directly related and proportional. If something has twice as much matter or mass, it will weigh twice as much and vice-versa. The mass or amount of matter or material of an object can be found by weighing it. See the topic of WEIGHT shown below. 1 amu is sometimes called a nucleon since it is part of the nucleus of an atom.

1g = 1 gram. This was initially defined as the weight of 1 cubic centimeter volume of water due to Earth's gravity force. Anything that weighed 1g therefore had the same about of substance or matter.   1 gram of cotton has the same weight and mass (ie., matter, substance) as that of 1 gram of iron.   Using volumes or containers to measure a true amount of substance has various problems such as randomly sized air gaps between the substance, such as between apples, grain, cotton, fabric, etc., and it creates a doubt as to what amount of matter or substance is actually in that volume. Surely a solid cube of gold weighs less than a hollow cube or box of gold that is the same size.

1g = the (corresponding) weight of 1cc of water or water mass = 1cm^3 of water = 1mL of water.

1g is also the base unit of the amount of mass, matter or substance, and without any influence of gravity.
    1g of mass has 1g of weight displayed on a common weight scale, and this allows finding an amount of mass by using a common weight scale calibrated to display the corresponding mass value of an object or substance.
    1g of mass, matter, material of any substance or element will have the same weight. This is since it will have the same total number of fundamental mass particles or atomic mass units (amu) which are protons and neutrons.
    1g of water is actually 1 gram of water molecules which are composed of hydrogen and oxygen atoms, therefore, a fraction of that gram of mass or material of water is hydrogen and another fraction of that gram is oxygen.
    Since atoms of different elements have different mass, the weight of those atoms is different. An atom of hydrogen does not weigh the same as an atom of oxygen. A mol count of hydrogen atoms does not weigh the same as a mol count of oxygen atoms.   Still, a gram of hydrogen atoms does weigh the same as a gram of oxygen atoms.
    A water molecule is composed of 2 hydrogen atoms and 1 oxygen atom. Hydrogen has 1 amu per atom, and oxygen has 16 amu per atom. 2 hydrogen atoms will have a mass of 1amu + 1amu = 2amu. A water molecule therefore has 2amu + 16 amu = 18 amu per water molecule. Since 16 amu / 2 amu is 8 times more, the weight of the oxygen atoms in any amount of water will be 8 times more than the weight of the hydrogen atoms, yet each water molecule has 1 oxygen atom per 2 hydrogen atoms, and that there is not 8 times more atoms of oxygen in a water molecule.   Mass does correspond directly to the number of (amu) particles, but not to atom or molecule sized particles which contain various amounts of amu or mass depending on the specific element or substance such as a molecule, and therefore these particles will have a different (total) mass. An oxygen atom has more amu per atom than a hydrogen atom, and therefore it has more mass and weight than a hydrogen atom.

1g = 0.035274 oz = ~ 3.5 hundredths of an ounce ,   1 oz = 28.3495g, a "Troy ounce" is 31.1035g, a "food ounce" is standardized to be simply 30g of mass. Since a pound (lb) is 16oz, a "food pound" is therefore: (30g)(16) = 480g

1g = 1(10^-3)kg = 0.001kg   : a gram is 1gram / 1000 = "a gram is one-thousandth of a kilogram"

1g of mass actually weighs = force = (mass)(acceleration) = (0.001Kg)(9.81m/s^2) = 0.00981Newtons

1 milligram = 1 mg = one-thousandth of a gram = 0.001g.
1 microgram = 1mcg = 0.000,001g = one-millionth of a gram = one-thousandth of a milligram = 1mmg = 1 ug = 1mcg

A slug is a unit of mass in the (older, previous)) US Customary or US Common, and British measuring systems.
This unit is based on forces and-or weight of a substance, whereas today, mass is no based on an amount of substance. **This unit has become outdated and is nearly obsolete**, but is useful to know about for any possible reference and-or conversion needed. The word "slug" is from same word used in reference to piece of bulk-metal or "bullion" yet to be processed into smaller metal items.

1 slug = 1 slug of mass = a mass of 1 slug

Torque = (force)(lever-arm) = (force)(radius distance from center) =
Torque units are (lb)(ft)   = lb-ft

WEIGHT: **Weight is equivalent to the downward force of a mass due to the influence or force of gravity (g) of the Earth being applied to and pulling downward on that mass and giving it acceleration, and even when the object appears still, and not in motion, and this can be observed by the constant weight value indicated for object on a weight measuring scale.**

Weight = a Force = (mass)(acceleration) = F = ma or= mg

Some units of weight are outdated, but are still needed for past reference and conversions.

In the S.I. or Metric System, weight = force due to gravity = (mass)(acceleration) = (kg)(9.81m/s^2) Many modern "(metric) weight scales" will automatically convert the weight of the mass to just its mass value. For example, an object that weighs 9.8N = (m)(a), has a mass of 1kg and will be displayed on the scale as 1kg.   This type of scale is actually a (user friendly) "mass scale" since for many people would rather have the (universal) mass value,rather than its corresponding force or weight value.   1Newton of force or weight is defined as = 1N = (1kg)(1m/s^2) = ma = f

1N = 1 Newton of force and-or it's equivalent weight = 1N = (1kg)(1m/s^2) = (mass)(acceleration) = force = ma.
   The velocity of a 1kg mass or object will change or increase by (1m/s) and for each 1 second of time if this amount
   of force (ie., the application of energy, inducing or causing a transfer of energy) is still being applied to that object
   and increasing its kinetic energy.   In the presence of Earth's gravity force which applies a higher acceleration of
   9.8m/s^2 near the Earths surface altitude (or practically less than a few hundred miles high), the mass would be
   less:

1N = force or= weight = (mass)(acceleration) =~ (0.1kg)(9.81m/s^2) = (100g)(9.81m/s^2) ~= 0.981N
1N = 101.97g =~ 102g   :or equivalent gram-weight (force) on Earth = **roughly 100 g ~= 0.1 kg   ~=   0.25 lbs**
1N = 3.5969452724 oz =~ 3.6oz
1N = 0.22481 lb   : noted as about or roughly 0.25lbs = a quarter of a pound, shown above
9.81N =~ 10N =~ 1kg , or more correctly: ~ 1kg in gravity = weight of 1kg of mass = 1kg weight = 1kgw

1 lb = 1 pound of force. 1 pound of force or simply "1 pound". An amount of mass that happens to weigh 1lb on a
   weight = force scale, could be said as being the weight of a mass of: 1lb-mass = "1 pound of mass" = 1 mass-lb,
   or a mass having a force of one pound in Earth's gravity force and acceleration of about: 32.2ft / s^2 = 9.81m/s^2
   Weight and mass are proportional, and if one changes by a factor, the other will also change by the same factor.

1 lb = 1 pound = 1 lb-weight = 1 lb-force = 1lbf = 1 one pound of weight in the avoirdupois system of weight = 7000
   average center grains of barley. A barley grain is a seed of barley and is therefore called a "barleycorn". The
   corresponding mass which would weigh or have a force or weight of 1 lb in the influence of Earth's gravity is called
   a pound-mass = lbm. The pound is used in the US Customary and (British) Imperial Measuring system, however,
   the metric or SI (System International, or International System) measuring system use kilograms for mass,
   Newtons for force, and meters for distance are now widely used. The English word and unit of weight called a
   "pound" is based on the Latin word "poundus" which is a unit weight. A phrase sometimes still heard is "pound it
   down" or "pound it in", which basically mean to force or press something down or in. So that people do not have to
   have barley seeds and to count them all, equivalent or reference weights of 1 pound and other weights are
   available and are commonly used, such as for calibrating (checking, adjusting) weight scales so as to be accurate.
   The (previous in history, ancient) Roman libra (L) or pound (lb) was equivalent to about 0.75 pounds today.

1 lb = 16 oz. of weight or force = corresponds to a mass of: 0.45359237kg = 453.59237 grams = about **454 grams** in
   the influence of Earth's constant gravity force causing an acceleration of (a=g=9.8m/s^2) and causing a
   force = weight = (mass)(acceleration) = ma = mg.   1lb =~ 45% of a kilogram, or vaguely as 0.5kg = half a kilogram
   Since 453.59237 grams = 453.59237 cc of water, taking its cube root we have a side length of: 7.683432 cm. =

1 carat (ct) = 1 "carat weight" = 200mg = 0.2g = 0.007055oz , hence 1g = 5 carats , such as for the mass of precious, rare minerals, stones and metals, such as a diamond. This word is derived from the word carob seed used as a common reference weight.

A karat = a fractional (here equal to 1/24) measure and unit for the percentage of pure gold in a metal alloy. 1 karat means that the substance is or contains (1/24)% pure gold. (1/24)% = 4.16$\underline{7}$% , 14 karat is (14/24) = 50% pure gold , 24 karat is (24/24) = 1 = 1.0 = 100% pure gold.   1 karat means that roughly 4% of the total mass and-or weight of a substance will be pure gold. 10 karat will mean roughly (4%/carat)(10 carats) = 40% of the total mass and-or weight will be pure gold.

Some common reference and-or calibration weights, and lengths:

When you do not have a known reference or calibrated weight or mass, such as 1oz or 1g, you can use some common items which are known to have a certain weight, mass or volume. You can use multiples and-or combinations of the known weights or masses so as to obtain or sum to a larger reference weight and-or mass. Many of the items listed below will require a fine (weight to equivalent mass) scale which can measure, say a gram to a high precision, say 2 decimal places such as (1/100) = 0.01 of 1 gram, and if you want to verify its mass and-or weight. If the item(s) you are weighing need a container to hold them in, be sure to first measure the weight of this container so as it can be subtracted (ie. to "tare", remove) from the total weight measured or displayed. Some electronic scales have a tare button which will set the scale to a value of 0.0, and to be pressed or activated after the empty container is placed on it.

1 **U.S. penny coin** = 1 cent = (1/100) of 1 dollar (1 USD $1 = 100 cents), monetary value, and base unit of the USA monetary system. If this coin was made in 1981 or earlier, it is mostly copper and with a mass of 3.11 grams = (3.11g)(1oz / 28.34953g) = 0.109702 oz.. For 1983 and later, 97.5% zinc, weighs 2.5 grams = 0.088185oz. The mass of two modern "zinc pennies" will be 5 grams. Its diameter is 0.75 in. = 3/4 in. = 12 sixteenths of an inch = 6 eighths of an inch = 19.05mm = about 19mm. Rim thickness is defined at 1.52mm. 4 of these penny coins side by side in a line, will have a total length of 3 in. 16 of these would have a total length of 12in = 1ft.   2 modern pennies would have a mass of: 2.5g + 2.5g = 5g = the mass of 1 U.S. nickel coin.
1 **U.K. pence coin** = penny coin = 1p = 1 cent or (1/100) of 1 pound monetary value. 1p = 0.01 GBP (Great Britain Pound). From 1992 and later, the coin is made with copper coated steel. mass = 3.56g, diameter = 20.3mm , thickness = 1.65mm.
1 **Euro cent coin** = (1/100) or 1 cent of a European Union dollar = E0.01 or $0.01E. Since 1999, it is composed of steel with a copper coating. Its diameter is 16.25mm, its thickness is 1.67mm, and its mass = 2.30g
1 **Rupee coin of India** = stainless steel since1992 , mass = 3.76g , diameter = 21.93mm , thickness = 1.45mm,
1 **U.S. nickel coin**, 75% copper, weighs 5 grams = 0.17637oz. , width is 0.835in = 21.21mm , rim thickness is 1.95mm
1 **Canadian nickel coin**, year 2000+, weighs 3.95grams = about 0.139oz , diameter = 21.2mm , thickness = 1.76mm
1 **Australian nickel or 5 cent coin** = 75% copper, 25% nickel. mass = 2.83g, diameter = 19.41mm , thickness = 1.3mm
1 **U.S. dime coin**, nickel clad, pure copper interior with a nickel-copper plate, 91.67% total copper, weighs 2.268 grams Diameter is 0.705 in = 17.91mm = ~ 0.7 in.
1 **Australian dime or 10 cent coin** = 74% copper, 25% nickel, mass = 5.65g, diameter = 23.6mm , thickness = 2mm
1 **U.S. quarter coin**, nickel clad plate, 91.67% copper, weighs 5.67 grams = 0.2 oz. A 1932 to 1964 silver quarter is 90% silver and 10% copper, weighs 6.25 grams = 0.22 oz. Diameter = 0.955 in = 24.26mm =~ 1 in
1 **Euro dollar coin** = E1 or $1E, As of 2002, diameter = 23.25mm,   thickness = 2.33mm, mass = 7.5g , various metals
1 **Australian dollar coin** = 92% copper, 6% aluminum, 2% nickel , mass = 9g , diameter = 25mm , thickness = 2.8mm
1 **U.S. Dollar Bill** = 75% cotton and 25% linen [flax plant] = 6.14in =155.956mm long = about 156mm, and 2.61in=66.294mm wide. Its diagonal length is about 6.912in long = 175.56mm = about 7in. Its weight is designed to be 1g, but could vary by a few hundredths of a gram depending on the moisture in it. This 1g defined weight is for the paper being in an environment that is said to have about 50% or average air humidity.

Craft wire, steel metal, ~0.5mm thick, ~35mm long = 3.5cm long = ~ (22/16)in = 1in + (6/16)in = 1.375in , weighed 0.1g Since 0.1g is to 1.375in, we can express this as: 0.1g/1.375in , Multiply num. and den. by 10, we have 1g / 13.75in 3.5mm of this wire mathematically corresponds to a mass of 0.01g = a hundredth of a gram

The scale shown has units of millimeters. The coins, from left to right are: quarter, dime nickel, and penny. As an additional reference, a common AA size battery is typically a total of 50 mm = 5 cm long, and 14 mm = 1.4 cm wide. 1 cm = 10 mm. The image above has been magnified, and should not be considered as the actual physical size. The 4 reference coins shown in the image have a total width of about 81.5mm = 3.201in The combined width of the penny and the nickel is 40.25mm =~ 40mm = 4cm. A strip of paper 4cm long (L) can be folded in half so as to have 2cm = (L/2) or half-length, and that can be folded in half so as to have a 1cm = (L/2)/2 = L/4 or quarter-length. This 1cm length of paper can be divided into two 5mm= (L/4)/2 = L/8 or one-eighth-length. For a 1/3 length of a line or one-third of any length= L/3, one method to find this length is to cut a strip of thin paper that is the total length in question, and then bend it into an S curve shape, and then a Z shape, and then flatten this structure so as to have three equal lengths, and with each being (L/3).

1 fluid ounce volume of water weighs approximately 1dry or solid ounce of weight. [technically, 1.040843 ounces].
1 U.S. fluid ounce volume of water weighs 29.57353 grams = 29.57353 cc of water ~= 30 "food, fluid, grams" = 30cc
1 fluid ounce of volume of water is about 29.57353 mL of water = 1.80469 in^3
1 British or "Imperial" fluid ounce = about 28.41mL , and weighs 1 oz = 1/16 of a pound
1 cup volume of water = 8 fluid oz, volume, of water =~ 8 dry or solid ounces, weighs 236.588 grams = 236.588cc of water
1 cc volume of water weighs 0.033814 dry or solid ounces = 1 gram of mass
1 liter volume of water = 1000cc of water = 1000g = and weighs about 2.2 lbs = 33.814 dry oz of weight =~ 2.113375 lbs
1 U.S. gallon volume of water = 16 cups volume of water =128 fluid ounces volume of water weighs 8.326744 pounds

A much less common today, U.S.A. "customary measuring system" used the slug as the unit of mass, and a pound as the unit of force (such as weight). 1 slug = 14.5939 kg. A mass of 1 slug has a weight, due to the force due to gravity, of 32.2 pounds. 1 pound of constantly applied force can accelerate a mass of 1 slug by 1 ft/s^2. From: force = weight = (mass)(acceleration) , 32.2 pounds = (1 slug)(32.2ft/s^2). Hence 1 slug of mass of any substance(s) will weigh 32.2 pounds. In this system, which often uses Imperial (an older British system, before the metric system) units, the units of pressure is = force / area = pounds / ft^2 , and-or pounds / in^2. Torque in this system is torque = (force)(distance) = (force)(radius) = pound-inch = lb-in or in-lb. Here, the radius or "lever-arm" value is essentially a "force multiplier" value. Another common torque unit is the foot-pound or pound-feet.

If the acceleration is: a = $\underline{\text{32.2 ft}}$ = $\underline{\text{9.81m}}$: = g , Here, this is the acceleration of a object due to the
                                    1s   1s   constantly applied force of gravity applied to it..

F = N = (m)(a) , and due to gravity, (a)=(g) and: F = N = weight = ma = mg

9.80665N = (1kg)(9.80665m/s^2) = (2.2046244 pounds-mass) = 2.2046244 pound-force = "pounds"
                                                        = weight in Earth's gravity force of: 32.2ft/s^2

Dividing both sides by 9.80665 , we find **1N =~ 0.224809 pounds** of weight = pound-force or= weight-force

Force = 1 Newton = ma = 0.22481 pounds ~= one-quarter of a pound =~ 0.25lb such as weight, solving for
                                                    pounds by dividing each side by 0.22481 :

Force = **1 pound ~= 4.44822162 Newtons**   : a measure of force

Work = (force)(distance) = (Newtons)(meters) = N-m Joules of energy   , and since: 1m = 3.28084 ft
                                                                                0.3048m = 1ft

Work = (force)(distance) = ((0.224809) (pounds)) N x ((3.281) (feet)) m   Joules   ,   when N=1, and m=1:

Work = (Newtons)(meters) = 1Nm ~= (0.7375622)(pounds)(feet) = 0.7375622 lb-ft   J = energy

Work = **1N-m = 0.738 lb-ft**   Joules of energy   , dividing both sides by 0.738 :
                                                    : this can be thought of as lifting 0.738 lb a distance of 1 ft,
                                                    or lifting 1 lb a distance of 0.738 ft.
Work = **1 lb-ft = 1.35582 N-m** J   : 1 pound-foot =~ 1.356 Newton-meters

Finding mass and weight from each other. You can find the mass of an object by dividing its weight by (g).

From: force = (mass)(acceleration) = weight   , we mathematically have:
        mass = (force / acceleration) = weight / g   : mass is essentially the weight value with the gravity factor
                                                        and multiplier removed, and here its divided out

Force has units of Newtons (N). Weight (a force) is actually a measure of an object or mass in a gravitational field
such from Earth. Gravity is constantly pulling (attracting) on the object with mass and causing it to accelerate. This will
cause the object to have a force, and this force is called the weight of the object. Weight, being a force, is also
measured with units of Newtons.

Mass (amount matter, and not its force) and weight (a force) are directly related and proportional to each other having
a constant of proportionality of (g):

(g) = (weight / mass) = (force / mass) = (mass)(acceleration) / mass = (acceleration),

Mass and weight can be calculated from each other if the local force of gravity is known which causes an attraction and
acceleration of a nearby mass. Technically, given two masses, their gravitational attraction or pull on the other mass is
the same value for both, and this concept is called "equal an opposite (direction) forces".

A 1kg mass will have a downward force or weight equal to: (1kg)(9.8m/s^2) = 9.8N or weight.
Therefore, to find the weight or downward force corresponding to this mass with Kg units, simply multiply it by 9.8

The corresponding mass associated with 9.8N of weight is from: mass = (weight / acceleration) = (weight / g) =

By filling a container with water, you can find the volume of it by the corresponding volume of water or the mass of it.
For example, if a contain was filled to some level by 400 grams of water, and knowing that there is 1g water / 1cc:
Setting up a proportional or equivalent fraction equation: 1g water / 1cc = 400g water / Xcc , Xcc = 400cc

1L = 1000 cm^3 = 1000 mL
1 liter = 1L = 33.8140227 fluid ounces = 2.1133764 pints = ~ 2 pints = 1 quart = 1 qt.
　　　The weight of 1L water at it maximum density at 4°C weighs 1 Kg = 35.274 dry oz = about 32 oz or slightly over
　　　2lb =~ the weight of 1 quarter (ie., 1 quart = 2 pints) gallon of water
1 liter = 1/1000 of the volume of a cubic meter.   (1m)^3 = 1m^3 = 1m x 1m x 1m = 100cm x 100cm x 100 cm
1 liter = 0.264172 gallons = roughly or slightly more than a quarter of a gallon.
　　　　1 gallon = 1 gal = 128 fl oz = 128 foz.   and   1L / 1gal = 33.814 foz / 128 foz = 0.264172
1 liter =~ 1 quart (quart or quarter =1/4 of 1 gallon) ~= 2 pints ~= 4 cups =~ 32 fluid or volume ounces
1 liter = 1.05669 quarts (a quarter, one-fourth) of a US gallon
1 liter = 61.0237 cubic inches , and the cube root of 61.0237 gives a side length of about 3.9372 in = 10cm , or ~4in =
　　　=~ (4 in)^3 = 4^3 in^3 = 4^3 cu.-in.   3.9372 in = 3 in + nearly 15/16
1 liter = 4.22675 cups , =~ roughly 4.25 cups = 4 cups + 1/4 cup , and since 1 cup of water = 8 fluid ounces of water,
　　　1L = (4.22675cups)(8oz/cup) = 33.814 floz

1 fl oz = a volume defined as the volume corresponding to 1/16 of a pound of water, but is then used for any liquid
1 fl oz = 1L/33.814 = 0.029573549 L = ~ **29.57355mL = ~ 30mL** =~ 30cc which is how it is defined in the U.S.A. food
　　　industry as a "food, fl. oz". 1fl oz in the S.I. or British "Imperial" system is: 28.41306mL = 0.02841306L .
1 fl oz =~ 29.574 cm^3 = a cube having sides of the cube root of this value, and which is about: 3.1cm in length
1 fl oz = 591.471 drops   =~ 600 drops
1 fl oz = 1.8046875 in^3 , A cube of this volume will have a side length of ~ 1.2175 in =~ 1in + (7/32)in
　　　Solving for 1 in^3 = 1 cubic inch by dividing both sides by 1.8046875:
　　　1 cubic-inch = 1 in^3 = 0.554112554 fl oz = slightly over half a fluid ounce

1 mL = a thousandth of 1L = (1/1000) L = 0.001 L = 0.033814 fl oz = **1cc** = 1g of weight if the substance is water.
1ml = 1cc = 20 drops

1 cc = 1 cubic centimeter = 1 cm^3 = **1 mL** = one-thousandth of a liter = (1/1000) Liter = 0.001L , 1cc of water
　　　will weigh 1 gram on an (equivalent mass) scale. 1 cm^3 = 1cm of length x 1cm of width x 1cm of height =
　　　1cc = 1 cm^3 = 10mm x 10 mm x 10 mm = 1000 mm^3 . **1cc = 1mL = 20 drops**
1 cc = 0.0610237 in^3
1 cc = 0.033814 (US) fluid ounces
1 drop = a volume = 1mL / 20 = (1/20) mL = 0.05 mL = "five one-hundredths of a milliliter" = 1/20 cc = 0.05 cc =
　　　0.05 cm^3 in volume. If this volume was a cube shape, it would then have a side length of the cube root of
　　　0.05 cm^3, and that equals 0.3684 cm = 36.84mm = 0.145in = slightly over (1/8) in = 0.125 in. The volume of 20
　　　drops would be equal to: (20)(0.05mL) = 1mL = 1cc. Since 1mL = (1/1000)L = 1 cc of water is 1g, 1 drop of
　　　water weighs = (1/20)g = 0.050g = 50mg = 0.0016907fl oz =~ 0.0017 fl vol. oz. Volume sphere, such as a drop
　　　= Vs = (4/3)(pi)(r^3) , r = cube-root of: ((3/4)Vs / (pi)) = cube-root of (0.2387324 Vs) =
　　　cube-root of ((0.2387324)(0.05 cm3)) =~ 0.22854cm = 0.09 in =~ 0.1 in =~ (1/10) in. =~ 2.3mm = (1.449 /16) in

　　　1 drop = 1 mL / 20 = (1/20) mL = 0.050mL = 50uL , hence there are 20 drops / 1 mL = **20 drops / 1 cc**
　　　1L = 1000 cc = 1000 mL ,   and the ratio of : 1L / 1 drop = 1000 mL / 0.05mL = 20000 ,
　　　　　　　　　hence mathematically, there are **20000 drops / 1 L**

　　　1 drop = 0.05 mL = 5 (10^-2) mL = 5 (10^-2)(10^-3) L = 5 (10^-5) L = 0.00005 L = 50 uL = 50 mico-Liters
　　　1 drop if water = 0.05 mL = 0.05 mg = 5 (10^-2) mg = 5 (10^-2) (10^-3) Kg = 5 (10^-5) Kg = 0.00005 Kg
　　　1 drop / Liter = 0.05 mL / L = 0.05 mL / 1000 mL = 0.00005L / L , if water: = 0.00005 Kg / L

1 cup = 8 fluid (volume) oz = 14.4375 cubic inches (in^3) =~ (2.435 in)^3 cubic inches = (6.1849 cc)^3 = 236.59 cc
　　　1 cup fluid volume is commonly considered as weighing 8 dry oz of weight if water (only), but is technically
　　　8.3214oz of weight in the United States, U.S. customary measurement system = 236.59 grams of weight since

came later in 1905. Aluminum and plastic barrels, containers, "kegs" or "casks" of various volume sizes and
shapes are also used and it depends on the product placed within it. Barrels are used to hold and protect
goods during storage and shipment. The shape of a barrel often resembles that of a cylinder, and often with a
height that is 1.5 times greater than its diameter. A US 55 gallon barrel has a 22.5 in. diameter and a 33.5 in
height.

1 Cord = Usually a volume of cut and dried wood, and with pieces of it to be routinely placed in a stone and-or metal
fireplace to heat a home. 1 cord is typically in these dimensions: 24 ft. long, by 4ft high, by 1.333 ft = 16 in
deep. Multiplying these dimensional values together yields a volume of: 128 cubic feet = 128 ft^3. A solid cube
of this size will have a side dimension of the cube root of this value, and which is: 5.04 ft =~ 5 ft.

1 Dozen = not a volume, but a (linear) count of 12 items. A Gross (meaning large, huge, many) is twelve dozen, hence a
dozen of a dozen, or a dozen dozen, and the number of items in a gross unit is (12)(dozen) = 12(12units) =
144 units or items.

For fluids, such as water, a volume (ie. space or 3 dimensional size, and not a weight) measurement:

1 fluid or volume ("food") ounce = 30ml = 0.030L , a food (weight) ounce is also considered as 30g of weight = 30ml water
1 cup = 8 fluid oz (ounces) = 0.240L = 240ml = 240cc   : = 240g = 1/2lb = 8 dry oz. of weight if the substance is water.
1L = 4.1667 cups   : this considers fl. oz as "food ounces" with 30mL per ounce, and (8)(30mL) =
                240mL per 8oz cup = 0.240L , otherwise,   1 cup = 0.23659 Liters   and   1 Liter = 4.22675 cups

Fluid (fl) ounces, fluid or liquid measurement is actually a volume or space (as in size) measurement, however
1 fl. oz of water actually does nearly weigh, and is considered about 1 oz of weight or "dry weight" (non volume,
but rather weight due to gravity). Consider that a cup full of plastic foam particles will weigh less than a cup of
water, and that a cup of lead metal will weigh more than a cup of water. All these substances of matter can
occupy the same volume, but these same volumes of the substances have different weights, and this is due to that
their density (amount of mass per unit volume) each, has different value. Fluid or liquid essentially means having
no discernible space between the particles, and can flow and fill any shaped volume. Fluid essentially means a
flowing liquid or substance. or liquid or substance that can flow.

Here is a list of the common fractions of a cup, based on using: 1/8 cup = 30cc = 1fl.oz, = 30 gram "food ounce"

3/4 cup = 6 fl.oz = 180ml = 180cc   ,   2/3 cup = 5.33 fl.oz = 160ml = 160cc
1/2 cup = 4 oz fl. = 120ml = 120cc   ,   1/3 cup = 2.67 fl.oz = 80ml   = 80cc
1/4 cup = 2 fl.oz = 60ml   = 60 cc
1/8 cup = 1 fl.oz   = 30ml   = 30cc volume = 30g of weight if the substance being measured within is water only.
1/8 cup = 2 tablespoons (tbs) = 6 teaspoons (tsp) = 1 fluid ounce (oz) volume = 1 ounce of weight if water only

To find the basic equivalent amount of ml or cc given a volume of fluid ounces, multiply the fluid ounces by 30cc.

1 teaspoon of (fluid) volume = 5mL = 5cc (typical), substance weight could vary widely depending on the substance
        properties.   Note that some "home teaspoons" may vary in the volume that they can hold, and may not be 5mL.
1 level teaspoon of water is 4.929cc ˜ 4.93 grams weight = about 5cc = 5 grams of water weight = 1/6 oz = 0.1667 oz
1 teaspoon of water actually weighs ˜0.17337 oz when not rounded to 1/6 "food ounces" = 30g / 6 = 5 grams
1 level teaspoon of water is equal to 1.33... fluid drams, which today is a unit of measurement that is not used much.
3 level teaspoons of water will equal the volume of about a 1 level tablespoon of water.
1 level tablespoon of water is about 3 level teaspoons of volume ˜ 15mL = 15cc = 15 grams if the substance is water
1 tablespoon volume of water weighs 0.52011 oz = about 1/2 fl oz, hence 2 tablespoons = 6 teaspoons is about 1 fl. oz
1 cubic centimeter = 1cc = 1 cm^3 = 1mL = 1milli-Liter = (1/1000) L = 0.001L = 0.0610237 cubic inches
1 cubic centimeter of water weighs 1 gram = 0.0352717 oz = 0.002204481 lbs
1 cubic inch = 1 cu. in^3 = 16.3871 cubic centimeters, and would be 16.3871 grams of weight if the substance is water
1 cubic inch of water weighs about (8.345404lbs/gallon) / (231cubic inches/gallon) = 0.0361lbs/in^3 = 0.578oz/in^3 =
        16.386grams/in^3.   Hence, a cubic inch of water weighs a little over half an ounce of weight.

1 cubic yard = 1cu-yd = 1cyd = 1yd^3 = 1yd x 1yd x 1yd = (3ft L)(3ft W)(3ft H) = 27 ft^3 = 27 cubic feet
1 cu. m. = 1 cubic meter = (1m)^3 = (1m)(1m)(1m) = (100cm)^3 = ((10)(10cm))^3 = (10^3)(1000 ccm) = 1000 Liters =
        = 35.314<u>5</u> ft^3 , and if this volume was a cube, it would have a side length of: ˜ 3.280835<u>5</u> ft = 3ft + (9/32) in

Substance	Weight / U.S. Gallon	Substance	Weight / cubic yard = 27 ft^3	Substance	Weight / cu. ft
alcohol	6 lbs	dry sand	2600 lbs = 96 lbs / cubic foot	oils	55 lbs
gasoline	6.2 lbs	concrete	4000 lbs = 150 lbs / cubic foot	water	62.427<u>2</u> lbs ˜ 28.342kg
oils	7.7 lbs	water	168<u>6</u> lbs = 765.234kg	steel	490 lbs =˜ 500 lbs
water	8.34 lbs = 18.37kg equivalent mass value			lead	708 lbs
liquid paint	11 lbs (typical)			gold	1206 lbs

For any same volume chosen and filled with a material, such as for example: a cubic foot of gold, and a cubic foot
of steel, the ratio of the weight of gold to the weight of steel is: (1206lbs/cu.ft) / (490lbs/cu.ft) = 2.46   It could be said
that gold is 2.46 times heavier than steel, or roughly 2.5 times heavier than steel. It will be also shown below that gold
likewise has about 2.46 times the mass (the measure of real, physical substance) as that of steel, and that is why gold
is also heavier. For reference: 1 gallon = 3785.312 cc = 3.785.312 L of volume , 1 lb = 454.592g = 0.454.492 L =
454.492 cc if water , and   1 cu. ft = 1 ft^3 = 28316.8466 cc = 28.3168466 L of volume

The density of gold is 19.28 g/cc, and the density of steel is 8g/cc. The ratio of the density of gold to that of the density
of steel is:is about : (19.28g/cc / 8g/cc) = 2.41 . This is about the same ratio of their corresponding weights, hence
weight and density are directly related and proportional.   **Since weight is directly related and proportional to
mass, mass is also directly related and proportional to density. Given the same volume of two elements or
substances, the ratio of their weights, and the ratio of their masses are the same.** For the above example, the
radio of the density of gold and steel is 2.41, and the ratio of their corresponding weights is:
(1206 lbs/cu.ft) / (490 lbs/cu.ft) =~ 2.461 , If 500 lbs/cu.ft was used for steel, the result is: 2.41

Gold has 19.28g/cc , Dividing both num. and den. by 19.28 so as to make an equivalent fraction, we find that
                gold has 1g/0.0518<u>7</u>cc   or using the reciprocal of this we find: 0.0518<u>7</u>cc / 1g = ~ 0.52cc / 1g

Steel has 8g/cc. , Dividing both num. and den. by 8 so as to make an equivalent fraction, we find that steel
              has 1g / 0.125cc   or by inverting (ie., reciprocal) this expression, we have: 0.125cc / 1g

Since steel has a lower density than gold, therefore, steel has less mass and weight per same volume than gold does,
and it will take more volume of steel so as to have an equivalent mass and-or weight of gold such as 1 gram, or
1 ounce of weight, and therefore: **density and volume are inversely related.   Note that 1 gram of steel has the
same mass as that of 1 gram of gold, aluminum, or any other element. A gram is not related to the density,
volume or size of a mass, but it is actually a specific amount of mass (ie., real substance, matter or material)
for any and all elements. A gram of any element will also weigh the same, and for the common weight-to-mass
or weight-to-grams calibrated scales, a gram of any element will be said as "weighing" 1 gram.**

**Given a certain volume of a specific substance or element, if the volume of it changes by a factor of (n), both its
mass and weight will also change by that same factor of (n), but its (natural, at standard air pressure) density will
still remain the same value.   It is of note that gasses can be easily compressed by using a force and-or pressure
so as to increase the density of it, such as to make liquid air or oxygen.**

Alcohol, the natural form. from say wood, is **methanol or methyl-alcohol** is C1H3OH. C1H3 is methyl, and OH =
O1H1 is an extra hydroxy or hydrogen-oxygen molecule. This type of alcohol is **poisonous**. Its molecular weight or
"molar mass" is about 32 grams per mol-atoms. A wood alcohol molecule has 32amu. Each carbon atom has 12amu,
each oxygen atom has 16amu, and each hydrogen atom has 1 amu.   The density (mass/volume) of methyl alcohol is
0.792g/cm^3 = 0.792g/mL = 792kg / L. . This will float on liquid water since water has a greater density of 1g/mL.

Water has a molar-mass of 18.01528 g/mol since each water molecule = H20 contains 2 hydrogen atoms and 1 oxygen
atom, and that each hydrogen atom contains 1amu, and each oxygen atom contains 16amu. The total sum of amu in a

V2 = (d2)^3 = (d+1)^3 = (d+1)(d+1)(d+1) = (d+1)^2 (d+1) = A2 (d+1) =
 = (d^2 + 2d + 1) (d +1) = d^3 + d^2 + 2d^2 + 2d + d + 1 =
V2 = d^3 + 3d^2 + 3d + 1 : V2 with respect to V1 when d increases by 1 , a cubic equation

The difference in these two volumes is how much the volume will increase:
(V2 - V1) = d^3 + 3d^2 + 3d + 1 - (d^3) = d^3 + 3d^2 + 3d + 1 - d^3 =
(V2 - V1) = 3d^2 + 3d + 1 : a quadratic equation , mathematically:

V2 = V1 + increase
V2 = V1 + (3d^2 + 3d + 1) = V1 + 3d^2 + 3d + 1 =
V2 = (d2)^3 = (d+1)^3 = d^3 + 3d^2 + 3d + 1 : a cubic equation

Here is a somewhat or related note about the mass, gravity, and volume of an object. If the volume of a substance or material doubles, the mass of that substance or material doubles. If the length (such as a diameter or radius doubles) of an object doubles, its area (A = l x w) doubles: A = l x w. If l or w doubles, the total area is now: 2A = (2l x w) = 2(l x w), but if the object was a sphere or cube, its volume increases by the cube of this factor (here 2) value, and this is equal to a factor of: (factor)^3 = 2^3 = 8 applied to the initial volume.

Vcube = l x w x h = l x l x l = l^3 .   : In a cube, l=w=h. If the side length of a cube is doubled:
(2l) x (2l) x (2l) = (2l)^3 = (2^3)(l^3) = 8(Vcube)   : the volume is 8 times more, and volume is said to change
                                                               or increase rapidly as the linear dimensions change

From this, if a volume is 8 times more, its dimensions are the cube root of 8, hence 2 times (twice) that of the reference mass volume. In general: linear dimension increase = cube root of volume increase factor.

If only one dimension of a volume doubled, then the volume will only double:

    Ex: If v1 = (l)(w)(h) , and if h doubles: v2 = (l)(w)(2h) = 2 ((l)(w)(h))= 2(v1) = 2v1

        As a simple thought example, if you had a cylinder filled with a liquid, and if the length of that cylinder doubled, the total volume is doubled. This is also the same total volume of two of the original cylinders. If the mass of an object increases by (n), its weight, volume and gravity will increase by (n).

        Ex. A volume 1 liter of water = 1L has a mass of 1kg of water substance, and will weigh 9.81N.

           If this volume is doubled:

            A volume of 2L of water will have a total mass of 2kg, and will weight 19.62N.

TIME

Physical changes and processes are not instantaneous and are said to need an amount time or duration to complete. An object can't be located at two positions simultaneously or at the same instant, and if it were so, it would make two objects from just that one object. If timed stopped, there would be no further processes or changes possible, and it would be as tings were essentially frozen or halted in position.

Time is a measure of and between past (already happened, "time ago"), present (now, currently), and-or future (to happen later) events (happenings, realities, other time points or reference times). Time elapsed is much like a total length duration and number of (accepted) time units of measurement. In general, time is often how long a certain event or process takes, hence it is also the difference between the ending time value and the starting time value. The basic units for time measurement are usually seconds (s), which is a fraction (1/60) of a minute, which is a fraction (1/60) of an hour, which is a fraction (1/24) of a day, which is a fraction (1/365) of 1 year of standard "(24 hour) clock time". The Earth rotates (spins) on its (imaginary) polar axis of rotation, once a day unit of time which is equal to 24 hour units of time. The Earth revolves

which is also known as Polaris. People living south of the solar plane are blocked from viewing the north star by Earth's horizon and land being in the way. The angle to the star Polaris is equal to the latitude of the observers location, and this angle can be used in construction some types of sundials. The 6pm hour line will then be automatically aligned in the true south direction of the Earth. The sundial will only function during the local 12 daytime or daylight hours of the day; from 6AM to 6PM, and if the sunlight is not blocked by clouds. Again, for the sundial as shown below, the rod is vertical, and the numbers and lines are placed horizontally on the ground or a stand. [FIG 212]

Basic Sundial
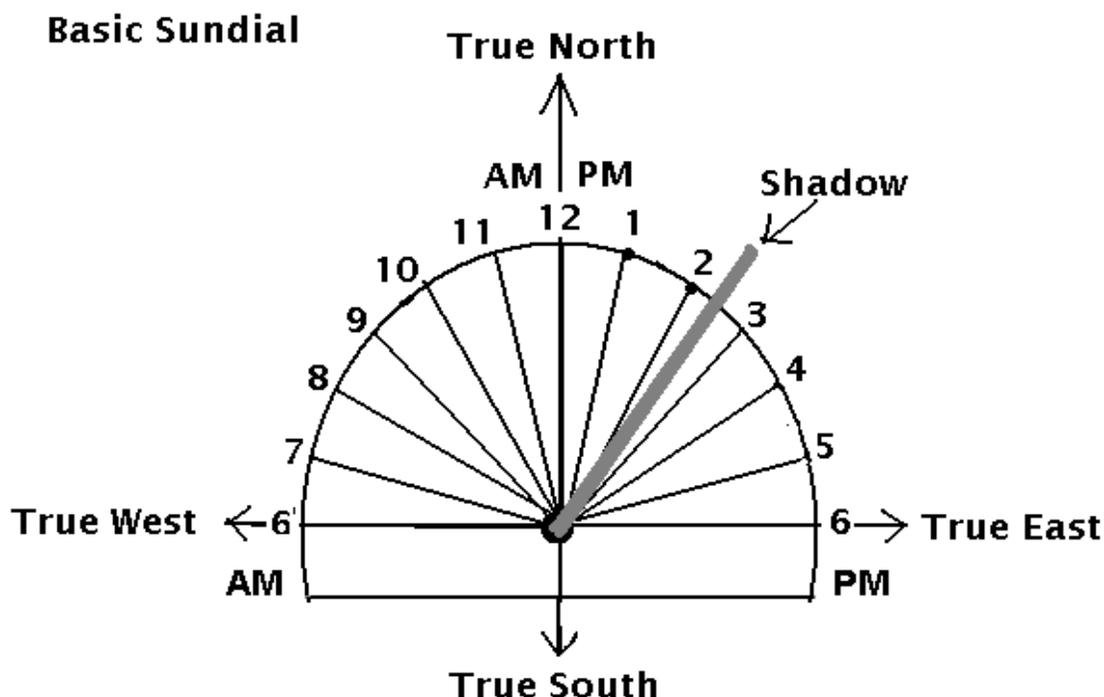
The shadow is indicating the time is about 2:30pm

On the spring (~ March 21) and fall (~ September 21) equinox day of the year, the Sun will appear to "rise" at the distant horizon in the Eastern direction, and the shadow of the vertical pole on the sundial will indicate 6AM in the morning. If you were at the equator, the Sun is generally directly overhead at noon or 12pm, and actually so on the two **equinox** ("equal day and night") days of the year, and any shadow created is very small using a pole sundial as shown in the above image. At solar or Sun noon, the Sun is directly in line with your longitude, and the shadow of the pole will be the shortest length throughout the day, and it will point to the true north direction. At the equinox days of the year, the equator is in line with the solar (ie. Sun) plane of Earth's orbit around the Sun, and the day time will be 12 hours long, and the night time will be equivalent at 12 hours long.

Another type of sundial has its pole tilted at an angle equal to your local latitude. The top edge of this structure is in line to the star Polaris. This discussion in this book about sundials and the displayed time is for an ideal sundial, such as if placed on Earth where the the solar plane is currently at, and informally always considered at the equator, and where the hour lines of the sundial are separated by 15°, but there are more precise equations for these hour line positions and which depend on your latitude from the equator. The more your latitude, the greater your tilt away from the solar plane and-or Sun, and the longer the resulting shadow created. There is much literature, books, clubs or "societies" devoted to sundials and clocks. For simplicity, a vertical stick in the ground, and 15° hour lines is acceptable so as to have a common time within the same region or area on Earth.

If the weather is cloudy, and-or at nighttime, a water-clock or sand-clock can be used. It is even very possible to tell the time by the position of the stars throughout the night as they (apparently) revolve about the north star (Polaris).   After

A Year or 365 days is typically divided into 12 month (named after the word Moon) units of time, with each month being (1/12) of a year which is about (365/12) =~ 30 days of time duration or length, and this amount roughly corresponding to the length of one Moon cycle or orbit around Earth. Each month has 4 units of time called a week = (1/4) of a month unit. 1 year = 52 weeks. Each week has a 7 day cycle with each new day called: Sunday (Day of the Sun, Sun-day), Monday (Moon-day), Tuesday (Mars-day), Wednesday (Woden's-day or Mercury-day), Thursday (Thor's-day or Thunder-day, Jupiter's-day), Friday (Frigga's day, Freya's day or Venus-day), Saturday (Saturn-day). These names are generally of ancient Roman and-or northern-European origin, often in reference to some god that was once considered.

 MONTH OF THE YEAR , DAYS IN THE MONTH , DESCRIPTON OF THE MONTH

1. January   31   [named after Janus, a Roman god for times and changes, February is from "Februa" = purification]
2. February   28   [a "leap year", ex. 2020, is a time correction every 4 years, and February will be 29 days long]
3. March   31   [1st month in Roman calendar, named after planet ("wandering star") Mars, and a Roman god of War]
4. April   30   [means "second"]
5. May   31   [named after Maia, a Roman fertility goddess, and possibly means "major" and older]
6. June   30   [named after Juno, a Roman queen god, and possibly means "junior" as meaning younger and youth]
7. July   31   [named after Julius Caesar, a famous Roman leader, born ~100BC, died 44BC, about 56 years old]
8. August   31   [named after Augustus Caesar, first Roman Emperor, born 63BC, died 14AD, son of Julius Caesar]
9. September 30   [name derived from "seven", since it is the seventh Roman month from the start of March]
10. October   31   [name derived from "eight"]
11. November 30   [name derived from "nine"]
12. December 31   [name derived from "ten" or "deca" , "decimal" means pertaining to 10]

The etymology or word origin of the words "Moon" and "month" are generally based upon each other. A lune is an ancient word for a crescent (ie., crest, having a peak) shape, such as that made by two circles that are not aligned. The Moon is sometimes called as Luna. Things pertaining to the Moon are sometimes noted using the word "lunar". The word "lute" for a string musical instrument, often having a half-round-like shape, seems to be based on the word "lune" and "flute".

Typical or common time indicating, notation or expressing formats:

    Date (within the Year)   Time (within the day)

    month / day / year   :in U.S.A   hours **:** minutes **:** seconds
    day / month / year   :in Europe, etc.   [Standard or Civilian Time uses two 12 hour periods: 1 to 12 for hours.
                                       Military Time uses numbers 0 to 23 for indicating the hour,]

"Time duration", "time length" or "time change or= the change in time" is the difference between two time points or measurements. If a process started at 10s, and ended at 30s, the time duration or length of that process is:

  time duration = ending time - starting time   and for this example:
      20s   =   30s   -   10s

Typically, on many "count up" (duration) timers or manually, the starting time is considered as 0s.

If a new process is said to take twice (2) as "long" as another, its time length, duration, or simply time, will be twice as much or "twice as long":

The new process, length, or duration time can be expressed with multiplication as:

    old duration x (time or duration factor) = new duration   , or simply:
    old duration x factor = new duration   or:
    (old time) x (factor) = new time   which can be expressed as:

$$(\text{new total time of the process}) = \frac{1}{1.5\ (\text{effective speed of the process})} = (0.667)\frac{1}{(\text{effective speed of the process})}$$

For both sides of the above equation to be in balance, the (effective speed of the process) must also be multiplied or increased by a factor (1/1.5) = 0.667 . This can be expressed as:

(new time of process) = 0.667 (old time of process)

In short, a process is said to be (x) times faster or quicker, the amount of time needed for that process is a fraction of (1/x) that amount of time. If a process is said to be (x) times longer, the amount of time needed for that process will a factor of (x) times more.

For a **mechanical clock** that uses physical force to move (ie., rotate) the gears so as to rotate the attached time indicators (ie. "hands", or pointers), the passage of time is indicated on a circular disk surface. For the hour hand, pointer or indicator, each quarter hour = 60minutes/4 = 15minutes, corresponds to a 90° of rotation. For the minutes hand or indicator (or "pointer"), and-or the seconds hand or indicator, each 15 minutes or 15 seconds, respectively, of time passage corresponds to 90° of rotation of the corresponding hand or indicator. This amount is one-quarter or 1/4 of a full circle and-or rotation (ie., turning). 15 is 1/4 of 60. 60/15 = 4. 15min/60min = 0.25 = 1/4 , 15sec/60sec = 0.25 = 1/4. and 360° / 4 = (360°)(1/4) = (360°)(0.25) = 90°. Each 1 second and-or minute corresponds to: 360°/60 = 6° Each hour of time corresponds to 30 degrees of pointer rotation: 12hours / 1 rotation = 12 hours / 360° = 1 hour / 30°. If the hour hand corresponded to 24 hours (a full day) instead of 12 hours (a half of a day), each hour would then correspond to 15 degrees of rotation per hour. Extra: The rotation of the Earth on or about its polar axis, and-or the apparent movement of the stars across the sky, is also 15° / 1 hour.

A mechanical clock is much like a mechanical counter with a numeric base of 60. When the seconds hand rotates once after 60 seconds of time has elapsed, the minute hand is incremented or increased by 1. When the minutes hand rotates once after 60 minutes of time has elapsed, the hours hand is incremented by 1. When the hours hand rotates 24 times, a day of time has passed. When 60 units of seconds and-or minutes have been expressed on the clock, the values "rollover", "back to" to 0, and to "start over", advancing and-or incrementing again.

In the summer months, the northern hemisphere is effectively tilted toward the Sun during the day, and it will receive more daylight time, and shorter night time. During this same time of year, the southern hemisphere is effectively tilted away or back from the Sun during the day, and it will receive less daylight time, and will be in its winter season.

ANGLES AND ROTATION:

Degrees and radians are units of angles and-or rotation. Degrees are man made practical units made before the concepts of radians was known, whereas radians are units derived from the natural and true concepts of circles and rotation. Degree units are still commonly used everywhere for practical, non-scientific uses.

1 degree is a fraction of a full rotation that is 360 degrees about a point. 1° / 360° = 0.002777...
1° = (1 part of a full rotation / total parts of a full rotation) = (1 part / 360 parts) = 0.002777... parts of a rotation
1 degree = 0.0174532925199433... radians

1 radian = 57.29577951308231... degrees

A steradian is a "solid radian angle" shape that is naturally based, and is a radian angle of 1 rad rotated about its center axis, like a (3-D) cone. Energy transmitted from an ideal (spherical) infinitely small point source can be thought to have this type of cone shape as it travels away from that location and outward and into a larger circular area.

rps = revolutions per second = revs / s = rot / s
rpm = revolutions per minute = revs / min. = revs / 60s = rot / min

1 rps = 1 rev / 1s , and after multiplying num. and den. by 60:
1 rps = 60 rev / 60s = 60 rev / min = **60 rpm** , multiplying both sides by N:
N rps = N 60 rpm

rpm / 60 = rps , Ex. 120 rpm = (120 rpm / 60) = 2 rps , 600 rpm = 600 rpm / 60 = 10 rps

For the linear velocity of a point on a circle:

For a circle, circumference = C **= 2(pi) r =~ 6.28 r** : circumference (C) is a distance , (r) = radius of the circle

From : speed = distance / time = v = d / t , for the speed of an object rotating about the center point of a circle,
the distance traveled for each rotation or orbit is equal to c:

$$v = \frac{d}{t} = \frac{C}{t}$$: the velocity when when the distance traveled = 1C = 1 revolution
: here, v **=** equivalent rotational velocity at point (r) from the center, rather
than angular velocity = w , and:

Total distance = (C)(revolutions) = (C)(revs) = 2(pi)(r) (revs) = (6.28 r)(revolutions)

angular velocity = w = $\frac{(amount\ of\ rotation)}{time}$ = $\frac{\cancel{c}}{t}$, for each revolution: $\cancel{c}$ = 6.28 radians , **w = $\frac{6.28}{t}$** using substitution:

$$v = \frac{d}{t} = \frac{C}{t} = Cf = \frac{6.28\ r}{t} = w\ r$$ **: linear velocity of a point on a circle or in a circular orbit, and in**
terms of the angular or rotational velocity.   Also:
d = vt = Cft = C (revolutions / second) (seconds)

Ex. A vehicle wheel has a circumference of C = 2m, and the vehicle is moving at 7m/s, what is the frequency of the rotating wheel? From v = C f , we have: **f = v / C** = (7 m/s) / (2m) = 3.5 (1/s) = 3.5 / s = 3.5 hz = 3.5 cycles / s =
= 3.5 revs / s = 3.5 hz

$$w = \frac{v}{r} = \frac{radians}{s}$$ **: rotational or angular velocity in terms of linear velocity and**
the radius of rotation

distance = d = (speed) (time) **= v t** = (w r)(t) = **wrt** : a point at distance (r), from the center of rotation,will travel
this (equivalent linear) distance in the given amount of time

Ex. An automobile (ie.,a "car") is traveling at a velocity or speed of 20 miles per hour = v = d / t = 20 mi / 1hr.
Since: 1 mile = 5280 feet and 1 hour = 3600 seconds , for the distance traveled in 1 second:

$\frac{20\ mi}{1\ hr}$ = $\frac{(20mi)(5280/ft/mi)}{(1hr)(3600s/hr)}$ = $\frac{105600\ ft}{3600s}$ = $\frac{29.33\ ft}{1s}$, to find how many revolutions per second = rev/s = rps :

From: distance = (velocity)((time) , velocity = speed = distance / time = (d / t)

An angular velocity or rotational velocity of: w = (1 revolution / time) = (6.28 radians / time) = (6.28 / time) ,
and this corresponds to a linear velocity of: v = (circumference / time) = (6.28 r / time) : r = radius

If the wheels of that car have a radius of 1ft: C = 6.28(r) = (6.28)(1ft) = 6.28 ft
1 rev corresponds to a distance of 6.28 ft, and x rev corresponds to 29.33 ft? , using a proportion equation:

EARTH, SUN, MOON, AND STARS ASTRONOMY

The diameter of the Earth is about 7926 miles at the equator, hence about 8000 miles wide, and is about 27 miles less in distance from the north geographic pole to the south geographic pole and is due to that Earth is spinning and the amount of (outward) centrifugal force is causing it to bulge slightly wider by about 27 miles at the equator region which is perpendicular (ie., 90°) to the axis pole, and located midway between it. The circumference of the Earth at the equator is: C = (pi)(diameter) = (2)(pi)(radius) = (2)(pi)3963mi = 24900mi =~25000mi. This circumference value is then also slightly less along Earth's polar axis. The radius of the Earth is often noted as (roughly) 4000 miles, and its circumference as 25000 miles.

An ancient Greek, multi-scientist named **Eratosthenes** (of Cyrene, a city in Libya) worked at the Library of Alexandria, and in about the year 240BC, he had two people measure the angles cast by the shadow of two vertical sticks on the same day and time (local noon) at two distant locations (Alexandria and Syene, Egypt) that are located directly north and south with respect to each on Earth's surface. Today we would say that these locations directly north and south of each other are located on the same longitude line and have the same time such as solar noon time (ie. true time, rather than "timezone time") where the Sun is highest in the sky, and also the same sunrise and sunset times.. He then (closely) calculated Earth's circumference using basic geometry and trigonometry. Today, his scientific method and result are considered highly accurate, and his was done at a time before he knew of the decimal number system. From this circumference value, Earth's diameter and radius could then be calculated also. These values eventually helped the calculation of other important astronomical sizes and distances such as for the Moon and Sun. The Earth was thought (ie., a theory) to be a probable sphere before this time, such as by Aristotle in about 300 BC. and due to various astronomical reasoning such as the Moon being round or spherical in shape and therefore, it orbits a (theoretical, not proven yet) round Earth. Eratosthenes also conceived the concept of what we call today as latitude and longitude lines, and hence their intersections (ie., coordinates, address, location) so as to map and-or find locations and-or their distances between them on the Earth's surface. Before Eratosthenes, it was well known, such as by the people of ancient Egypt who studied the Sun and its motion throughout the year(s), and that there were 365 days in a year. Eratosthenes made a more accurate calculation of 365 days plus a quarter of a day, hence 365.25 days. After Earth orbits the Sun in 1 year of time, the nighttime stars (which always do not appear to move with respect to each other, and appear fixed in position due to their great distances away) and-or constellations will be in the exact same position as the year before. Modernly, the number of days in 1 year has been adjusted to be: 365.2421875 days. In common and-or non-scientific use, a year is still considered as simply being 365 days of time duration or "time length".

Constellations or known groups of stars that are within +,- 8° of the ecliptic are called the zodiac. Long ago, ancient astronomers divided the ecliptic into 12 constellations called the zodiac, and each is a month long in apparent duration and-or alignment. During the day at noon = 12PM, the Sun is in line with one constellation of the zodiac, while at the same time on the other side of the Earth at midnight = 12AM, we can see the constellation that is 180° or 6 constellations away from the Sun's and Earth's apparent circle of travel though the night sky and zodiac throughout the year. As the Earth travels counter-clockwise about the Sun, during the year, the Earth and-or Sun will appear to enter a new constellation among the zodiac about every 30 days or month.   Ancient astronomers called and used the zodiac as "signs" of yearly events, and would then predict things that will happen during that time of year.

[FIG 212A]

To North Star, Polaris

N 50° 40°
40° 50°
vertical pole
observers location
40° Latitude
Observer's local horizon line
50°
40°

What inspired Eratosthenes to determine if the Earth was round? It was probably several known facts. One fact is that in the Northern directions from certain locations (ie., near the Equator) on Earth, and in the Southern directions from certain locations (ie., near the Equator) on Earth, the average temperature was lower. This hinted of the land at the pole regions which do not get much direct sunlight energy, and due to being more curved away from the direct sunlight. One other fact is that the Sun was sometimes directly above in certain locations (ie., near the Equator), but never directly above in many other locations, and rather at a maximum angle that is less than 90° with respect to the local horizon. The locations were it is directly above are also the hottest locations. Another fact is that people on small boats could not see the low laying shoreline areas, and this is due to the curvature of the Earth. It is also possible that the round Moon shape gave Eratosthenes an idea that Earth may also be round, and particularly, a sphere shape. The Moon (light and shadow) phases throughout a month of time have a curved shape, and which indicates that the Moon may be a sphere. Eratosthenes new that locations directly south or north or each other had the same (solar) time of day in those locations. For example, in those locations the Sun would be at its highest angle throughout the day. Though in these locations, the shadow lengths of similar poles would be at different lengths, and today we know that this is due to the different latitudes, the indicated time on a solar clock, even if just a stick in the ground, would point in the same direction, and so as indicating the same time of day. The stars appear to move eastward to westward like the Sun does. The stars also appeared to travel about the star Polaris throughout the night and day of each year, and that the star Polaris is always at the same angle in the night sky, and this angle depends on the observers (latitude - north and south) location on Earth. To Eratosthenes, all these things mentioned seemed to indicate that the Earth might be a sphere, and which rotated from westward to eastward throughout the day and night on an axis.

Extra: Notice that at each latitude, that the circumference or circle around the Earth at that latitude is less the further north from the equator, and less further south of the equator. Using the above figure, the radius (Rlat) of this circle at a certain latitude is equal to the bottom leg of the large right triangle shown in FIG 212B. Clat = 2 (pi) Rlat .

Ex. The distance around the Earth at 40° Latitude is: Clat = 2 (pi) **[Rlat]** = (6.28) **[** (Cos Lat ¢)(Radius Of Earth) **]** =
= (6.28)(Cos 40°)(4000 mi) = 19243 mi

Another important note is that the longitude lines get closer together and converge at the poles. At the Equator, each degree of longitude corresponds to a distance of:

northern Russia called Siberia, north Mongolia (ie., just south of northern Russia, and perhaps a mixed race including northern China) which are all generally a northern Asian race of people. Supposedly during the last "Ice Age", theses "Paleo people" cam to the Americas, near Alaska, on an (53 mile wide) ice bridge at the Bering Sea in the northern Pacific Ocean, and between Asia and North America continents. Columbus took his discoveries and news back to Europe. and where many ships and people would soon make the journey to the "new world". By 1493 Columbus was beginning to create some small colonies in the Dominican Republic (a Hispanic [Spain territory] island) such as the first one called Isabella. On the American mainland, Jamestown (now in Virginia state) was the first colony of Europeans and was began about 100 years after Isabella, and in 1607. Plymouth (now in Massachusetts state of the United States of America - USA) was the next significant colony started in 1620. The natives were both afraid and in wonder of some of the abilities of the people from across the Atlantic ocean to the east, such as having products made of metal which was both hard and durable and much less brittle, and therefore, it was considered as an extremely useful and valuable item to them. Some natives, particularly in the South America regions, already knew of and-or had gold metal products created from melted gold nuggets, but gold is relatively soft and rare and is not practical for tools. Columbus also made some wonderful discoveries such as new varieties of plants and foods to take back to Europe, and to grow them if possible. The people of Ireland are well known for growing potatoes, but relatively few know that the plants and seeds for them were brought from the "new world" that Columbus discovered. It did not take too long for the Europeans to realize that the native people that Columbus encountered were not from the country of India, and therefore, Columbus did not sail around the Earth to prove it was a sphere. Some of his ship crew, and later the Europeans, probably theorized that the Natives were of some far-east Asian lineage and not from India. The colonies of the United States declared independence from European rule on July 4, 1776 when the Declaration Of Independence was signed, and the United States of America, Constitution was created on September 17, 1787 in the city of Philadelphia, Pennsylvania. This was nearly 300 years after Columbus. Most politicians and-or representatives signed the Constitution, but some did not because of a lack of a written, specific Bill Of Rights. The Bill Of Rights began to be placed into the Constitution as Amendments to the Constitution in 1789.

The land of the Americas is very long southward and naturally impassable by a ship, westward or eastward, until **Ferdinand Magellan** (1480-1521), from Portugal, and who was later associated with business of the Spanish empire, sailed very far south along the east coast of the Americas and made it to the Pacific ocean to the west. Magellan actually called this vast water as the Peaceful Sea after sailing through the rough seas he encountered in the southern Atlantic Ocean. Today, the Pacific Sea is commonly known as the Pacific Ocean. The Pacific Ocean is the largest ocean on Earth. Magellan made it to the Philippine Island areas in the south-western Pacific ocean, and of which he had once previously sailed near to and knew the geography fairly well, and then he finally knew that the Earth was a sphere shape. These areas area also modernly known as the South-East Asia Islands, and of which are northward and north-eastward of Australia. These South-East Asia Islands are also called the Polynesian Islands and of which are inhabitant by the Polynesian people which are an ancient mixed race of people from India and Southern Asia. The people of the Philippine Islands are generally considered as the original or main source of the Polynesian or "Pacific-Islander" people(s) that inhabit the islands which are mostly to the south and south-eastward of the Philippine Islands in the Pacific ocean. One of Magellan's fleet of ships with its captain and crew made it back to Spain in **1522**, and thereby verifying and proving to others that the Earth was a sphere shape, and that the Sun and Moon were probably spheres also. **Marco Polo** (1254-1324), from Italy, had previously explored this east, far-east, and south-east asia areas in after 1270 and brought back several discoveries to Europe, such as pasta (ie., noodles, a food made from wheat, and which did not spoil easily) and printing by using wood blocks as the print pattern. Polo went to find new and better routes, people, and items for trade. Polo traveled on land in 1271, and eastward along the "silk road" trade route toward China, Mongolia and South-East Asia in 1271. His father and uncle had previously explored some of these trading routes. Some of the other countries he explored were Israel, Iraq, Iran, Afghanistan and Pakistan, After reaching China, he met **Khubilai Khann** (1215-1294), from Mongolia, who was the grandson of **Genghis Khan**. Khubilai became a famous ruler in the Mongolia area and later in China. Khann made Polo a diplomat and so as to help travel about and explore the far-East areas. In 1292, he began to his sail back around south-east China, south-east Asia, and the continent of India. He landed in Iran and traveled back to Europe on land. He had traveled 24 years on this journey. He did not sail around Africa, and the Suez Canal in Egypt was not yet built. Polo wrote a book called "The Travels Of Marco Polo" about his journey, and which was then useful for many other explorers and merchants (business people, traders). The first from the "west" (ie., Europe, before the discovery of the New World - Americas in the Western hemisphere of the Earth) to discover going around the southern tip (Cape of Good Hope) of Africa as a trade route to Asia was **Bartolomeu Dias** (1450-1500), from Portugal, in about 1448.

Aristarchus in about the year 270BC, and later improved by another Greek named Hipparchus in about 129BC. Clearly, the Moon is closer to the Earth than the Sun due to that the Moon always eclipses or blocks the Sun and its light during a solar (ie., Sun) eclipse. The shadow of the Moon will then be on the Earth. A lunar eclipses is when the Earth blocks the sunlight from reaching the Moon, and the shadow of the Earth is then seen on the Moon. Another astronomical fact is that the apparent visible angle (or "width") of the Moon and Sun is the same part of the overhead sky (180°, horizon to horizon if no hills or mountains are in the way), however, this does not mean their distances are even close due to that the Sun could possibly be, and it is rather much farther away and still appears as the same visible angle, width or size as the Moon. During the Apollo Program manned moon landings (July 1969AD to December 1972AD), a laser reflector was placed on the surface of the moon so as to measure, calculate and verify its known distance from Earth using a high power laser light from Earth that was aimed at the Moon so as to have a small but usable amount of it reflected back to a receiving station on Earth after an amount of time of travel has passed. Clearly, the distance to the Moon takes half this measured amount of time at the speed of light: Distance to Moon = (speed)(time) = (speed of light) (time in seconds / 2). This distance of separation will also cause a constant, noticeable time delay or "gap" in radio communication such as with people speaking to each other, however it is a low value of only about 1.3 seconds to the Moon, or to the Earth from the Moon.

Aristarchus, of Samos, Greece, in about 200BC, believed (proposal, theory) the Earth and other planets and stars orbited the Sun in a heliocentric ("Sun-centered") universe. **Nicolaus Copernicus** (1473-1543), from Poland, calculated and published a verification of this theory is in 1543AD.   This was also verified by **Galileo Galilei** (1564-1642), from Italy. in about 1610. In 1610, after being inspired by the recent invention of the telescope, he made the first practical high power (about 20 times magnification or "power", ie., 20x) telescope at that time and made many important astronomical discoveries using it.   The (refracting = light bending, usually with glass lenses) telescope was a newly recent patent applied for and-or construction (in 1609) by **Hans Lipperhey (aka Johann Lippershey)** (1570-1619)**,** of Denmark, and it had a low magnification or "power, and it was also known previously that two magnifier lenses held at a distance apart could magnify a distant object. In 1610, **Galileo** discovered with his telescope that planet Jupiter had 4 visible moons of its own, and which revolved around it periodically (at a specific duration or amount of time), and that Earth's Moon had craters and mountains, Venus was seen to have (sunlight illumination) phases (like the Moon about Earth) as it orbited the Sun, sunspots, and planet Saturn's ring structure. **Johannes Kepler**, in about 1609AD, found that the planets orbit the Sun in a elliptical orbit with the Sun at one of the foci (ie., end-region) of their elliptical orbit, and this is one of his celebrated planetary motion (orbit) laws. Later, in about 1667, **Isaac Newton** created the celebrated mathematical laws for gravity and motion. Newton is also credited with making the first practical Newtonian or reflecting telescope in 1668, and of which a curved parabolic shaped mirror is used as the main "objective lens" or primary mirror to gather a large amount of light coming from an object, and then aim (direct, reflect) it all to be concentrated (and brighter, more intensity) and focused together at the eyepiece lens. Not long after the Newtonian telescope was invented, the famous (calculus) mathematician **James Gregory**, from Scotland, previously conceived the Gregorian telescope in 1663, and which was later built by the famous **Robert Hooke** in 1673. Gregory and Newton knew each other very well and influenced and assisted each other. The Gregorian telescope effectively replaced the eyepiece mirror with a small circular, concave focusing mirror to direct the light back through the center of the main (objective) parabolic mirror to an eyepiece lens at the end of the telescope. This increased its focal length and magnification to about twice that of a Newtonian telescope. A (Laurent) Cassegrain telescope is similar to a Gregorian telescope, except that the secondary mirror is convex instead of concave. For a basic telescope, the formula for its image magnification or "power" is the focal length (Fo) of the objective lens divided by the focal length (Fe) of the eyepiece lens. Basic lens telescope power = Fo / Fe. To increase the power of a telescope, an objective lens with a longer focal length should be used, and-or an eyepiece lens with a shorter focal length should be used. With modern images sensor with a pixel count of many millions, this lens power can be effectively increased so as to have a greater resolution where smaller parts of the object can be discerned.

The diameter of the Sun is about 865,000 miles, and this was first roughly calculated by Aristarchus. The circumference of the Sun is then about 2.72 million miles. The distance to the Sun is about 93 million miles away, and this was also first roughly calculated by Aristarchus. At the Sun's equator, **sunspots** (dark regions on the Suns surface) will appear go west to east across and around the Sun in about 25.4 Earth days, and about 35 Earth days at its poles. Therefore, the Sun also spins in a counter-clockwise direction as seen from above its north pole, and once on its polar axis at an average of about 30 Earth days.   At the equator region of the Sun, a point, sunspot or ejecta (matter released, thrown off outward) on the Sun has a horizontal velocity across or on the Sun of about 2.72 million miles / 30 days = 90582 miles / day = 3774

complete) sunlight, and 13.66 days of (some to complete) darkness. I mentioned the word "some" because the full (lit) Moon or new (dark, unlit) Moon only lasts for a few seconds in reality as the Moon is still constantly rotating on its axis, and appears to the average person to last about 2 or three days of time length. But for example, as soon as a full Moon happens it will then soon begin to develop a thin dark crescent which will start to "wax" or cover over the Moon as the light "wanes" or is removed away. Because the Moon does not have much of any appreciable amount atmosphere gases, there isn't much, if any, of a "twilight" (brightening and-or darkening) to be seen in a theoretical (Moon) sky, but rather a more abrupt changes in the light levels. Because the Moon lacks an atmosphere, it is a great location for astronomical viewing.

For the same seen "Moon phase(s)" (ex., new moon to full lighted moon and back to new moon, etc.) as seen from on Earth, and the Sun in the same sky location as seen on the Moon (ie., a "moon-day or rotation on its polar axis) as it also orbits the Sun as part of the Earth-Moon system, takes about 29.53 Earth days of time. 29.53 days is nearly 30 days of time which is a theoretical, roughly and commonly spoken "month of time".

The Moon (slightly elliptical) orbits about the Earth at: 360°/ 27.32 days = (average of) 13.17716° / day = 13.17716 deg / 24hrs = 0.549048 deg / hour of movement (orbiting) about or around the Earth. Because of this motion, the Moon rises in the sky and-or is at the same position in the sky every night about 53 minutes later than the day before. In short, this is actually due to the extra time needed for the Earth to then "catch up" to the moving or advancing Moon about the Earth, and have it appear in the same location in the sky.

From above the surface of the Moon at its north pole, the Moon will be seen to revolve counter-clockwise on its axis, and also have a counter-clockwise orbit around the Earth, and this motion is exactly timed so as to eventually make the Moon appear as to not rotate on its axis, and always having the same side of it facing Earth. Setting up some proportions type of equations:

60 min / 15° = X min / 13.17716° , Xmin = about 52.71 min   : the Moon appears "late" by this much every day

13.17716° / 24 hours = X° / 1 hour , X° = 0.549048° / 1 hour   : how much the Moon appears to move in the sky per hour eastward or "backwards" with respect to the apparent motion of the stars. Ex. In the 12 hours of the night, the Moon will appear to travel about: (0.549048°/hr)(12hr) = 6.589° eastward with respect to the position of the stars.

[FIG 213]

right-angle trigonometry:

Tan ¢ = opposite side of angle / adjacent side of angle = 8000 mi / 240000 mi = 0.33333

¢ = ArcTan (Tan ¢) = Arctan (0.33333) = 1.9092° =~ 2° visually.

SOME COMMON MATH FORMULAS

The following formulas and discussions are presented so as the reader may have at least a basic, acceptable or minimal awareness, knowledge and understanding of some of the typical scientific concepts they may possibly encounter at some point in various fields of study and-or in everyday life. For most people, this would be enough to have available and to consider, but these topics when fully explored could easily fill volumes of other books that are more needed for those who have focused on these specific topics or fields of study, experimentation, and work.

distance = (speed) x (time)   : The most common equation of these three variables. We also see that to know distance, we need to first know what speed is. Mathematically:

speed = distance / time   :after the division is performed, speed is essentially the rate of an amount of distance units traveled with respect to and during one unit of time . Speed is "how fast" or "how steep, intense or rate" (ie. a rate called "speed"), and its value is defined as how much distance is changing with respect to the change in time. This equation essentially reduces to how much distance is changed or how much an object traveled in 1, "per 1" or per unit of time. Speed (s) is formally called velocity (v) in science studies.

    Ex. 100miles / 2 hours = 50 miles / hour
    Ex. 200 miles / 4 hours = 50 miles / hour , hence the same distance and time rate
                         as the last example.
    Mathematically:

time = distance / speed   : here, time can be mathematically described and-or equal to as the ratio of the change in distance with respect to the amount of speed

Examples of the (mankind made) units of measurement for distance are units of length such as: feet (ft), miles (mi), kilometers (km), etc.

Examples of the units of measurement for time are: seconds (s), minutes (min.), hours (hr), months (mo), years (yr), etc.

  Two times values can even be expressed as a ratio, and for example: "days per year" = days/year =
  = 365.25 days /1 year, and the reverse ratio of this is: 1 year / 365.25 days. Both of these fractions include 1 year unit.
  Dividing the numerator and denominator values of the first fraction by 365.25, we have these equivalent fractions
  which include 1 day unit and the equivalent amount of year units: 1 day / 0.00273785 yr = "1 day is to 0.00273785
  years" and-or "0.00273785 years is [equivalent] to, or corresponds to 1 day".

Examples of the units of measurement for speed or velocity, and which is a combined unit of distance divided by a unit of time such as: (miles/hour) = "miles per hour"   , (feet/second) = "feet per second".

                When speed is not an average value, but for a specific point in time during the travel, it is often called the (current or instantaneous) velocity. Distance is a length value. Units, such as for example: speed = meters/second, or miles per second, are sometimes called "derived units" that are composed of other basic units of measurement. velocity is another scientific word for speed.

It the total time of travel was 20s, t=20s , the total distance traveled is:

d = v t = $\dfrac{1m}{(1s)}\dfrac{(20s)}{(1)}$ = 20m

If an objected is constantly increasing (ie., changing) its speed or velocity by 1 meter of distance per second after each second of time, it is said to have an acceleration of (a) = (change in velocity / time) equal to (1 meter per second) more, after each second of time, hence:

a = $\dfrac{(change\ in\ velocity)}{1s}$ = $\dfrac{1m/s}{1s}$ and this is mathematically equal to: a = $\dfrac{1m}{s^2}$

We also see that the (change in velocity) mathematically equals:

(change in velocity) = (acceleration) (time) = at   , and if the initial velocity v1 was 0m/s:

(v2 m/s - 0 m/s) = v2 = **v = a t**

If the initial velocity is 0m/s, after 10s, the (accelerating or increasing) velocity will have a value of:

v = a t = $\dfrac{1m}{1s^2}\dfrac{(10s)}{(1)}$ = $\dfrac{10m}{s}$: here, at v1=0m/s, a = $\dfrac{v}{t}$

For simplicity of some of these expressions, let the delta symbol of: Λ   represent the change in a value:

(change in a value) = (difference in the values) = (value2 - value1) = Λ value

a = $\dfrac{(change\ in\ velocity)}{(change\ in\ time)}$ = $\dfrac{(v2 - v1)}{(t2 - t1)}$ = $\dfrac{Λv}{Λt}$,   If v1 = 0, and t1 = 0, then: a = $\dfrac{v2}{t2}$

Λ v = (v2 - v1) ,   therefore: v2 = v1 + Λv   ,   Note, if there was no change in velocity: Λv =0, there
                                                        was no acceleration: a=0, and vice-versa

Λ d = (d2 - d1) ,   therefore: d2 = d1 + Λd   ,   if d1 = 0 , d2 = Λd

Λ t = (t2 - t1) ,   therefore: t2 = t1 + Λt   ,   if t1=0 ,   t2 = Λt

Since: d = v t

Λ d = Λ (v t)   ,   to have a change in distance, a change in velocity and-or time is needed:

Λ d = Λ v Λ t = a Λ t Λ t = a (Λ t)^2   : to have a change in (v), a change in (a) is needed,
                                                        and which is caused by a change in force.

If a velocity, say v2, is 2m/s faster than its previous value of v1, the change in velocity is: Λv1 = 2m/s:
new velocity = old velocity + change in velocity = v2 = v1 + Λv1 = v1 + 2m/s. If the time of
traveling at this new velocity is 3 seconds, the increase in distance that corresponds to, or is due to this
change in velocity is:

Λ d = Λ v Λ t

negative sign, and therefore the acceleration value would have a negative sign which essentially indicates a negative acceleration = opposite of an acceleration = a deceleration.

Acceleration or deceleration is a measurement for how the speed or velocity of an object changes over, or with respect to time (t) due to a force acting upon that object. It is the rate of the change in speed per unit of time.   If an object is going very fast but maintains its speed, no forces acted upon that object to change its speed, and therefore, it did not accelerate to a higher or lower speed (velocity). It had 0, or no acceleration.

acceleration = (change in speed) / (change in time)   , or after simplifying the fractions: acceleration = (change in speed) / (1 time unit)

The formula for acceleration is actually be expressed as a rate of another rate that is also with respect to the same variable that is time (T), and because of the definition of acceleration, we can express acceleration in relation to just distance and time, and with this formula for acceleration, it will include a squared variable of time (T)..

Since **speed** or velocity = (D / T), and using substitution into the above formula for acceleration:

acceleration = (change in velocity) / (change in time)   , and this reduces to:
= (change in speed) / (unit time) = (change in (D / T)) / T =
= (change in distance) / T^2   =   D / T^2

Ex. 3m/s^2 = An acceleration of "3 meters per second squared" = "3 meters per second, per second" = (3m/s) / s . A change in velocity of (3 meters per second) for each second of time considered.

Ex.: If the acceleration was noted or calculated at: "five meters per second squared" = = 5 (m / s^2) , this could be thought of as the objects speed or velocity increased by 5 meters per second of time for (per) each second it was accelerating or increasing in speed.   If an objects initial velocity was 100 meters per second of time, and the time of accelerating was 10 seconds, the objects new or total velocity would be:

First note that mathematically from equation for acceleration:

(change in velocity) = (acceleration) (change in time) = AT   : a multiplication or product

total velocity = original velocity + change in velocity   : all values with respect to time

Vt = Vo   +   (acceleration)(time)

Vt = 100 m/s + (5 m/1s^2)(10s)
Vt = 100 m/s + 50 m/s   combining like terms:
Vt = 150 m/s

Also of note as shown above is:

(acceleration)(time) = (a speed value) when the change in time was from an initial time value considered as 0 seconds, and therefore, the change in time is equal to just the total time value that the acceleration was applied.

energy applied to a mass. A higher amount of energy can apply a higher amount of force. A higher amount of force can apply more energy. Also from this itself, we have: Energy = J = mad = Fd .
Shown in this book in this book is that: Kinetic Energy = mv^2 / 2   Joules = Fd Joules = mad Joules

A force, such as gravity, that is constantly applied to an object will cause that object to have an acceleration or change in speed and-or direction. An acceleration due to gravity is called gravitational acceleration and has a constant value since the mass of Earth and its gravitational force is constant. Earth's gravitational acceleration is often given the symbol of (g). **(g) = 9.81 m/s^2**. Expressing this in the general force equation: f = ma = mg.

The weight of an object is the mass of that object being influence by the force of gravity, causing it to accelerate and effectively creating a force equivalent to the weight of that object: **weight = f = ma = mg**
With this equation, the mass of the object can be found, and the gravitational acceleration:

m = f / a = f / g = weight / g   and   **a = g = f / m = weight / mass** : **the higher the acceleration, due to a higher force applied to the mass, the higher the weight and-or the (weight per mass).**

Earth's gravity (attraction) force is due to Earth's large amount of mass (material), and it effectively extends outward from its center and causes an ("gravitational" [force]) induced acceleration of: g = 32.2ft/s^2 = 9.81m/s^2 upon any sized mass or object in a vacuum (no air or other matter, and zero internal pressure), but is nearly the same value for objects in an air medium when the object is smooth and has low air resistance (ie., air collision, "drag" or "drag force" in the opposite direction, reducing its kinetic energy) due to it colliding with air and slowing it down by an effective application of a force in the opposite direction. Another way to consider this is that the object essentially looses some of its energy as it compresses the air beneath it and causing the air to have some pressure or force in the upward direction, and the object will have difficulty gaining any more kinetic energy due to gravity.

velocity = initial velocity + change in velocity
v = vo + v1
v = vo + (g)(t)   : see (*) below for a derivation of velocity = (g)(t)
v = vo + (9.8m/s^2)(t)   : time (t) will also have units of seconds (s).

(*) When the initial velocity of the object is 0m/s, as when simply released by a persons hand at a height above the ground, and the time is considered as 0s:

acceleration = (change in speed) / (change in time),

a = (final velocity - 0) / (final time - 0) = velocity / time = v / t   , therefore, mathematically:

v = at   , since we are dealing with Earth's gravitational acceleration (g), this equation can be expressed as:

Ex. If the initial velocity is 0m/s, and an object accelerates at: a = 1m/s , the velocity of the object after 10s is:

v = initial velocity + change in velocity = 0 + change in velocity = a t = (_1m_) (10s) =   _10 m_
                                                                                                                                (s^2)   s

v = at = gt   : the units of t = time are usually seconds (s), unless otherwise noted.
v = (g)(t) = (9.8m/s^2)(t) = 9.8m / s   :this has the form of: velocity = distance / time , from: distance = (speed)(time)

Here is an algebraic example of acceleration that uses the gravitational acceleration (g) constant. Shown is how to find the distance an object has traveled after it has been dropped from a height and after an amount of time has elapsed:

a = acceleration = change in speed / time = (S1 - S0) / t   mathematically:

force = (mass) x (acceleration) Force is the application of energy to an object. Energy used up, transferred or applied is called and measured a work or "work done". For the units of measurement of force:

F = (mass)(acceleration) , having units of: (kilograms)(meters/s^2) = Newtons = N
1N of constantly applied force could be thought of as that needed to accelerate
1 kilogram at 1meter per second, for each second that force is applied. If you
know an objects acceleration during a certain amount of time, you can find its
velocity at a certain point or instant instant in that time using: v = at

work = a force applied for a distance, its units of measurement are equal to that of
the units of energy which is Joules (J) since this is how much total energy
was used up, applied and-or transferred.

work = (force)(distance) = (mass)(acceleration)(distance) =
= (ma)(Meters) Having units of: (kilograms)(meters/s^2)(meters) =
= (kilograms)(meters^2) / (s^2) = kg (m/s)^2 = kg (v^2) joules

The rate of using energy or doing work is called power (P), and its units are watts (W):

power = Pw = work / time = energy / time.
For the units of power: **W = watts = joules / second**

For example, consider that a battery may have a lot of available energy that was slowly
accumulated over some time value, and stored within it, and if only a small amount of it is
used during each unit of time, the power value and-or energy usage rate will be a low
value. For example: 5 watts means 5 watts of power used per second = 5 joules of
energy used or drawn from that energy supply (ie., an electric battery) per second =
5W = 5J/s.

Due to an applied force, an object will then change speed which is called **acceleration**.
(accelerate = change from one speed to another) and-or direction. If the object is
caused to go faster, it has gained (**kinetic=movement**) energy which is an ability to do
work such as applying a force to something else. It is possible to define energy as
stored force. For example if you compress a spring with your hand, it will store the force
used to compress it, and you can feel an equal and opposite (directional) force or
pressure applied to your hand as the spring tries to release it's new stored energy. As
another example, a rechargeable battery can be charged by the force of falling water at a
distant hydroelectric electric generating station. The battery with stored energy is also
said to have "(electric, charge) potential (potent, future ability, available to be applied)
energy". Since water has a much higher density than air, moving or flowing water at the
same velocity as that of air will contain or have much more moving matter, and the total
kinetic energy and-or or power per second (ie. watts) will be much greater. Water is
about 816 times denser than air. The density of water is typically: 1g/cm^3, and the
density of air at sea-level is typically 0.001225g/cm^3. Density of water / density of air =
(1g/cm^3) / (0.001225g/cm^3) =~ 816. In general, a hydro-electric generator will
produce more power or electricity than a wind-generator when the velocity of the water
and wind are the same. A high dam can be used so as to increase the velocity, energy
and resulting force of the water striking the turbine of the electricity generator and
producing a torque (rotation force) so as to rotate it and the rotor of the generator
connected to it.

Force can be thought of as like a **pressure**, influence, ability, cause or driver upon

is the same on both the Earth and the Moon. Basically, weight is like an effective or enhanced force due to gravity acting upon the object or mass. A mass and its value will not change, and is "universal", and will be the same value anyplace in the universe.

A force applied to an object (mass) will continue to give it acceleration until that force is released or removed to 0 applied force. The amount of force that was applied to the object can therefore be indirectly measured by the effects of it after its application to the mass and the change in velocity or acceleration that the mass had. The measure of the force applied is directly related to both the mass and acceleration of it. f=ma. That is, acceleration of the mass is rather directly related to the force applied and indirectly related to the mass: a = f / m.   It is also more difficult to accelerate (or decelerate) a larger mass, and to give it more kinetic energy. As can be seen in the formula, mass is indirectly related to acceleration and vice-versa.   Given a fixed and constant value of force applied to a mass, if the mass is larger, the acceleration is then smaller.

We found that the value of acceleration is inversely related to mass, and that the smaller the mass given a certain force, the greater the acceleration of that mass due to the applied force upon it. In other words, a force will have more effect on a smaller mass. Larger masses are said to have more **inertia** or "resistance" to change in acceleration and hence more of a (natural, property of the mass alone) resistance to a change in speed (ie. motion).   It could also be said that larger masses resist change in their total energy, hence it will therefore take a longer time for them to gain or loose their stored energy, and that they have a greater ability to do work as compared to smaller mass. A simple illustrative analogy example would be a large or massive volume or pot of hot water, and a small cup of hot water. The large pot of hot water has a greater heat capacity or "heat-momentum" and will take a longer time to initially gain, such as heat up, and then release its acquired energy into the surroundings and cool off. Another illustrative example of inertia would be like trying to change (increase or decrease) the same depth of water in a container with a large surface area of water as compared to that of a small container with a small surface area of water. In short, more massive or larger things have a greater capacity to store energy, but also have a greater resistance or "slowness" to a change in some value or physical property such as gaining or loosing energy, such as temperature and-or speed (ie. kinetic energy).   A larger battery is said to have more "energy capacity" or power ability than a smaller battery of similar construction.

For a "thought example", consider an object traveling in a direction in space at some value of speed, and a faster moving object comes straight behind it and headed in the same direction. The faster moving object will now effectively have some resistance to its free motion in space when it collides with the slower moving object. The faster moving object will essentially cause the slower moving object to increase its speed due to the force of the collision. The faster moving object essentially transfers some if its (kinetic) energy to the slower moving object. The faster moving object will be reduced in speed and (kinetic) energy, and the slower moving object will increase in speed and (kinetic) energy.   The total (kinetic) energy, before and after, of this system of two objects is still the same since energy cannot be created or destroyed, and this is a part of the "conservation of energy" concept.

Pressure is a measure of force applied to a unit of area.   : It is essentially a measure of the concentration of force, or the force concentration in a given area.

pressure = force / area   : then force = (pressure)(area)

For force, the units for the measurement of mass are kilograms in the metric system. It is

and other machinery. It is possible to compress air that it will become dense enough to be liquefied.

Momentum = Mass x Velocity   = mv

Momentum is a measure of an objects ability to keep moving at the same velocity and direction, hence it is technically a vector quantity having both a magnitude and direction. It is also much **like** a measure of the total (moving or kinetic) potential energy or force available due both the object's mass and its speed. **Momentum can be thought of as a measure of an objects motion** at a given moment in time.   A small particle having a low mass going at high speed can have the same value of momentum as a very large mass going at a slow speed.   Also, consider that there is a law of motion stating that an object in motion will continue or keep that speed and direction until some outside force acts upon it, and this could be described as part of the concept of momentun. Momentum is rather a measure of its ability to keep its motion, and the more mass and-or velocity an object has, the more energy needed to alter its direction and-or speed, hence it has more momentum or ability to keep its motion, including if its was initially at rest where it has the same mass, but 0 velocity or speed.

If a first object with more momentum strikes a second object, it applies a force to that second object, and vice versa, since that second object is also then applying some force to that first object, and this is an example of the concept of equal and opposite forces. Some of the energy or momentum of that first object will be transferred to and gained in that second object with less momentum. Basically, what is happening is that the object with more momentum (and has more "potential influence" upon another object) is using up, loosing, transferring, or applying some of its KE energy and momentum to a second object which gains that amount of transferred energy or momentum. In particular, since when the masses remain the same, an energy or momentum increase is indicated by a velocity increase in that second object. The first objects mass will still be the same, but it will have a reduced velocity. The total sum of the KE energy or momentum of both masses will remain the same after a collision of two masses or objects.

The more the mass or "massive" an object, the more inertia (resistance to changes in velocity which is a resistance to acceleration) it will have, and therefore, the more difficult (ie., more energy needed) to change its speed (ie., motion).

One easy to understand example of momentum is when water is flowing in a pipe, and a water valve is shut quickly. Before it is shut, that flowing (moving) mass of water had speed.   The water is said to have momentum (ie., a certain movement and ability to keep moving). When the valve (such as connected to a control knob to control the water flow into a sink and-or tub) is shut off quickly, the kinetic energy available or stored in the water then needs a place to go and be applied elsewhere, and the water pressure in the pipe and upon the side walls of the pipe will quickly rise, and it will cause the pipe to rattle and-or "knock" like someone hit it with a hammer and to make it move and produce quick pulse of sound. This effect is commonly known as the "water hammer".

It is said that for every force there is an equal and opposite force. The downward force of an object near the Earths surface is actually equal to the weight of the object. As the object sits on the ground or table, the object will be stationary and not moving, and this is because of the surface tension of the Earth prevents that mass from going further downward, even though the force of gravity is constantly applied to it.

A lower mass object has less inertia or resistance to an applied force, and its speed is therefore more affected by the (same) value of the force of attraction of two masses. It is

1J = 1Nm = newton-meters = (force)(distance) = (ma)(distance) = work = energy

Friction and-or collisions with other objects will cause an object to waste, transfer or loose some of its kinetic energy. It will slow down unless more force is applied to it. Since it will take energy to apply a force, force can be described as the application of energy.

Here is a verification to the kinetic energy formula shown above: [FIG 215]
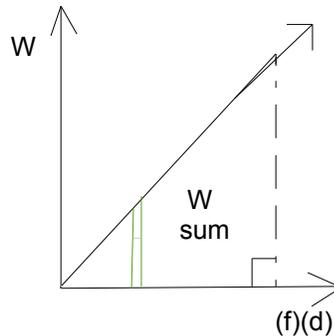
Work = (force)(distance).   A small bit of work is done when there is a small bit of distance the force was applied. The (infinite) sum of all these small bits, slices or rectangles =(force)(very small change in distance) of work equals the total work done, and that can be visualized by the area beneath the curve above, and clearly it is equal to the area of a right triangle having sides of (f) and (d). One theoretical slice or rectangle is shown between the green colored lines. Though this rectangle is slightly longer on one side, when the rectangles gets very small due to the change in distance being very small, this difference also gets very small and negligible or insignificant to the (much larger) total sum.

$$\text{Area} = W = KE = \frac{(f)(d)}{2} = \frac{(ma)(vt)}{2} = \frac{(m\frac{v}{t})(vt)}{\frac{2}{1}} = \frac{mv^2}{2} \text{ Joules}$$

If the velocity of an object increases by a factor of (n), its KE increases by a factor of n^2:

$$\frac{m(nv)^2}{2} = \frac{mn^2v^2}{2} = n^2\frac{mv^2}{2} = n^2 (KE)$$

   This also indicates that for example, to increase velocity by (n) will require (n^2) times more input energy. For example, to double (2) the speed of an object will require 4 times the input energy. In space, with no friction losses, once a desired speed or velocity has been reached, the energy applied to create a thrust force upon the object is no longer needed, and the object will continue to travel at that faster speed. The object will also have 4 times the kinetic energy as it did previously. To slow the speed of an object will require a force in the opposite direction to its forward travel. A force in the opposite direction will reduce its velocity and available kinetic energy. The object will decelerate (a negative acceleration) as long as the force is applied. The energy

creating magnetic fields in the wire coils of electric motors, and which the magnetic force is then used to give a (kinetic energy) movement or motion to the rotor (rotating) part of that motor and something such as an electric fan or wheels that are connected to it.

The units of current (the flow of electric charges or particles, such as electrons of atoms which have gained some kinetic energy, such as from a battery, to move and do work) are Coulombs of charge per second or= **Amperes** (A or sometimes (a)). A Coulomb of charge (ex. electrons) is a specific amount (6.28×10^{18}) of charged particles such as electrons. An amount of 1 Coulomb of charge flowing (like a current) or passing by a point in 1 second is called an (1) ampere in honor of **Andre Ampere** ("amp" = A) who made significant advances in the study of electricity, specifically electro-magnetism, and charge movement or flow. Ampere also invented the (electric to magnetic, or "electromagnetic" movement which is essentially a type of [non-360 degree rotating] electric motor or movement) ammeter (or "amp meter") to measure the amount of current flowing through a wire. This device with its internal coil of wire has similar properties to the later developed (current-to-magnetic [electro-magnetic] movement) device called a loudspeaker which transforms or converts electrical signals or pulses into mechanical audio signals or pulses.

Hans Christian Oersted (1777-1851), from Denmark, is credited to initially discovering electromagnetism in 1820 when he noticed a wire with electricity going through it would cause a (magnetic) directional compass to change its direction away from the magnetic north direction of the Earth. This indicated that electricity in the wire was causing some magnetism in the wire. By using a compass about the wire, it can be deduced by observation that the magnetic field is circular about the wire.

In simple terms, magnetism and-or a magnetic field is much like a gravitational field, but much stronger, and that a magnetic field has two poles like an electric field, and can be thought of as a starting pole and a ending pole for the (imaginary) lines of force. Studies of the atoms of a magnet have shown that a directional alignment of atoms and their electrons will create a magnetic field. A permanent magnet requires no electricity to create a magnetic field. When a piece of iron was made to be hot and in an external magnetic field, and then cooled, the alignment of atoms and their electrons remained fixed or permanent. Passing current through a wire will cause the electrons to be aligned and flowing in the same direction, and this will cause a magnetic field, and which is at a right angle to the electric field created by the electrons.

In1830, **Johann Schweigger** (1779-1857), from Germany, found that if the wire was placed into a coil shape, that the magnetic field created would be concentrated, and therefore stronger. A device he made is an (electric) galvanometer, but unlike just sensing static electric charges and-or fields, his device could measure the flow of charges or electric current. Later, **Michael Faraday** discovered magneto-electricity, whereas a magnet (either a non-electric permanent-magnet or an electro-magnet (sometimes called a solenoid) made from a coil with current going through it) can induce a voltage and current in a wire, especially a coil of wire where the effect is essentially larger (ie., more produced voltage and-or current) and more noticeable and practical. In 1821, Faraday also invented a primitive device that created a consistent and circular motion motion due to the applied electricity, hence he created a primitive type of electric motor, and it inspired others.

Ampere soon took great interest in Oersted's discovery of electric current creating a magnetic field in a wire, and basically he noticed that the larger the battery voltage, the greater the magnetic field produced, and this indicates that the amount (ie., current value) of electricity in the wire was now larger. Surely more electricity (ie., modernly called current or current flow) in a wire would also make the wire a higher temperature and we

(attractive or repulsive) force upon an electrically charged particle at that point or location in that field. Here are some related equations, and the reader may skip over them, and review them when ready and-or needed:

E = Electric field strength = $\underline{\text{force}}$ = $\underline{\text{F}}$, with units of: $\underline{\text{Newtons}}$ = $\underline{\text{N}}$
$\qquad\qquad\qquad\qquad\qquad\quad$ charge $\quad$ Q $\qquad\qquad\qquad\qquad\quad$ Coulomb $\qquad$ C

$\quad$ Mathematically: F = EQ $\quad$: note, the amount of charge of a single electron is:

$\qquad\qquad\qquad\qquad\quad$ (1.602564)(10^-19)C , since 1electron =
$\qquad\qquad\qquad\qquad\quad$ 1 particles and charge out of
$\qquad\qquad\qquad\qquad\quad$ (6.24)(10^18) particles and charges = 1C

$\qquad\qquad\qquad\qquad\quad$ 1 charge / 1C =   1 / (6.24)(10^18 charges =
$\qquad\qquad\qquad\qquad\quad$ 1 charge = 1 electron / (6.24)(10^18) charges =
$\qquad\qquad\qquad\qquad\quad$ 1 charge = charge on 1 electron = 1.602564 (10^-19) C

$\quad$ Or in brief: 1C of charged particles or electrons = (6.24)(10^18) charges or electrons, and if each side is divided by the factor of (6.24)(10^18), we find the charge of 1 single single electron to be: 1.602564 (10^-19) C

Since force is the application of energy upon something else, work is essentially a measure of using or applying energy for a physical distance or length, Therefore, work can also be measured with the same units used for energy, and those are in Joules.   Note that since energy from a source is needed to do something, after that energy is used up with its application or doing work, that amount of energy has been removed from the source and-or system and transferred elsewhere during the work.

$\quad$ W = Work = (force)(distance) = Fn Dm $\quad$, with units of Newtons-Meters = Joules

$\quad$ Mathematically: F = EQ $\quad$ and $\quad$ W = (EQ) Dm = FnDm $\quad$ and E

Voltage (V) is technically defined as equal to the amount of work or energy required to move or force charge across or between two points. Since work is essentially equal to an amount of energy, voltage is also an amount of energy and is often called as "voltage-potential energy" or simply a voltage potential (ie., ability to move current so as to transfer energy (via electrons) and apply power or work). A higher voltage has a greater ability or potential to induce and move electric current. Consider this form of Ohm's Law where: (current) = (voltage / resistance) = (I) = V / R. Current is directly related to the voltage inducing, forcing or generating it.   If the voltage increases by a factor of (n), current will increase by the same factor (n).

Work is a measure of using or applying a constant force (mass x acceleration) through a distance, and it could be said as being how much total energy was required and used, and the units of work therefore contain the same units for energy, which are Joules.

Voltage = Work / Charge $\quad$ or = Joules / Coulomb $\quad$: "energy per coulomb of charge"
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **V = J / C**

From the above equations, we can mathematically express an electric field strength using voltage and the distance between two points in an electric field (E):

$\quad$ V = J / C = W / Q = FD / Q = (EQ) D / Q = ED $\qquad$ mathematically:

(hp) was commonly used as a unit of power that is used for comparison of the power ability of various machines that eventually replaced horse and water-wheel power. The most common unit of power as of the year 2020 is called the **watt**. James Watt also invented an improved version of the steam (hot, pressurized water gas) engine in 1876 which helped the industrial (manufacturing) revolution which reduced time and costs and improved the availability of many things till this day although most steam engines have now been replaced by high power electric motors. Watt's steam engine considered the previous steam and (first practical) piston (a cylinder with a disk in it which pressure can move that disk and rods attached to it) engine developed in 1712 by **Thomas Newcomen**, (1664-1729) from England. The density of water at just below boiling temperature; 100°C = 212°F, is about 0.96 g/cm^3. The density of steam at just above boiling temperature is much less dense at about 0.0006 g/cm3, and it is a highly kinetic or energized gas. If 1 gram of water (which has the dimensions of 1cm^3) was converted to steam, it would theoretically occupy a space or volume that is about (0.96 g/cm) / (0.0006 g/cm) = 1600 times greater than 1 cm^3. 1600 cm^3 = a volume of 1.6 Liters

1 foot-pound = 1 ft.lb = the energy needed to move a weight or apply a force of 1 pound through a distance of 1 foot. **1ft.lb = about 1.35582 joules**. 1ft.lb / 1s or = 1lb ft / 1s = about 1.35582 watts

For torque, 1 pound-foot = 1lb.ft = about 1.3558 Nm , Nm = Newton meters , hence the units of a force value . This is a pound of force applied at a (perpendicular) distance of 1 foot from the pivot, axis, center or turning point, and is essentially the length of a radius and-or lever-arm.
Ex. For tightening or un-tightening bolts to a certain recommended and safe value.

As mentioned above, Pw = (V)(I), and here are some other mathematical equivalences based on Ohm's law: :

Pw = V x I = (I x R) x I = I^2 R = V x (V/R) = V^2 / R

Ex. 0.5 amps of current is flowing through a 100 ohm resistor: Pw = (0.5^2)(100) = 25w

Ex. If a 100 ohm resistor has a maximum power rating of 1W, what is the maximum continuous (ie. dc = direct current) current that can be allowed to flow through it when the resistor is being electrically (electronically) operated at it's maximum power handling rating?

From Pw = I^2 R , Ia = sqrt (Pw/ Ro) = sqrt (1 / 100) = sqrt (0.01) = 0.1A

If this amt. of current was flowing through that resistor, the voltage applied across it would be: Vr = Ir Rr = 0.1a x 100o = 10 volts, but this is not necessarily the maximum voltage that can be across it, for higher voltages can be across that resistor if less current passes through it so as the actual power applied to that resistor can still be equal to or less than the maximum power rating of that resistor.

$$Pw = \frac{energy}{time} = \frac{J}{s} = \frac{(\_J\_)(Qc)}{(\ s\ )(Qc)} = \frac{(J\ )(Qc)}{(Qc)(\ s\ )} = V\ I \ \ watts$$

$$Pw = \frac{energy}{time} = \frac{J}{s} = \frac{work}{s} = \frac{(force)(distance)}{s} = \frac{(mass)(acceleration)(distance)}{s} = \frac{Nm}{s}$$

J = Nm = (Newtons)(meters)

Since P = (voltage)(current) , if given a certain amount of power, and if the current is reduced to a very small value, the total (kinetic, motion) energy and power delivered by those charges would correspondingly

local force of gravity, such as the gravity value of a planet such as Earth. This constantly applied force of gravity will cause an object to accelerate, therefore giving it energy which can be used to apply a force such as its weight. Weight = F = ma.

weight = force = (mass)(acceleration)   : the units of force are Newtons (N)
                                                                                          Units of mass is Kg, and units of acceleration is m/s^2

Many formulas use the mass value of an object rather than its weight value. If given an objects weight value, to convert it to its corresponding mass value, then divide it by the force of gravity (g) = 9.81m/s^2 = 32.2ft/s^2 since from F = weight = ma = mg   , mass = weight / g :

mass = weight / acceleration = N / g   : the common units of mass is the kilogram (kg)
                                                                                    N is Newtons, the units for forces.
                                                                                    g is Earths gravitational acceleration constant of ~= 9.8 m/s^2

Since 1 gram = 0.001Kg = 1(10^-3)Kg , the weight of 1 gram of matter is:

weight = force = (mass)(acceleration) = (0.001Kg)(9.81m/s^2) = **0.00981N**   : **weight of a gram of mass**

Most common scales will not display 0.00981N for when a mass of 1g is placed upon it, but the scale is calibrated to display the corresponding mass value of that weight sensed or influencing the scale. The scale will display 1g. This weight scale therefore is a mass scale. Since a kilogram is 1000 times more mass than 1 gram, its weight will correspondingly be 1000 times more than the weight of 1 gram:

 weight of a Kg of mass = (weight of 1 gram)(1000) = (0.00981N)(1000) = **9.81N**   : **weight of a Kg**

Density = Mass / Volume   : Density is a measure of the amount of mass or substance (material such as the atoms, or
                                               atomic particles) per, or in a, three dimensional area of space or volume. Density can also
                                               be thought of a s the concentration or amount of matter within a given volume or space. A
                                               denser substance can be said as being more concentrated.

                                               The units of mass are grams. If the units of volume are cubic centimeters, as commonly
                                               used in density tables. The units of density are derived units composed of the units for
                                               mass and volume, and for here, they are: (g/cm^3) = "grams per cubic centimeter".

                                               If the mass of an object is unknown, it can be found from the objects weight since:
                                               weight = force = (mass)(acceleration) = ma = mg   ,
                                               and m = mass = weight / g = (weight kg) / (9.8 m/s^2)
                                               Most common "weight scales" display the equivalent mass of that weight, such as grams (g).

                                               The amount of material or substance of water in 1cc volume is defined as the reference and
                                               unit mass of 1 gram.

                                               1 gram of mass of water per 1 cubic centimeter (cc = cm^3) volume (space) is defined as
                                               has a density of 1g/cc = 1g/1mL at 4°C. This is also the max. density of water. This is the
                                               common reference (of measurement) and or unit of mass (ie.,the mass unit) when it comes
                                               to comparing densities of other elements and materials.

                                               1 gram of water is also indicated as 1 gram of weight on most scales of which are
                                               calibrated so as actually express or display its corresponding mass values. This value is then
                                               more correctly called a "gram-weight" or "mass-weight" value. The weight or force of 1 gram
                                               of mass is called 1 gram-weight or 1 gram-weight unit, and commonly (and incorrectly) as 1

Water has a density of 1g / 1cc, hence the density of this oil is less than that of water, and this oil will float on water. Because oil has less density than water, it has less mass and weight per same volume as water. Dividing both num. and den. by 0.928 , and inverting the fraction, we find the volume of 1 gram of this oil substance: volume per gram = 1.0776 cc / 1 gram

Because the oil is less dense than water per volume, it has less mass in the same volume or space, and it will take more volume of oil so as to have an equivalent amount of mass as 1 gram of water. Here is an image of the density test: [FIG 215A]
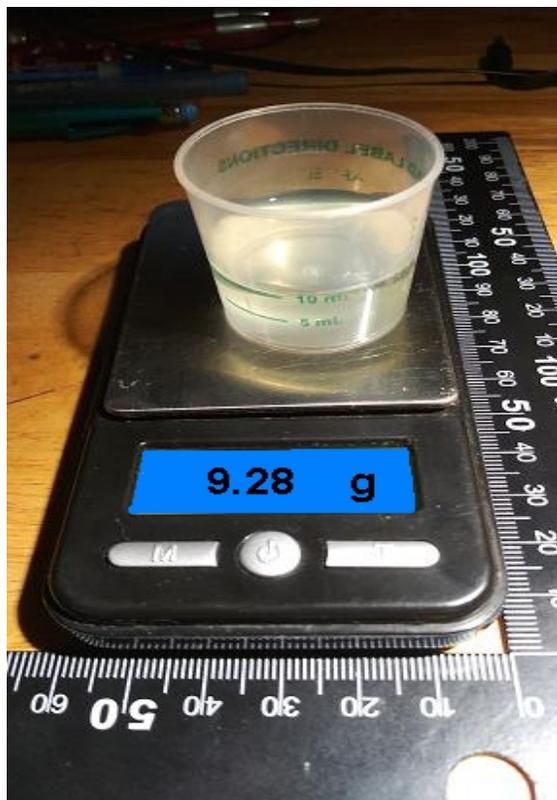
The length ruler is shown as a size reference, and each graduation or step is a millimeter.

If the substance in the measuring cup was 10mL = 10cc of water, it would have a mass of 10g since each 1cc of water has a mass of 1g. This is actually a good way to first calibrate a measuring cup and-or determine if its volume is gauged or marked correctly before measuring the volume of substances which have the same volume but different mass and its corresponding weight. To calibrate a weighing scale, you can purchase or use an object with a known calibrated mass and-or weight. The (average) mass of some objects such as coins is given in this book. A 1983 or later U.S. copper coated, zinc penny coin has a mass of 2.5g. Four of these coins weighs about 4 (2.5g) = 10g.

lead is considered highly dense and is therefore heavy, but unlike gold, lead has a fairly low melting point or temperature where it turns from a solid state, phase, structure or form, to a liquid form where the atomic bonds necessary for crystallization are much weaker. The atomic bonds are weaker due to the heat energy absorbed and this increase in kinetic energy (ie., atomic movement) prevents crystallization, solidifying or bonding of the metals atoms.   For comparison, gases have very weak or no atomic bonding.

To convert grams per cubic centimeter to grams to cubic inch (ci = in.^3):

From: 1 cubic inch = 1 cu. in^3 = 16.3871 cubic centimeters = 16.3871cc , we can multiply a given number of grams in the grams-per-cubic centimeter value by 16.3871 so as to find the number of grams per cubic inch (ci = in^3).

Water, being 1g/cc will be (1g)(16.3871) = 16.3871g/ci .

Steel, being 8g/cc will be: (8g)(16.3871) = about 131.1g/ci , and since mass per unit volume is 8g and is (8g/1g) = 8 times more than that of water, that amount of mass will be both 8 times more and also weigh 8 times more than that of an equivalent volume (here, 1gram) of water.

To convert grams to pounds: Since 453.592g = 1 lb = 16 oz,   divide the number of grams by 453.592 to convert to pounds. From this we can mathematically find that:

To convert grams to ounces: Since 28.35g = 1 oz,   divide the number of grams by 28.35 to convert to ounces.

For when 30g = 1 "food ounce", then simply divide the number of grams by 30 to convert to "food ounces".

If a density is noted as 8 grams per cc of volume: $\dfrac{8g}{1\ cc}$

The reciprocal of this is: $\dfrac{1cc}{8g}$

Dividing both the numerator and denominator by 8, we can find the unit ratio of the amount of volume per 1 gram unit:

$$\dfrac{\frac{1cc}{8}}{\frac{8g}{8}} = \dfrac{0.125cc}{1g} \qquad \text{or:} \quad 1g\ /\ 0.125cc\ =\ \text{"one gram per 1/8 cc"}$$

process called photosynthesis. Carbon dioxide is denser, and therefore heavier than common air. Air also contains traces of some other elements. Air is normally 0.870 g/cc at 1 atm (14.7psi). and can be liquefied. This can be done by what is called the Linde process which condenses (increases its density) and this will then be used to reduce the heat energy of it when it is passed through a heat exchanger or radiator system.

lithium 0.53 [a lightweight, soft metal at room temperature, a small amount is used in rechargeable batteries]
alcohol [pure, non-poison ethanol 0.789 , isopropyl 0.786 "rubbing" alcohol, poisonous typically used for antiseptics]
          [methanol is ethanol that is "denatured" with poisonous methanol, typically used for fuels and solvents]
potassium 0.862 [a soft, and very low radioactive (radiates or emits energy and-or particles) metal, many people
          are low in potassium and it is found in some foods and-or some salt-substitutes].
food oils   0.91 to 0.93 is typical [ex. olive oil 0.917, corn oil 0.93, will float on water since oil is less dense or heavy]
sodium 0.971 [a soft metal, melting point of only 208°F, it is highly reactive with water, Table-salt contains sodium]
water (solid, ice) 0.9167 [ice is cold, crystallized water, and is less dense than liquid water because the crystal
                    solid expanded into a larger volume, hence some of that material, here ice, is outside
                    that reference volume, here 1cc = 1 cubic centimeter = 1 cm^3]. Water freezes at 0°C
water (liquid) 1.0 [comparative reference density, defined at 4°C, slightly above freezing, and max. water density.
                    Water, H2O1 = H2O, as a liquid, is essentially many condensed (ie., now denser, more of a solid)
                    molecules of hydrogen and oxygen gas. Sea-Water is about 1.03g/1 cc . Liquid water can be
                    thought of as condensed or concentrated water vapor (humidity and steam), and ice can be
                    thought of as condensed or concentrated liquid water. When water vapor looses its thermal
                    energy, it can condense and form into snow (fine ice crystals, at high altitude, where its very
                    cold) and-or rain drops when the temperature is not water freezing (at or below 32°F = 0°C)]
                    **Though water feels and is much softer than hard ice, water is actually more dense than
                    ice in terms of the amount of mass per volume = mass density = mass/volume.**

milk 1.3 [typical, for whole milk containing some fats (oils)]
PVC 1.38 typical [Polyvinyl Chloride, a plastic, often used for electrical and water pipes. Made from petroleum (ie.,
          crude oil). PVC pipes and material can be softened (at about 200°F) with hot air such as in an oven or hot-fan
          blower, and then bent into shape before cooling. **HDPE** is High Density Polyethylene, a relatively soft plastic
          made from petroleum, and can be recycled and-or melted (about 350°F) in an oven and made into products,
          including bulk plastic material available for other smaller projects. Being softer, the density of HDPT is lower
          than that of PVC, and is only about 0.95 g/cm^3 and will float on water which has a density of 1g/cm^3 = 1g/mL]
honey 1.42 typical
calcium 1.54 [a soft metal, used in some alloys (mixes) of metals, melting point of 1548°F , a bio-nutrient]
table sugar 1.59 [a food sugar, can be produced to have a crystalline, solid form]
magnesium 1.74 [a lightweight brittle metal, used to strengthen some alloys, highly reactive (burns) in water, can
                    also be found in Epsom Salt (magnesium sulfate=MgSO4 which contains about 450mg of Mg per
                    5g plants so as to do photosynthesis. many people are magnesium deficient which can cause
                    some health problems; various nuts and seeds like pumpkin and sunflower seeds are high in
                    magnesium, and seeds can be ground and added into other foods, such as soups, as flavorings].
phosphorus 1.82 white, 2.34 red, 2.69 for black [white, highly reactive, a **non-metal**, soft, waxy, used in fertilizers]
beryllium 1.85 [a rare brittle metal, a nickel or copper-beryllium alloy is used to improve heat and electric transfer]
carbon dioxide 1.96 ["dry ice" is cooled (frozen) and solid (condensed) carbon dioxide, with a density of 1.5 typical]

sulfur 2.07 [a **non-metal** element, it is also a necessary nutrient for life, such as for making amino acids and-or
                    proteins]

table or food salt 2.165 [a compound, NaCl, composed of 40% sodium ions with a net positive charge, and 60%
                    chloride atoms (chlorine ions with a net negative charge, gained an electron from the sodium) by
                    weight. These ions electrically attract and bond together to form each molecule. This is a ratio of
                    (40% / 60%) = (0.4 / 0.6) = (0.2 /0.3) = (2 parts sodium to 3 parts chloride). That's also a total of
                    5 parts, and (2/5) = 0.4 = 40%, and (3/5) = 0.6 = 60%. Sea water which has a typical density of
                    about 1.03g/cc, and is about 3.5% salt, hence it is slightly more dense and heavier than pure water.

to 50 ug per liter of seawater. Evaporating the water out of **sea water** is one way to extract the minerals in seawater. 1L of seawater will contain "sea salt" (sodium-chloride, a compound of elements) weighing at 10 grams of sodium and 20 grams of chloride, plus other lesser quantity of dissolved minerals (elements) such as magnesium chloride, sulfate, calcium chloride, calcium, potassium, sulfate, bromine, iodine, etc. In the United States, the (min.) RDA of iodine is 150 ug/day. Note that evaporating seawater may actually "gas-off" the (volatile, fragile) iodine into the air. Some sea plants, such as seaweed and kelp absorb iodine and have a much higher, less volatile, concentration of it].

germanium 5.3 [a semi-metal or metalloid, brittle, similar to silicon and used for semi-conductor electronics]
radium 5.5 [a highly radioactive and dangerous element, many more times naturally radioactive than uranium]
gallium 5.9 [a soft metal, brittle when cold, liquid at about 85°F, used for some semi-conductors and mirrors]

vanadium 6.11 [a rare, soft and ductile metal, its oxide resists corrosion, and is used in high strength alloys]
zirconium 6.52 [a hard metal, malleable, high strength, non-reactive]

neodymium 7.0 [a "rare-Earth" metal, soft, reactive, used in some high magnetic field strength magnets and lasers]
zinc 7.134 [a metal with a melting point of 787°F, often used in alloys, and for an additive in other products]
chromium 7.15 ["chrome", hard, non-reactive, used for a stainless, protective and shiny plating for other metals]
tin 7.29 [a metal with a low melting point of 750°F that can be reduced further if mixed with bismuth and-or zinc]
manganese 7.3 [a hard and brittle metal , used in some alloys for magnets , glass making, a reaction catalyst]
iron (steel is iron with some carbon) [7.87 for low carbon mild hardness steel, 8.0 for grade "316 stainless steel"]
    [Sometimes called ferrum (Fe), ferrous or ferrite, iron melting point is 2700°F, steel melting point is 2500°F, its name may mean "metal from the hot fire or furnace", "fire metal" or "furnace metal". Iron oxide is surface rust due to air-oxygen contact, The body uses a small amount of iron, as atoms, to carry oxygen through the body. Iron-pyrite (FeS2 = 1 iron atom to every 2 sulfur atoms) or iron-sulfide is iron with traces of sulfur in it, and it occurs naturally, atom by atom in the presence of flowing water containing the supply of atoms, sometimes as a beautiful crystal form. It has a yellow color as that of shiny brass and is somewhat rustproof. The density of iron-pyrite is less than that of plain iron or steel, and is about 5gm/cm^3. Pyrite is a semiconductor with an electrical (for conduction) "bandgap" of about 0.95eV and could be used as a part in a "crystal" (ie., sold-state, radio).] Forged steel is compressed or "hammered" steel when it is red hot, and is stronger due to the increased size of the metal crystal structures. Many tools and machine parts are made from forged steel. Some iron artifacts have been dated to about 2000 BC, but before about 1200BC, smelted iron was generally not available to most people, and hammered iron from meteorites was sometimes used in a very limited basis. Bronze was in much use by the time iron products became available. The first iron production and use is usually thought to be in south-eastern Europe, and India. Every piece of iron and-or steel was surely valuable at those times and until to the "industrial revolution" beginning at about 1800AD where many products began to be "mass (ie. massively, many) produced" and became much cheaper and available due to advances in science, technology and transportation. The ore form of iron is very abundant in the Earth.

brass 8.6 typical   [an alloy of varying proportions of mostly copper, and with zinc, and is golden-yellow in color]
cadmium 8.65 [a soft and malleable metal, toxic, used in some rechargeable batteries]
bronze 8.5 to 8.9 typical [a hard alloy of mostly copper, and with tin, typically has a reddish-brown color]
        [brass is much less of conductor of electricity than copper, and bronze is even worse than brass]
cobalt 8.9 [used in some blue pigments, alloys, magnets, a very small atomic part of the B12 vitamin compound]

nickel 8.91 [a hard, malleable, an nonreactive or non-corrosive metal, a potential toxic health risk from its dust. The melting point of nickel is about 2650°F]
copper 8.96 [an orange-red colored metal, has very good thermal and electrical conduction. The melting point of copper is 1984°F]

bismuth 9.81 [a dense, heavy metal with a low melting point of 520°F, an alternative to using toxic lead metal]
molybdenum 10.3 [used in some stainless steels and other alloys]
silver 10.5 [a metal with excellent heat and electrical conductivity, but higher in cost than copper. silver reflects

join, bond and remain near each other and form the (atomic, molecular))crystal structure of ice. The atoms in a gas have a higher kinetic energy. Gases have a low density, while solids have a high density. The air that we breath is a gas at standard temperature and pressure, and is mostly a mixture of nitrogen (78%) and oxygen (21%). Ammonia (NH3) is a gas that is a nitrogen and hydrogen compound, and it is used in fertilizers and cleaning products. Some ammonia is produced when protein is processed in the body. Ammonia has a "sharp", strong pungent smell and is found in urea and "smelling salts" that are sometimes used to awaken people or make them more alert.

Plasma is a state of matter that is a very hot gas and is high in kinetic energy, and its atoms have broken apart into protons (positively charged) and electrons (negatively charged). An example is a gas light such as a neon ("new gas", a natural trace element found in air) or fluorescent light which is basically a glass tube filled with a gas that gets turned into a plasma by applying electricity through it. Fluorescent lamps are coated on the inside by a material that will fluoresce, which means when struck by invisible high energy light radiation, it will radiate visible light radiation.

One definition of a gas is that a small unpressurized amount of it will fill or expand outward into any (empty) container of which it is placed into. When a gas spreads out into a larger volume, the density of that gas will decrease since there is now less matter (here, gas) per unit volume. A gas spreads out because the atoms or molecules are not joined together as like that of a liquid or solid, and may repel each other electrically, and-or have thermal, kinetic energy.

One definition of a liquid is that is will fill any volume it is placed into, but due to its density and the gravity (force) of Earth, it will only fill the lower (ie. in height) volume of that container.   Unlike for the gas, the volume and density of the liquid will generally remain the same when pressurized. A liquefied element or molecules is more dense than the gas(s) that compose it. A liquid can be defied as the state of a substance when it is between its solid and gas form. A liquid is usually less less dense than a solid.

Matter that is less dense is more compressible. Gases are more compressible than liquids. Liquids, though only slightly compressible, are more compressible than solids. In brief, gases can be converted to a denser or liquid state by using the Linde-cycle or process invented by **Carl Linde** (1842-1934) from Germany in 1895. In this process the gas is compressed and this increases the pressure of it and effectively increase its density. A side-effect is that the temperature of the gas will also rise when it is compressed. The hot, compressed gas is then let to cool off (release much thermal and-or kinetic energy and then it reduces in temperature (getting colder, more dense) and pressure) by thermal conduction (ie., transfer) through the wall of the smaller volume holding it, and-or through a heat exchanger, and then allowed to then expand and reduce in pressure, and the cycle is repeated until the gas is liquefied. This is also basically how a modern food refrigerator works. Liquid oxygen, nitrogen, liquefied natural gas (LNG, methane), and liquefied propane gas (LPG) are common examples of liquefied gases. Once liquefied and cold, if warmer air then comes into contact with liquid oxygen or nitrogen, some of that cooler liquid at the surface and boundary of the warmer air will then increase in temperature boil off into the warmer, higher energy air and make a visual steam or "fog" appearance of the less dense gas.

We see that the phases or states of matter (or atoms) are mostly due to the influence of external physical conditions such as temperature and pressure applied to that matter and its initial state.

A solid metal, usually with a hard crystalline atomic or molecular structure, can become soft when heated to a hot temperature, and before it is melted to a liquid state or status. The high heat will weaken its hard crystalline structure. When a metal is soft, it is more pliable (bendable, malleable), and more "plastic-like", and can be more easily **forged** or pressed into a shape using a high amount of force-and-pressure, such as with a hammer. This is usually how coins are made, and also to have some design or inscription placed on them by another carved piece of hard metal.   A person called a "black smith" can make many types of metal (usually iron or steel) items by hammering and-or cutting the hot metal piece against a larger piece of metal called an anvil. The metal is then allowed to cool naturally and-or is dipped ("quenched", heat extinguished) into water or an oil so as to harden it more quickly and to give the metal some specific property such as hardness or some pliability.

The amount of energy needed to change a specific element from a solid state to a liquid or melted state depends on the amount of mass of it and the type of element since each element has a various atomic bond strengths. This amount of energy to change between a liquid (near the solid state) and solid (near the liquid state) is called the **energy or heat of**

```
Darmstadtium , Ds ,      110 ,        281 ,         34
Dubnium      , Db ,      105 ,        268 ,         29
Dysprosium   , Dy ,       66 ,        162.50 ,       8.54
Einsteinium  , Es ,       99 ,        252 ,          8.84
Erbium       , Er ,       68 ,        167.26 ,       9.07
Europium     , Eu ,       63 ,        151.96 ,       5.24
Fermium      , Fm ,      100 ,        257 ,          9
Flerovium    , Fl ,      114 ,        289 ,          14
Fluorine     , F ,         9 ,        18.998 ,       1.7    : a gas at STP
Francium     , Fr ,       87 ,        223 ,          1.87
Gadolinium   , Gd ,       64 ,        157.25 ,       7.9
Gallium      , Ga ,       31 ,        69.723 ,       5.9
Germanium    , Ge ,       32 ,        72.64 ,        5.3
Gold         , Au ,       79 ,        196.966 ,      19.28
Hafnium      , Hf ,       72 ,        178.49 ,       13.1
Hassium      , Hs ,      108 ,        269 ,          41
Helium       , He ,        2 ,        4.0026 ,       0.0001785   : a gas at STP
Holmium      , Ho ,       67 ,        164.93 ,       8.8
Hydrogen     , H ,         1 ,        1.00784 ,      0.00008988 :a hydrogen atom has 1amu, 1mol of hydrogen is ~1 gram,
Indium       , In ,       49 ,        114.82 ,       7.31                                   hydrogen is a gas at STP
Iodine       , I ,        53 ,        126.9045 ,     4.93   : a non-metal
Iron         , Fe ,       26 ,        55.845 ,       7.87
Iridium      , Ir ,       47 ,        192.22 ,       22.6
Krypton      , Kr ,       36 ,        83.798 ,       0.00373   : a gas at STP
Lanthanum    , La ,       57 ,        138.91 ,       6.15
Lawrencium   , Lr ,      103 ,        266 ,          16
Lead         , Pb ,       82 ,        207.2 ,        11.342
Lithium      , Li ,        3 ,        6.941 ,        0.534
Livermorium  , Lv ,      116 ,        293 ,          12.9
Lutetium     , Lu ,       71 ,        174.97 ,       9.84
Magnesium    , Mg ,       12 ,        24.305 ,       1.74
Manganese    , Mn ,       25 ,        54.938 ,       7.3
Meitnerium   , Mt ,      109 ,        278 ,          37.4
Mendelevium  , Md ,      101 ,        258 ,          10.3
Mercury      , Hg ,       80 ,        200.59 ,       13.53
Molybdenum   , Mo ,       42 ,        95.95 ,        10.3
Moscovium    , Mc ,      115 ,        289 ,          13.5
Neodymium    , Nd ,       60 ,        144.24 ,       7.0
Neon         , Ne ,       10 ,        20.179 ,       0.0009  : a gas at STP
Neptunium    , Np ,       93 ,        237 ,          20.4
Nickel       , Ni ,       28 ,        58.693 ,       8.91
Nihonium     , Nh ,      113 ,        286 ,          17
Niobium      , Nb ,       41 ,        92.906 ,       8.57
Nitrogen     , N ,         7 ,        14.0067 ,      0.0012506   : a gas at STP
Nobelium     , No ,      102 ,        259 ,          9.9
Oganesson    , Og ,      118 ,        294 ,          5
Osmium       , Os ,       76 ,        190.23 ,       22.5
Oxygen       , O ,         8 ,        15.999 ,       0.001429   : a gas at STP
Palladium    , Pd ,       46 ,        106.42 ,       12.02
Phosphorous  , P ,        15 ,        30.974 ,       1.82   : a non-metal
Platinum     , Pt ,       78 ,        195.08 ,       19.46
Polonium     , Po ,       84 ,        209 ,          9.35
Potassium    , K ,        19 ,        39.098 ,       0.862
```

The number of atoms per gram of an element in basic atomic theory is:

   atoms per gram of an element = (amu per gram of the atoms of an element) **/** (amu per atom of the element)
               or (amu per gram of the atoms of an element / 2) **/** (number of protons in each atom)

     : For hydrogen, atoms per gram = (amu per gram of hydrogen), and not divided by 2 since hydrogen does not
      have a complementary neutron for each proton. Also hydrogen has 1 amu per atom, and which is 1 proton.

The number of atoms of an element, per mol of amu is: (Avogadro's number) / (amu per atom of the element)

     Note also that 1 mol of amu is defined as having a mass of 1g, hence the value above is also equatl to:
     number of atoms of an element per gram = (Avogadro's number) / (amu per atom of the element).

It is believed, and expressed here in simplicity, that the "Big-Bang" (ie., a highly energetic energetic or "supernova") origin of the universe contained all the initial particles (electrons, protons and neutrons) of the atoms and mater, and that the first atom created shortly afterwords was the hydrogen atom. It would later take the gravitational forces of a large hot energized star to create (ie. due to a "fusion" of particles) most of the other "heavier (in weight)", more massive elements from the hydrogen gas and some other low density atoms or elements. To create all the other elements that are more dense than iron, the star needed to go supernova (ie., explode, creating tremendous pressures in regions of and near it) and-or collide with a small (about the mass of the Sun), very dense neutron star that is the remnant of a "collapsed star" such as from a supernova, and has an abundance of neutron atomic particles that do not have an electric charge. In general, and statistically, the greater the density of an atom or element, the less abundance of it and the less stable it is, and can   "decay" into, change or convert to, a lighter, less massive element and having greater stability.

Notice that most elements except the gases and a few other elements are metals. Carbon, sulfur, phosphorous and iodine are the most common non-metalic elements. Some elements such as the ones known as semi-conductors are sometimes called **metalloids**. The most common metalloids or semi-metals are silicon, germanium, boron and antimony. Metalloids have properties of both metals and non-metals, hence they are often semiconductors which do not conduct electricity very good. Metalloids are brittle, and this can be easily seen while handling a thin and brittle, glass-like silicon solarcell.

(The reader may skip over this discussion if it is too complex right now, and-or consider viewing it at another time.)

An example of comparing density and mass values of two elements:

Aluminum , Al , 13 , 26.982 , 2.698
    Iron , Fe , 26 , 55.845 , 7.87

Iron has twice the number of atomic mass particles in each atom as compared to aluminum:

   Calculated here using just the given number of protons for each atom:

    $\dfrac{\text{\# of protons in an atom of iron}}{\text{\# of protons in an atom of aluminum}}$ = $\dfrac{26}{13}$ = 2 : iron has twice the mass , or divide the amu in each element atom

Now comparing weight-volume densies, (g/cc):

    $\dfrac{\text{weight-volume density of iron}}{\text{weight-volume density of aluminum}}$ = $\dfrac{7.87}{2.698}$ ~= 2.917 =~ 3.0   : about three times as much per unit volume.
                                       A cubic-centimeter of iron will weigh about
                                       3 times more than a cubic-centimeter of
                                       aluminum.

It is reasonable to believe that if the mass, substance or material of something doubles, then the weight of that something

Given 1 atom of substance B, it will take 2 atoms of substance A so as to have the same amount of total (amu) mass and weight.

Weight is a property that depends on both the amount mass and the strength of the local gravity influencing it with a force and giving it that weight value. A mass value is universal, however weight values are not. For example, the weight (a force) of an object will be less on the Moon than it is on Earth, and this needs to be considered for various kinds of science research.

A gram was originally defined as a unit of weight, and specifically, it was defined as the weight of 1cc of water at 4°C. If some other substance weighed the same amount, then it has the same mass in terms of the number of amu, but not in terms of the number of atoms which for different substances or elements have a different number of amu per atoms.

For a given weight value, say an ounce of weight, when the amu per atom increase, such as for another element, there will be fewer atoms needed for the same weight.

The more amu particles, substance, or material in a unit volume, the more mass in that volume and the denser that material is said to be. density = mass / volume . As indicated before, given atoms with a larger mass, there will be fewer of them for a certain weight. This is an inverse relationship. For a given number of atoms, larger mass atoms will occupy a larger volume than that of lower mass atoms. More lower mass atoms can be put into a certain volume than larger mass atoms.

For learning more about matter and mass, please see the following discussion called: More About Matter, Mass And Mol,

Fundamentally, for pure samples of an element have the same number of protons and electrons, and without considering the possibility of a small percent of atoms being isotopes. An **isotope** is a **nuclide** (an atom of an element that has a different number of protons and-or neutrons. "ide" is a word suffix with a general meaning of a combination of things) with the same number of protons and a different number of neutrons than the average, common or fundamental atom of the same element:

Atomic Mass (matter) = (2)(Atomic Number) = (atm) of each atom of that element
= (2)(number of protons)
= (number of protons) + (number of neutrons)

When the probable or actual number of neutrons are considered, the atomic weight or mass of an atom is, at minimum, twice the atomic (proton) number plus some small percentage of that atomic number due to a high probability of some isotopes (have extra neutron(s)) of that element that may be present in the sample. This small increase generally increases as the atomic number increases. Though an atomic (proton) number of an element is constant, the more accurate atomic mass is an average or expected value that can be considered. The mass of an electron is very negligible and is generally not considered for the atomic mass (or weight) of any atom. The mass of an electron is sometimes noted as about (1/2000) that of a proton or neutron. The sum of the mass of all the electrons in an atom is still negligible and does not even come close to the mass of one proton or neutron.

Ex. The most common or pure form of carbon has an atomic (proton) number of 6 has an atomic mass of 12 and is identified as C12 ("carbon-twelve") and has a fundamental atomic mass of 12amu (atomic mass units = protons + neutrons). When considering a small percent of those carbon atoms in the sample being probable isotopes, such as C14 ("carbon-14") which has 2 extra neutrons and which increases the total number of atomic mass units in that atom, and the "weighted" or average mass is slightly higher than 12, is considered as 12.0107amu. Though this is only a small percent increase, but when considering many millions of probable isotope atoms in the sample of the element, this small percentage may need to be considered. Certainly, a pure (all similar atoms) sample of C14, having slightly more mass than C12, weighs slightly more than C12 of the same volume.

Ex. Iron atomic (proton) number is 26, and its atomic mass is therefore (2)(26 amu)=52 amu, however due to isotopes

1 proton for its nucleus.   It can be said that there are 12 grams of weight / mol of carbon atoms.   The atomic mass or molar mass (matter) number or rating of C12 in grams per mol = **grams / mol** = 12 grams / mol   or 1 mol / 12 g . Dividing both the num. and den. by the atomic number we can find the number of mols per gram, or matter per gram weight = **mols / gram,** which is essentially the reciprocal of the atomic weight: (1/12) mol / 1g = 0.0833 mol of carbon atoms / 1g.   The larger the atom, the less the number of them in order to make 1 gram of matter.

The (weight, not mass) density of carbon is: 2.26g / 1 cc . If 1 mol of carbon is to 12 grams, how many mols of carbon are in just 2.26g of it?   First, 1 mol carbon atoms / 12 g = (1/12)mol carbon atoms / 1g : after dividing both num. and den. by 12 so as to create an equivalent fraction.   The result is: 0.083333 mol / g   , then: (2.26g)(0.083333 mol / g) = 0.18833 mol   In general:

    mols / gram = 1 / (grams per mol)   = 1 / atomic weight rating   **: mass per gram**
                                          atomic weight = molecular weight

    x mol = (grams of an element) (mols / gram)

If a formula said to use 0.2217 mol of carbon, how much would that amount of carbon weigh? Since 1 mol of carbon is to 12 grams, and creating a proportion type of equation:

$$\frac{1 \text{ mol carbon atoms}}{12 \text{ grams of carbon atoms}} = \frac{0.18833 \text{ mol carbon atoms}}{x \text{ grams of carbon atoms}} \quad \text{mathematically:}$$

$$x \text{ grams of carbon atoms} = \frac{(0.18833 \text{ mol carbon atoms})(12 \text{ grams carbon atoms})}{(1 \text{ mol carbon atoms})} =\sim 2.26 \text{ grams}$$

    In general:

    x grams of an element = (mols of the element) (grams / mol)   : converting mols of an element to grams

Due to that a substance can be compacted and-or compressed to a smaller volume, and that also increases its density for that unit volume, such as with a powdered form of a substance such as carbon dust or flour, it is generally not recommended to determine the weight of something by using a volume of that something. Depending on how much it is compacted or not, 1cc of carbon powder may not weigh exactly 2.66 grams.   It is usually best to always use a substance's known weight so as to measure the weight or mass of a substance, rather than use the volume of it. If known measurements were previously made and if you do use a consistent substance and volume of it in a repetitive manner to measure mass or weight, then it is possible to get practical, but not necessarily precise or scientific results with that method of using a volume of a substance.

Since: density = (g/cc)   and   molecular weight = (g/mol)   , multiplying the left equation by the reciprocal of
                                                      the corresponding sides of the right equation,
                                                      or= dividing the corresponding sides:

$$\frac{\text{density}}{\text{molecular weight}} = (g / cc)(mol / g) = (mol / cc) \quad \text{: mol per cubic centimeter of an element}$$

Some other helpful formulas to remember for finding some needed values are :

                  density = weight / volume   : "weight density"

                  density = mass / volume   : "mass" or "matter" density

                  atomic mass rating of an element = mol / gram   : "atomic or mol density"

example of this is when the air temperature inside a piston goes to a high temperature value when it is being compressed and has a high pressure. This initial compression, to essentially "start" (begin the cycle of combustions) the self-running, chain-reaction of compressions and combustions was used with the first type of automobiles manufactured. Here, a person would start a car (automobile) by manually (physically) turning a crank lever with some force so as to move (or "turn") and compress a piston(s) in the engine so as to begin an initial combustion inside that engine of the vehicle. Later, by 1911, geared, high torque, mechanical power and-or force, electric "starter motors" began to be used to "turn-over" (move, cycle) the engines pistons and start a vehicle.

Combustion (ie., oxidation) can also happen slowly ("slow combustion") such as with aging wood or paper after a long time, and it will get darker and have a higher percentage of carbon as the water evaporates. Combustion can happen very quickly as during the explosive combustion with "fireworks" displays. The rusting on iron is due to oxidation. Rust particles are iron-oxide (a molecule of iron and oxygen) and it usually has a brown-red or dark-orange color. A candle or a fire with burning sticks is a form of combustion, and the smoke is a steamy mix of hydrocarbon (mostly water and carbon) with some other trace gases and minerals. It is even possible to collect this "fire gas" substance(s) so as to produce a high energy liquid fuel, and which may be called something like "wood-gas".   Charcoal is wood that has been already been "cooked" or processed so as to have a higher carbon content, and then it is used as a portable and popular fuel to cook food without much smoke, however, in the process of making charcoal, much potential energy of that wood slow combustion is lost into the surrounding air, and charcoal is then a questionable item in terms of being energy wasteful. The price of charcoal purchased at stores is relatively high as compared to unprocessed dry wood, and therefore it is only purchased occasionally by most people who use it.

Oxygen is needed for combustion to occur, and sometimes some substances are added to a fuel to increase its combustion efficiency, and these are called oxygenates, and are said as increasing the "octane" of a fuel.

Mohs Scale = The Mohs Scale Is a type of a "molecular hardness" (resistance to visible scratching) scale invented in 1822 by **Friedrich Mohs**, (1773-1839), from Germany, who studied rocks and geologic landforms. The values could be thought of as a measure of the amount of force needed to scratch a sample piece, however, the scale is not exactly linear, and is more of a slight exponential-like scale or mathematical relationship of the amount of force needed to scratch the element.   Note that hardness and density are not necessarily mathematically directly related or linear in value, for example, lead is very dense and heavy, but is also soft and melts easy and has a low Mohs Scale rating.   For an element to be visibly scratched (technically a cut or plowed groove), it must be done by another element with a higher Mohs Scale rating.   Generally, elements and materials that have stronger atomic bonds will have a higher Mohs rating, and it could be said that those elements are more crystallized at the atomic (atom) level.
The Mohs scale was developed to determine what element a rock may be composed of.

When a Mohs Scale value for an element is greater than 1, the amount of force needed to scratch it is roughly proportional to the square or cube of that value. Here are some typical Mohs Scale values:

Potassium = 0.4 , Sodium = 0.5 , Lithium = 0.6   : low hardness = relatively soft metals

Talc ("soapstone", which is similar to plaster ("paste, layer", a pre-word for "plastic")
            calcium-sulfate="gypsum") , chalk (calcium-carbonate) = **1.0**

Lead , Gallium , Calcium , Tin = 1.5   : lead is quite dense, but has low hardness (ie. "soft") , lead
                                                              melts at 621°F , gallium melts at 85.6°F , tin melts at 450°F
Alabaster = a denser, crystalline calcium-carbonate stone = 2 , crystalline, Alabaster calcium-sulfate = 1.5
Bismuth = 2.25 has a relatively low melting point of about 521°F
Zinc , Magnesium , Silver , Gold , Table-Salt , 2.5   : gold is quite dense (mass/volume), but only as
                                                                         hard as zinc
      zinc melts at 420°F , magnesium melts at 1202°F , silver melts at 1763°F , gold melts at 1946°F
Aluminum = 2.75 , aluminum melts at 1220°F
Copper = 2.85 av. , copper melts at 1985°F

For the Rockwell C hardness scale = HRC, a diamond tipped cone, and with an applied force of 150kg is used.

The Rockwell Hardness of a material is then expressed as: HR = (N - hd) , where d is the depth in millimeters. For the Rockwell A scale: h = 500 and N = 100, For the Rockwell B scale: h = 500 and N = 130. For the Rockwell C scale: h= 500 and N = 100. This HR equation has the form of:

   Rockwell Hardness value = HR = (Arbitrary Initial Standard Reference Value) - (multiplier)(indentation depth in mm). The less the depth of the indentation, the higher the Rockwell Hardness value. The recommended scale to use depends on the specific material being tested.

Some of the following are mostly new book discussions, and it includes some brief, additional and-or alternate discussions of previous and-or further book discussions and definitions, and so as to help the reader have a further understanding of a concept and-or formula.

Torque = (**Force Applied) x (Lever-Arm Distance)** : this is not exactly the same as Work = force x distance, and as described below, torque is essentially the resulting or effective amount of force output after a change or transformation (ie. here, an effective amplification, or growth) of a given applied or input force. With the Work equation, the force is not amplified as it is with the torque concept and equation.

Torque is a measure of the resulting and effective rotating, turning or "twisting" **force** (ie., a force that will cause a rotation about a center point), and if that force is constantly applied, it will cause something to accelerate about a fixed point. An example is a long wrench tool being pushed or pulled so as to loosen or tighten a screw or a nut on a bolt. Another example is a rod, beam or lever which has a pivot (turning or rotation point) or "fulcrum" set at some distance along it, of which the rod, beam or lever is able to rotate on or about, including up and down vertically. Levers are used as simple (mechanical) machines to have a "mechanical advantage (gain, increase, amplification)" so as a small force or weight can lift objects that have a heavier weight, although requiring more time and-or distance to do so. Torque may sometimes be mentioned as "moment" or "moment force". Often, once a bolt and-or nut is "broken free" or "loosened" from applying its ("clamping") pressure to mechanically join things together, the torque needed to completely remove it will be much less than that used to tighten it.

Lever-arm is a term associated with torque forces, and is the distance from the point a force is applied on the lever, beam or bar, to the point of pivot or rotation of that lever. The lever material transfers the input force to the center and applies the output force. The Lever-arm distance is essentially like a radius or distance value from the pivot point as if that point was the center point of a circle. Since force is (mass x acceleration), we see in the torque equation that the lever-arm distance is essentially a multiplier to the force applied, and technically, this is because the lever-arm essentially increases or multiplies the acceleration. Given a constant value of applied force, if you increase the lever-arm distance, then the output torque (twisting force) will increase by the same factor the lever-arm was increased. The equation shown above assumes that the applied force is perpendicular (at a right angle, or 90° to it) to the lever arm, otherwise, when the force is not applied in a perpendicular manner, the applied force is actually less, and the more general equation to use for such conditions is:

Torque = (Force) (Lever-arm) (Sin ¢)   : where the angle is the angle between the lever and the applied force. When the applied force is perpendicular (90°), Sin angle is at it's maximum possible value of Sin 90°= 1, and all the force is being applied to produce torque and the equation simplifies to the previous one shown: T = F L

[SEE FIG 216 below]

Increasing lever-arm increases or amplifies torque (force) because the acceleration of a point (where the force is applied) on the lever is greater when it is farther from the (center) point of rotation, and that the distance traveled about the point

1lb of weight is 1lb of force = about 4.448$\underline{2}$N of force =~ 4.45N of force =~ 0.45359kg of force =~ 454g of force
1 kg of mass will have a weight of 9.80665 newtons of force = 9.80665N of force ~= 9.81N of force = 2.2046 lbs
1 lb = 0.45359 kg = 453g

Here are the derivations of the above conversions:

F newtons = (mass)(acceleration) = (m)(a) = $\frac{(kg)\ (\ m\ )}{(s^2)}$ = $\frac{kg\text{-}m}{s^2}$: N = newtons = force units

1N = (m)(a) = (1kg) (1m / 1s^2) = 1 kg-m/s^2

When the acceleration (a) = (g) = 9.80665 m/s^2 , such as for weight force:

9.80665N = (m)(a) = (1kg)(9.80665 m/s^2) = 9.80665 kg-m/s^2   = 2.2046 lbs of force

1kg of mass will weigh 9.80665N when in the gravitational field ("force field" or "field of force") of Earth.

If the weight of a mass or object is 1lb = 16 dry-ounces, the corresponding mass in kilogram units is 0.45359 kg:

4.448$\underline{2}$N = (m)(a) = (0.45359 kg)(9.80665 m/s^2) = 4.448$\underline{2}$ kg-m/s^2 = 1lb of force

100lbs =~ (4.45N / lb)(100 lbs) = 445N   , 1000 lbs = 4445 N , 1 ton = 2000 lbs = 8890 N

1ft = 0.3048m ~= 0.3m.

1lb-ft =~ 1.355817Nm   :derived from: (1lb)(1ft) =~ (4.44822N)(0.3048m)=~ 1.355817Nm , therefore mathematically:
1Nm =~ 0.7375624 ft-lb

Though kilograms are often thought of as a weight value, it is actually an amount of mass, and the corresponding weight or force of an amount of kilograms of mass is shown above as:

F = weight = (m)(a) = (kg)(m/s^2).

The reason for thinking a kilogram is a weight value is that for example, a 1cc = 1 gram of water is said to have a "weight" of 1gram or gram-weight. Most scales will actually display the corresponding mass value and it is commonly said or spoken as being a weight value, and which is a loose way of saying it was a measured value from a device (scale) that initially senses the weight of the mass. Most weight scales are calibrated to show an equivalent amount of mass of the object. Therefore, if a scale indicates 1 gram, you will know that the mass of that object is equivalent to 1 gram of mass. As mentioned several times in this book, weight and mass are mathematically proportional to each other, if one variable increases or decreases by a factor (n), the other will also increase or decrease by that same factor. A mass value of 1 gram of any type of atom (ie., element), object, or substance will also have the same weight, and this is how **the total mass of an object can be determined, and that is by measuring its corresponding total weight.** If you know the volume of the object, then its density = (mass / volume), and then you can find what (if consistent) element it may be from an element density list.   (weight / volume) = (force / volume) =(ma / V) = (mg / V) could be defined as a "weight density" of an object.

Since letting mass density = d = (m / V) ,   and letting: weight density = (mg / V) , we can see that if we divide the weight density of an object by (g = 9.81 m/s^2) , we will then have its corresponding mass density = (m / V) and this is very useful if you do not know the volume of an object.

Energy can be defined as the ability to do something such as work. Therefore, a measure of energy is a measure of this ability to do work.   The units of energy are Joules (=J). A force (ie., influence) can be used to apply and transfers energy

1hp = 748w = $\dfrac{746\ J}{s}$ = $\dfrac{746\ N\text{-}m}{s}$

Work is a measure of applying a constant force (F) or energy to something (object) over a length (distance) of movement. The units of work are the same as the units of the measurement of energy, and that is Joules (J). Since work is measured with units of energy, work is also a measure of the total amount of energy used and-or transferred to something else:

Work (W) = (Force) x (Distance)   :Joules (J) of energy , mathematically:

Note that here, distance is the distance of which the constant force was applied and the time needed to do so. Also, note that mathematically. and expressed like an average ratio or rate:

Force = Work / Distance = energy applied per unit of distance.

Force = $\dfrac{work}{distance}$ = $\dfrac{energy}{distance}$ = $\dfrac{J}{m}$ = N   : mathematically, force is the rate of energy used per unit distance, or the rate of applying energy over a distance, and for this example it is how much energy was used or applied for each meter. For an object to accelerate, the force must be constantly applied, and therefore, the energy used and-or needed must be constant.

Work = (force)(distance) = (Newtons)(meters) = Nm Joules of energy. , Mathematically:
Work = (mass)(acceleration)(distance) Joules
Work = (mass)(acceleration)(speed)(time) = mavt Joules   : acceleration = a = change in v / change in t,

  a = change in v / change in t , and if the initial values were: t=0, and v=0 , a = v / t = (d/t) / t = d / t^2

Since d = (speed)(time) = vt, and the initial velocity was 0, then that speed or velocity was not always applied during that amount of time (t), and the value of v= (d / t) is not then actually correct to use for (v). What value of (v) should then be used? The average value of the two velocities can be used: v av = (V final - v Initial) / 2 , and this assumes there was a steady increase in the velocity during that amount of time due to the (constant) acceleration during that time:

Work = $\dfrac{(m)(a)(v)(t)}{(2)}$ =

Work = $\dfrac{(m)(v)(v)(t)}{(t)\ (2)}$ Joules = $\dfrac{mv^2}{2}$ Joules = Which is a measure of the total energy needed to do that work and-or the gain in kinetic energy (KE) of an object in motion due an applied force and-or work upon it. mass (m) has units of kg, and velocity (v) has units of m/s.

 An object just moving in space at a fixed velocity and without accelerating (a=0), then the factors of acceleration and time are not influencing factors of its ("stored", "potential") equivalent energy value, and therefore are not even considered as factors in the equation. The object has energy due to just its mass and velocity. Because the object has mass and velocity, the object has momentum which is set as being equal to this value. Momentum p = mv = (mass)(velocity), and it is defined as a measure of an objects ability to keep moving and-or its resistance to changes in velocity and-or direction. The momentum of an object is directly related to the kinetic energy (KE) of that object. In short, it takes more energy or force to influence (change [increase or decrease]) the speed, and-or direction of a more massive object.   Since v = d / t, momentum can also be describe as a measure of a mass through a distance during an amount of time: mv = (mass) (distance) / t

 Ex. A mass of 1 kg which has a velocity of 3m/s^2 has KE = $\dfrac{(1kg)(3m/s)^2}{2}$ = $\dfrac{(1kg)\ (9\ m^2/s^2)}{2}$ = 4.5 Joules

Ex. Since KE = mv^2 / 2, if the speed of the wind (which has mass due to the air gases) doubles, its instantaneous kinetic energy for an instant of time will increase by 4 if it is considered as a single mass or object:

KE = mv^2 / 2 = 0.5 mv^2   , and if the velocity of a thin layer or mass of air or wind doubles:

KE = m(2v)^2 / 2 = m4v^2 / 2 = 2mv^2   and this is 4 times more KE, but more needs to be considered since we need to consider the KE in a volume of air that will pass a point and-or be available to the wind-blades of the wind electric generator every second.

Kinetic Energy (KE) of the wind is being used to power a wind- turbine so as to make electricity or power some mechanical equipment such as a mill or grinder, and the result would be 8 times the power if the wind speed doubles.   An instant of air mass can be considered as a thin, two dimensional layer of air, such as the bottom layer of a cube, however over time such as 1 second, this thin layer of air becomes a volume of air which has three dimensions, and the total KE of the air during that time would be the sum of all these layers of KE during.

Consider: power = energy / time = KE / time = joules / second

The thin layer or area of the air can be considered as an instance, bit or "differential" of a larger volume, and noted as dV. To find that volume, we can take the anti-derivative of it; and here in this example: dV = 2mv^2 (which is the energy in each thin layer of air and-or instantaneous or bit of volume of that air per second, and when the speed had only doubled) and get the total or sum of all those thin areas or bits of volume so as to have the full volume (V) of air, and the full KE value of that volume of air during 1unit of time (here it is 1 second).

Consider: Volume = a^3, and that the derivative of it is dV / da = 3a^2 = an area (A) equal to a thin layer (ie., an area) of, and instantaneous bit of that volume. From this it can also be inferred that a higher dimension is equivalent to and-or composed of an infinite number of bits or layers of the next lesser dimension.
Volume = a^3 = (aaa) = (a)(a^2) = (a)(Area). As a side note, Volume is mathematically equivalent to Area times the square root of that Area.   Ex. If Area = (2)(2) = 4, Volume = (Area)(Area^0.5) = Area^1.5 = 4^1.5 = 8 =
= 2^3 = (4)(2) = 8.

Also note mathematically: Volume / a = Area,   and that as mentioned above or taking an infinitesimally small change, called a differential (d), in each variable, we have:

d Volume / da = dV / da = d Area = same ratio value of Area = Volume / a   , or:
From: Volume / a = Area   , we have: Volume = (Area)(a) , and after taking the differential of each side:
d Volume = d ((Area)(a))   and   mathematically:
dV = d ((A)(a)) = d A da

The sum of each bit of volume = Sum of (dV) = equals the whole volume (V).

For our equation above of 2mv^2 , and if we take the anti-derivative (ie. summing up all these derivative, bit, component or instantaneous layers of area, which equals dV for each bit or layer), we have:
2mv^(2 + 1) / 2 = mv^3,   and if speed doubles, KE will increase by the cube of that factor value, and here it is the cube of 2, and it is: 2^3 = 8 times more KE per second, or power than the KE at the half the speed or velocity.

What is the mass of air passing a point in 1 second. The density = (mass / volume) of air is noted at about:
Ma = 0.001225 g/cc. Multiplying both the numerator and denominator by 1000, we have:

Ma = (0.001225 g / 1 cc)(1000 . 1000) = 1.225 g/1000 cc = 1.225 g / L .
A cubic meter = 1m^3 = 100cc x 100cc x 100 cc = 1,000,000 cc   , after dividing by 1000cc / L:
A cubic meter = m^3 = 1000L .
A cubic meter of air has a mass of: (mass / 1 L)(1000 L / 1 m^3) = (1.225 g / L)(1000L / 1 m^3) = 1225g / m^3 =

[This space for book edits.] If you are able, please make a kind donation to the author of this book.

If all the kinetic energy (KE) of a mass such as M1 is transferred to a larger mass at rest such as M2, the velocity or movement of M2 will be less, but its amount of KE will be still the same. When M2 applies force to M3, it will transfer its KE to M3. Since M3 is a smaller mass, its KE will still be the same, but its velocity or movement will be more:

Momentum in = Momentum out : conservation of momentum (p, here using P), theoretically 0 losses,
Pm1 **=** Pm2 = Pm3 full transfer of momentum. Whereas (KE) is a quadratic type of
(M1)(V1) = (M2)(V2) = (M3)(V3) relationship to (v), momentum is a linear relationship to (v).

Work in = Work out
Work due to M1 on M2 = Work due to M2 on M3
(force)(distance) = (force)(distance)

When M1 strikes M2, M1 will apply a pulse of force to M2. M1 will loose energy and decelerate, and this is also due to the concept of equal and opposite forces. The inertia of the large mass of M2 will be like a resistance in the path of M1, and so the force applied to M2 is as if it was "reflected" (by compression and expansion of the atoms) back to and upon M1, especially when the velocity of M2 is very low, such as 0, and it will take some time for it to accelerate to a velocity. Since this "reflected force" is in the opposite direction and upon M1, M1 will decelerate and loose velocity and momentum (ie. energy). The momentum and energy that M1 transferred or "lost" will be gained by M2.

An "elastic" collision or impact is when much of the applied energy or momentum is "reflected" back to the object(s), and their KE will be, or nearly be, the same as before the collision as if none was lost and-or transferred to something else.

Forces cause motion, but forces also often causes a very high speed compression (force, pressure or "shock") wave of matter (ie., atoms) that transmit the energy from one atom to the next via force, much like how sound is a compressed wave or vibration of air that transmits the pulse of energy, atom by atom through the air.

When M1 collides or impacts with M2, the force and available amount of energy is quickly transmitted in what is described as a pulse of energy and-or an impact force. Here, quickly transmitted means a very short, almost imperceptible, amount of time to do so. The change in velocity with respect to time is high, therefore, the acceleration (here, in the form of a deceleration) is also very high during that short amount of time.

a = change in velocity = Vtraveling - 0 : at impact the velocity is 0, especially if the object
change in time a fraction of a second that got impacted did not move much

Since: F = ma , and if (a) is a very large value, the impulse (a quick impact) force will be a very high value, but only briefly in time, and over a very short distance. Because the force is high, and force is the application of energy, the energy "release" or transfer (joules/second) will be a very high rate. We see that a quick acceleration and-or deceleration can effectively increase or amplify the force created to a very high value for a short amount or pulse of time. This high force value can cause metal to bend in a high speed collision, and especially mores so when the objects are traveling toward each other and with each having their own kinetic energy (KE=mv^2 / 2) and momentum (p=mv) which will increase the resulting decelerations and (impulse, quick-impact) forces during the collision. As a thought concept, when the objects area headed toward each other at the same velocity, if the velocity of one object was then twice as much, its KE increases by 4.

If an object collides with another object that is already moving, such as in the same direction and slower so as there is actually a collision, then only some of its KE will be converted to force upon colliding with that object, therefore, it will keep the energy that was not transferred to the object. The energy transferred to the object will increase its KE (joules) and momentum (movement, p = mv).

Since: Work = (force)(distance) = W = (F)(d) = (ma)(d) , we can mathematically derive:

F = ma = $\frac{W}{d}$ = $\frac{Joules}{d}$: force expressed as a rate of using energy per unit of distance,
and:

Ex. A moving bicycle has movement or kinetic energy due to the mass of the bike and rider, and the velocity of the
   bicycle:

   KE = mv^2 / 2   with units of Joules if the mass is in kilograms, and the velocity is in meters per second.
            1J = (1kg) (1 meter/1 second)^2

   A typical adult bicycle rider may weight about 140lbs, and is moving at about 8 miles per hour. If the rider
   quits applying or inputting power to the bicycle pedal and wheels, the bike will start to slow down due to the
   friction with the riding surface and some friction in the bearings and gears of the wheels of the bicycle. The KE
   of the bicycle decreases rapidly due to that the value of v^2 decreases rapidly as the speed decreases. Eventually,
   the bicycle forward motion will stop with a velocity of 0.

   As the bicycle with its kinetic energy coasts without any more input energy, it is possible to calculate the average
   amount of KE (of this reality and-or curve equation) lost per meter and-or per second due to friction, as the KE
   goes from the maximum value to 0 when the rider no longer applies input energy.   The distance the bicycle travels
   will have to be measured. It is also possible to create a speed and time graph. The curve is not liner, but a
   parabolic type shape. By knowing this amount of lost energy per second, this is equivalent to the amount of input
   energy per second needed to keep the bicycle going at the same speed for 8 miles per hour.   Rough estimates of
   about 100 watts = 100 joules per second of input energy or power is needed to keep the bicycle going at that same or
   constant speed of about 8 miles per hour.

   140lbs = about 63.5kg

   8 miles/hour = 12.875 km/hour = 12875 meters per hour = 12875m / 3600s =~ 3.5763 meters per second
   For some extra reference: **10 miles /hour = about 4.47 meters per second. 1 miles/hour = 1.467 feet per second =
   about 0.447 meters per second**, hence divide the units of miles by (8/3.5763) or= (10/4.47) =~ 2.237 to find meters
   per second. To convert meters per second to miles per hour, multiply the number of meters by 2.237   For the above
   example:

   KE = mv^2/2 = (63.5kg)(3.5763m/1s)^2 /2 =~ 406 Joules of kinetic energy is contained in the combined mass of the
   rider and bike at the given velocity. 406J = 406 watt-seconds , and about say 50w to 100w or= 50 J/s to 100 J/s of
   this will be lost each second due to wheels and riding surface friction, and it must be re-input by the rider each second
   so as to maintain the same speed or velocity. If the KE of the bike system is 406J, it will require that much initial energy
   to be used or input so as to have that value, plus some extra input energy is needed so as to over come friction while
   reaching the desired velocity and-or KE.

   If the rider stops inputting energy to over come friction with the wheels, the bike may coast for perhaps 10 seconds and
   then have a KE of 0J. This is an average energy loss of: 406J / 10s = 40.6J / s = 40.6w. This is also an estimate of
   the amount of input energy from pedaling the bicycle that will be needed to keep it moving at 8 mph. This value of
   joules can be converted to calories. **1 cal = 4.187 J =~ 4.2 joules** , **1J = 0.239 cal**   , **40.6J /s = 40.6(.239cal) / s =~
   9.7 cal / s**

For the relationship between energy and momentum:

   KE = _mv^2_   and **p = momentum** = mv , we have v= p/m and by squaring both sides: v^2 = p^2/m^2 , we have:
         2

   KE = m v^2 / 2 = m (p^2 / m^2) / 2 = p^2 / 2m = (momentum)^2 / 2 (mass)   , solving for p = momentum:

   p^2 = 2 m KE   , taking the square root of both sides:

   p = $\sqrt{2\ m\ KE}$ = 1.414 $\sqrt{m\ KE}$

Calories = An indicated calorie number or level is basically the total energy available, and in these modern times, it is usually in reference to food(s) substances only, and for the specific food(s) in question since the total calorie value will vary for each different food substance or mixture and the amount of it being considered. This energy can be used by the body to do things, such as the fuel for moving muscles so as to move or do things which will require some energy (ie., fuel) to do. Any of the available calorie energy that is not used immediately by the body is either stored (perhaps as some fat, muscle or other tissue for a future use by the body) and-or eliminated from the body. 1 calorie of energy is formally defined (ie., its calculation and measurement) as the amount of energy needed to raise the temperature of 1 gram of water to 1°C higher than a given temperature. 1 calorie is equivalent to about: **1 cal = 4.187 J =~ 4.2 joules** of energy. Note, that to heat a gram of a certain substance or element other than water, it may take more or less than 1 calorie, and the specific value is called the thermal ability or thermal capacity of a substance. Also this equation assumes that all of the input energy to try to increase the thermal energy of the water or substance increased of the thermal energy of the water or substance by that input amount, and this is generally not the case due to the specific situation and its inefficiencies of the energy transfer. For example, with a flame heating a pot on a stove, some of that energy is wasted around the side of the pot and wont be used to heat any water in the pot. **1J = 0.239 cal**

The average human adult at rest, say sitting and-or lite activities, will use about 100 watts of energy = 100 Joules of energy per second for powering all their body processes (blood flow, digestion, breathing, etc.) of which much is eventually converted to heat and keeping the body's internal temperature at a comfortable 98.6°F = 37°C, and of which some of this heat energy is constantly being lost to the cooler air that is in contact with the skin, and the body needs to then keep reheat itself. 100 W = 100 Joules of energy / 1 s =~ 23.88 cal / s =~ 24 cal / s , and this would total to 2063232 cal / day =~ 2000 kcal = 2Mcal = 8640000 J / day . It is often, though technically incorrect in terms of units, commonly said that adults should consume a scientifically estimated, **"2000 cal / day diet"** so as to replace this (estimated, averaged) amount of energy used up and converted to mostly heat energy. For a person doing moderate exercising and-or doing general work, this 100 W of power value can increase to say 300 W of power, and these people will then require three times more food, energy or calories so as to replace this used up energy and to sustain and power their work ability. Many animals have a similar or slightly higher body temperature than that of humans.

A calorie in terms of food: A calorie value or level is not an indication of a food's nutritional (nutrients= vitamins [elemental compounds] and minerals [elements]) content or value. **A food may be high in calories, but low in nutrition.** Many diets include some physical exercise and limit the number of calories you should eat, and then you will therefore need to get your proper nutrition (vitamins and minerals, and with at least the minimal daily amounts recommended) in that food you do eat, and assuming your body actually did absorbed all the available nutrition in what you did eat. Some people eat well to get their nutrition, but have digestion problems, and so that nutrition available may not actually be getting absorbed into them, and they should therefore consider deliberate supplementation (ie., a vitamin pill(s)) after consulting with their doctor. For people having some difficultly loosing weight, it is still better to at least not gain any more weight, and which is a very positive start of controlling their weight. Having food is comforting and life maintaining, however it can actually become detrimental to our health if it contains inadequate, low nutrition, and-or if we are already overweight and not controlling our weight gain. Try to always stay aware and consider these matters every time you shop for and-or eat foods. Too much sugar, oils (fried foods, chips) dairy (cow milk, cheese, sour creme, etc.), salt, and bread (wheat) products have been a problem for many people. Vegetables, some fruit, and a general variety of foods are usually recommended. Some foods are also more enjoyable, depending on how they were prepared and cooked. For example, finely shredded raw carrots are easy to eat, are somewhat sweet and make a wonderful addition along with some other available shredded vegetables into ramen ("quick-noodles" [cook quickly]) or a "veggie sandwich" with a small amount of your favorite salad dressing.

Ex. A certain bag of potato chips that has a total of 255g indicates that 1 serving for a person is 28g = 1oz, and that it has 150 calories. How many grams and calories would 3 servings be? 3 servings would be (3 servings) (28g/serving) = 84g. 150 calories is to 28g as is x calories is to 84g. 150 cal/28g = x cal/84g. After solving for x, we have x= (3)(150 cal) = 450 cal . A 252 gram bag of these chips will have: 252g / (28g/1 serving) = 9 defined or standard servings. This bag will also have at total of: (252g)(150 cal / 28g) = 1350 cal.

When atoms are made, tremendous kinetic energy and force is needed to force together, and essentially fuse together, the protons, neutrons and electrons of an atom. Even each neutron and proton are now understood (after research beginning 1964) to be composed of three quark (with mass a mas of 1/3 of a proton or neutron) particles each, and with gluons holding the quarks together with what is called the "strong atomic force". This energy used to force particles so as to be together as an atom, is stored in that atom as strong atomic or nuclear forces and this could be thought of as being like a compressed spring with stored energy, and if the spring is cut, the stored energy is released. The more particles in a particular type (ie., element) of atom, the more total energy need to make it, and the more stored, potential energy in it.

Relativity = Relativity takes into effect of how the observer (apparently) sees, experiences, and-or understands (or assumes) things. For example, if you were in some type of vehicle, such as a car going 50 miles and hour, or even going 1 million miles an hour, you would generally not even notice how fast that vehicle is going. With respect, reference, relative, or in relationship to you, it would seem or appears as that you and the vehicle and all the objects in it are going 0 miles per hour or not moving in reference to you and each other. This all happens because you and the vehicle are both going the same speed and direction, and there isn't any difference in speed between you and the spaceship your in. To a distant stationary observer on a planet, they would indeed see the spaceship vehicle with you in it traveling quite fast with respect or relative to them and their own corresponding position (location), speed and direction. It is often said that speed or velocity is "relative", or "relative to the observer (and their own speed)". Due to this fact, is not to difficult to then image that distance and time are also relative to the observer. Now consider the possibility that gravity is also relative to the observer, surely, for two astronauts or objects falling at the same rate inside a spacecraft orbiting about the Earth, they will sense no downward force of gravity upon them, and it is as if g=0m/s^2 , but it is the relative or apparent value to them. It will still require energy to make a force and apply it (the energy) so as move an object in "weightless" space, but once it is applied, the object will keep moving unless another force acts upon it. Extra: While on this subject, due to the mass value of an object, it will still have an inertia (ie., effective resistance to the change in motion, and-or that an amount of time may be needed to transfer the applied or input energy) value, and its value is relative to the amount of mass. If an orbiting spacecraft were to reduce speed and-or stop in its orbit, it will begin to feel the effects of gravity and it will be pulled closer to the Earth by not having enough velocity to maintain the altitude of orbit, and the spacecraft will reduce in altitude, such as needed for landing the spacecraft.

Some conversions for energy, work and power

BTU = British Thermal Unit. If the temperature of 1 pound (the weight of 16 fl or volume ounce of water) of water is
increased by 1 degree Fahrenheit, it is gained 1 BTU of (thermal) energy.   1lb = 1 pound =~ 453.592374g =
= ~ 454g =! 0.454kg.   1°F = 0.555°C ,   1°C = 1.8°F . 1 BTU = 0.293 watts, therefore 1 watt = 3.413 BTU

Calorie = A (thermal, energy) calorie or cal is a unit of (thermal energy) measurement when the temperature of 1 gram (= 1cc) of water increases by 1 degree Celsius or Kelvin. 1 cal =~ 4.184 Joules of energy.

Foot-pound = A unit for the amount of energy to lift or move a weight of 1 pound upward by 1 foot of distance. In terms of
torque or twisting force, 1ft-lb is the (perpendicular applied) force of 1 pound on a lever arm that is 1 foot
long. If the force is still 1 pound, but the lever arm is now 2 feet long, then the torque or twisting force is
2 ft-lbs, and this is also equivalent to a weight of 2lbs on a lever arm of 1 ft. Torque = (F)(L).

1 Joule of energy = about 0.000948 BTU = 0.239 calories = 1 watt-seconds = 1Nm =
= 1 Newton-meters , (ie., the energy = work of 1 Newton of constantly applied force to an object through
1 meter of distance).   1BTU =~ 1054.85J =~ 1000J = 0.293w

Note that force = (mass)(acceleration)
1N = (1kg)(m/2^2) = the force that can accelerate 1kg of mass 1 meter per second/per second that it
is applied.

1 Joule in terms of electricity is the amount of energy needed or used to force or move a total of 1 ampere or 1 coulomb of charges / 1 second of current through a resistance of 1 ohm for a duration of 1 second. `V = J / C , therefore: J = VC =

Total Power used or needed = Total Energy
Total Time

Power = Energy = Joules / time = Work / time = (force)(distance) / time = (force)(velocity)   : d=vt , v=d/t , t=d/v
time

Force = Power = (mass)(acceleration)   ,   velocity = Power = Joules / s = Joules
velocity Force Force (m)(a)(s)

time = __Total Energy Used__ = Joules   : = Ts = time to deliver, acquire or use an amount of power
Energy Use or Rate Power

velocity = distance / time = meters / second

acceleration = change in velocity / time = (change in d/t) = (meters / second) / second = meters / second^2 = m/s^2

Force = (mass)(acceleration) = (mass)(meters)   ,   mass = __Force___ = Force = Force s^2
(s^2) acceleration m/s^2 distance

Force = (mass)(distance) = (mass)(_d_) = (mass)(v) : (Force)(s) = (m)(v) = momentum, or change in momentum
s^2 (s)(s) s

An object having (n) times more mass will take (n) times the force and-or energy to move it the same given distance, for example lifting a mass and-or weight a height. For a given amount of mass, if the force and-or energy applied is reduced by (n), the acceleration is reduced by (n), and the time needed to move it a given distance will increase by (n).
If the mass increases by (n), Energy applied and-or stored / unit mass is still the same ratio.

Work = Energy = (Force)(distance) = lb-ft = Nm = (mass)(acceleration)(distance) = (m)(a)(v)(t) , Joules
= (m)(d/t^2)(v)(t) = mvv = **mv^2**
(**n**)Work = (**n**) Energy = (**n**) lb-ft = (**n**) Nm = (force) distance = (**n**)(mass) (acceleration)(distance) , Joules
= (mass) (**n**)(acceleration) (distance) , Joules
= (mass) (acceleration) (**n**)(distance) , Joules

Ways that thermal or heat energy can be transferred from one object or place to another.

1. **Conduction** = Direct contact, such as from the atoms of a hot object in contact with the atoms of a cooler object. The cooler object will increase in temperature, while the hot object will therefore decrease in temperature. Friction or mechanical resistance is an example where some kinetic energy is converted into thermal energy, friction is usually a sign of wasted energy, an energy loss. Fluids and substance blended or rated at high speed in a container can actually increase in temperature due to friction.

2. **Convection** = To convey or transfer of heat energy by a medium such as a liquid or gas. This transfer can be assisted by pumps and fans. The warm air from an electric heater is an example. Hot. low density and weight air rises upward like a flowing current, and heavier weight and density cool air falls downward (ie. sinking) like a "draft" (flow) of current. This warm air can then transfer its heat to another object via conduction or direct contact.

3. **Radiation** = Transmission through space. An example is solar (sun) heat (thermal) radiation (energy transmission). Thermal radiation is actually a form of light which itself is a form of electromagnetic radiation.. Invisible radio waves are another example of electromagnetic radiation and is often called "RF" or Radio Frequency energy. The energy in this radiation can then heat up objects. A solar air heater can convert solar radiation into warm air, heat convection after the heat is first conducted to the air at the collectors

Ex. Water has a high heat capacity of about (4.2 J / C°) per gram of mass , hence (4.2J / g) / 1°C = (4.2J / g-°C). It could also be said that water has 4.2J of energy per gram for each degree rise in Celsius temperature. It will also take 4.2 joules of energy to raise the temperature of 1 gram of water by 1 degree Celsius, and if there was no energy losses doing so. If there are (4.2J/C) / (1 gram), and if we multiply the numerator and denominator of this fraction by 1000, we have there are (4200J/C) of energy per kilogram of water. Water has a density of 1 gram / cc. This value of 4.187J =~ 4.2J is also defined as the **calorie** energy unit.

Ex. Copper, which is much denser than water has a low heat capacity of about 0.385 J / C° per gram of mass. Copper has a density of nearly 9 grams/ cc, and this is 9 times more than the density of water, and the ratio of their heat capacities is that water is about 11 times more, and-or copper is 11 times less.

Compared to other substances, water is fairly difficult to heat or change in temperature due to its strong atomic bonds, and that its density is fairly low at 1g/cc, and as compared to the much higher densities of metals, and yet it therefore retains heat very well, and is a fluid which can transport thermal energy in a practical manner through a pipe, etc. To much thermal energy applied to water will cause it to change its physical state to a gas called steam or water vapor when it boils. The higher the temperature of water, the lower its density.

If the water being heated is (N) grams, it will take (N) times more energy to raise its temperature by 1°C :

Total energy needed for 1°C in temperature in N grams of water = Ng $\frac{(4.2J)}{g}$ = N(4.2) Joules

The above value is for a 1°C change in temperature, and if the change in temperature is Tc times higher, it will take Tc times more energy to do so. This is so since the energy needed is the same for each degree change, step or increase in temperature, and the total energy is the sum of energy needed per degree change. This sum can be expressed as a multiplication by the number of degrees change in temperature = Tc.

Total energy needed for a change in temperature of Tc of N grams of water = Tc Ng $\left(\frac{Joules / g}{1 C°} \right)$ Joules =

= Tc Ng (4.2) Joules for water

4.2 Joules is also called a **calorie** of thermal energy. 1 calorie will raise the temperature of 1 gram of water by 1 degree Celsius. 1 calorie = 4.2 joules = 3.96567 **BTU** (British Thermal Units) ~= 4 BTU. A BTU is the amount of energy needed to raise 1 pound (about 16 fl. oz = 454 cc) of water by 1 degree Fahrenheit. 1lb = 0.454kg = 454g. 1J =~ 0.0009478 BTU , and 1 BTU =~ 1055.075 J : 1 BTU can be roughly considered as equal to about 1000J. It will take about 4220J =~ 4 BTU of energy to raise the temperature of 1kg = 1000g = 1000cc of water by 1 degree Celsius. 4BTU will also raise the temperature of 1 pound of water by 4F° , or 4 pounds of water by 1°F.

This above example also assumes all of the energy used to heat the water was converted to thermal energy of that water, and with no wasted energy or energy losses. The equation presented was theoretical, and there will be some loss of energy. For an object to retain its thermal energy, temperature or heat value, it must be insulated from its surroundings by something that does not remove or conduct heat or thermal energy away from it. This is the case as for thermal energy storage, where that energy is stored until it is needed. A Thermos (tm) or other similar device is a double walled glass container that is insulated by a **vacuum** between each wall, and can keep and-or cook foods if a process known as "thermal (retention) cooking". Here, hot water and the item to bee cooked such as noodles is placed within the container, and all that is needed is time to cook and soften the noodles rather than more energy. Likewise, in theory, a well insulated house would not have to use any more energy to be heated to a comfortable temperature once it was warmed up. If there are no energy losses in a container, the substance within it can be heated by using just a small amount of power, but it may take considerable time depending on its mass and-or volume: From: Energy input = Power = Joules / time = P = J / t, we mathematically have: J = Pt, and for heating purposes, this must be thermal or heat energy.

Sodium	0.97	1.2	
Magnesium	1.74	1.05	
Air	0.0013	1.012	: =~ 1.0
PVC	1.38	1.67	: Polyvinyl-Chloride
oils	0.93	1.75	: averaged, for food oils, petroleum is 2.1
clay	1.5	0.91	: for the denser clays
Aluminum	2.7	0.9	
Salt	2.16	0.88	: sodium-chloride = "table salt" = "food salt"
Concrete	2.3	0.88	
Glass	2.89	0.82	
Granite	2.7	0.79	
Carbon	2.25	0.71	
Steel	8.05	0.48	: doesn't store too much thermal energy, but is easy to heat
Copper	8.96	0.39	
Silver	10.49	0.23	: very easy to heat, its atoms have a high thermal transfer in it
Gold	19.3	0.13	
Lead	11.35	0.128	: easy to heat and melt (at 621°F) to a liquid

1g mass of water = 1cc volume = 1mL volume or water that has gained 4.186 Joules of energy will have increased 1°C higher in temperature. Likewise, it could then be said that if 1cc = 1mL of water was now 1°C higher in temperature, then it has gained 4.186J of energy. As seen above, metals with high density, dense crystal-like atomic (ie., atom) structure, can be heated relatively easily due to their thermal transfer from atom to atom. The above could be also stated as: (4.186J / 1°C) / 1g = (4.186 ((J / g) / C°))

From this information, if 1L = 1000cc = 1000mL = 1000g of water was 1°C higher in temperature, that water has gained: (1000cc)(4.186 J / 1°C) = 4186 J of energy.

If 1L of water changed by 10°C higher, that water has gained: First, 1L = 1000cc = 1000mL = 1000g if water: (1000g)((4.186 J / g) / 1°C)(10°C) = (4186 J)(10) = 41860J of energy.

For water: $\dfrac{4.186J}{1°C}$ is to   ,   then $\dfrac{1J}{0.2389°C}$ is to   ,   : after dividing both the num. and den. by 4.186

If the temperature of an amount of water decreased by 1°C, then it has lost 4.186J / 1cc = 4.186J / 1g if water.
If 1L = 1000cc = 1000g if water decreased in temperature by 1°c then it has lost: (1000g)(4.186J/g) = 4168J of energy

A **thermal conductivity** rating of a material is how well (ie. how fast) heat or thermal energy can be transferred or flow through it over a time period. It is the rate of heat or energy transfer within itself. A thermal conductivity rating is essentially based on the amount of energy needed per unit time which is power or watts, and so as to change the temperature within the substance at a distance of 1 unit away and by 1 degree increase in temperature.

Some metals have a very high **thermal conductivity rating**. Silver, copper and aluminum have a high thermal conductivity rating, and they also conduct electricity very well and therefore have a low electrical resistance to the flow of electricity (electrons, current). Air and-or insulation foam have a low thermal conductivity and is used help keep things (objects, a volume) remaining at a certain temperature, and to also minimize wasted heat and-or energy. A vacuum has no thermal conductivity and is the best temperature or heat insulator. Here are some approximate thermal conductivity ratings (W / m k°) of some materials, and they can be considered as considered relative values. For example, silver has a rated value of 410, and it therefore transfers heat 410 times faster than that of glass which has a rating of 1. Although water can store much thermal energy, it is not a good thermal conductor of thermal energy within itself. Hot water can rather be transferred to where it is needed via pipes such as to deliver thermal energy. Hot steam (ie., water vapor substance, gas), rather than its temperature can do much work due to its high (gas) pressure and-or kinetic energy.

If given two different metals with the same mass, say 1g, and if they are placed in an oven with a certain air temperature inside it, those two metals will absorb thermal energy via (heated air) convection until they reach the same temperature of the air in that oven. If the metals have a different density and associated thermal capacity rating, then the amount of thermal energy that each will then have or store in its atoms, will be a different value. Metals with a higher thermal capacity can store more thermal energy per gram. Metals with a higher thermal conductivity will heat up or increase in temperature at a quicker or increased rate with respect to time than a metal with a lower thermal conductivity. Silver will heat up much faster than steel will, and this is because silver has a higher thermal conductivity rating and this is determined by its density, atom and crystal structure. Note also that silver has a lower thermal capacity than that of steel. In general, it could be considered that silver rather conducts thermal energy better than it stores it.

A SIMPLIFIED THERMAL ENERGY TRANSFER EXAMPLE

In this example we will be given two different masses or objects of the same metal, say iron or copper, hence they will have the same density. If it helps, you can consider these pieces of metal as being cube shaped. We are to find the resultant or final temperature ("the thermal equilibrium temperature") of the combined masses after the two metals are joined together at a side, and considering no external heat energy losses.

Helpful considerations for when the masses are joined together:

1. If given an evenly heated mass or object of the same element and-or density, it has
   a total amount of thermal energy associated with it. Any portion of that amount of mass
   has the same portion of the total thermal energy of the total amount of mass.   For
   example, 10% of a mass will contain 10% of the total thermal energy of the total mass.

2. Heat or thermal energy will transfer from the hotter mass to the cooler mass. There must
   be a temperature difference for heat to be transferred.   A temperature difference can be
   considered as the resulting "heat pressure" which will cause a transfer of thermal energy.
   Hotter substances or elements have atoms and-or molecules that have an increase in
   (thermal) energy, and they are then moving and-or vibrating faster and will have a higher
   kinetic (ie., movement) energy, and which can be transferred to other atoms and-or
   molecules during a collision, and thereby loosing some of its kinetic energy and then
   getting cooler as the object it collided with gains (thermal and-or kinetic) energy.
   Remember that when and object collides with another other, each is effectively colliding
   with the other and will affect each other in some way.

3. The hotter mass will get cooler, and the cooler mass will get hotter, and the amount of
   temperature change for each depends on the amount of mass for each. If the two
   masses have the same temperature before being joined, the larger amount of
   mass tends to then have a slightly lower temperature, and a smaller amount of mass
   tends to change more and quicker in temperature. Over time, the rate of the change
   in the temperatures of the masses with respect to time will decrease, and then those
   two objects will eventually be at the same temperature or "equilibrium temperature".
   The equilibrium temperature will have a value that is between the two temperatures,
   and will be closer in value to the temperature of the largest mass.

4. The more area of surface contact of the two masses, the faster that the heat can transfer
   or conduct from one to the other. An example is with the construction of a heat "sink"
   and-or radiator, such as to cool a motor or electronic part, or to heat a room quickly. A
   heat sink will absorb heat from a device, and that heat can be transferred, radiated to,
   and-or absorbed by a cooler air mass. This will cause the heated air to become less
   dense and it will then rise upward and allow more (cooler) air to then come into contact
   with the heat sink which may sometimes be called a (heat) radiator.

The total energy of this system is the energy of mass1 plus the energy of mass2:

   Et = Em1 + Em2
   Et = (m1)(t1) + (m2)(t2) = (1)(0) + (4)(100) = 0 + 400 = 400   : at the both start and end at equilibrium
   : (m)(t) = "mass-temperature product" and can
   use (v)(t) = "volume-temperature product"

The total mass of this system is:
   Mt = m1 + m2 = 1 + 4 = 5

The total amount of energy at equilibrium temperature (Te):

   Et = (Mt)(Te)   or can use: **Et = (Vt)(Te)** :if you have measured volumes (V), such as water fluid.
   400 = (5)(Te)   , solving for Te:
   Te = Et / Mt = 400 / 5 = 80   considering each mass having this new temperature, their energy is:

   Em1 = (m1)(t1) = (1)(80) = 80
   Em2 = (m2)(t2) = (4)(80) = 320

As a check, the ratio of these two masses is:

   m1 / m2 = 1 / 4 = 0.25

Mass2 had 4 times the mass as that of mass1, and at equilibrium, mass2 will have 4 times the energy as that of mass1. The ratio of their corresponding thermal energy is:

   E1 / E2 = 80 / 320 = 1 / 4 = 0.25

The author has tested the above formula using 1 container of cool water at 63°F, mixed with 4 containers of hot water at 114°F, and the measured temperature was about 103°F, and the calculated temperature, using the above method, was about 103.8°F The slight error was probably due to both a slight measurement error and some minimal heat losses in the large combining and mixing container, of which may have absorbed some thermal energy.

Solving for the temperature needed for a volume and-or mass so as to have a certain equilibrium temperature of two volumes (V) and-or masses :

Ex. If V1 = 4, T1 = 100°F, and V2 = 1 , what is T2 if the equilibrium temperature (Te) after mixing these two volumes of water is to be 120°F ?

   Et = Vt Te = (V1 + V2) 120°F = (4 + 1) 120° = (5) 120° = 600°   : total thermal energy

   Et = Vt Te = E1 + E2 = (V1)(T1) + (V2)(T2)   , solving for T2:

   T2 = $\dfrac{Et - (V1\ T1)}{V2}$

   T2 = $\dfrac{600° - (\ (4)(100°)\ )}{1}$ = $\dfrac{600° - 400°}{1}$ = 200°F   : this temperature is close to water boiling at 212°F

How is the specific heat rating of an element determined? In brief, it is done by performing an experiment, often by immersing the metal in question into a volume of water and noting all the changes, and then making a calculation. The mass and temperature of both the water and metal are also measured before this takes place. A loss in temperature and-

"blue-jeans" (ie., dungarees, durable cotton material) are a poor choice to wear in cool or cold weather. Loose fitting "sweat pants" (ie., a thicker soft cotton) are then a better choice to wear, and two layers of pants can be worn if the temperature is very cold and windy outside. A winter or cold weather coat is also needed in cool or cold weather conditions. A "hoodie" coat is acceptable, but only if the person has layers of shirts on, hence a oversized coat may be necessary for a person to feel comfortable. The author has found that an indoor, dwelling temperature as low as 60°F = 15.5°C is livable and comfortable if people dress properly, and this will also help conserve energy and money needed for it. Keeping active also helps keeps us warm in cooler weather. .

When the air temperature gets lesser than say 72°F or "room temperature", the heat from our body is more easily transferred to cooler (lower temperature) air since the difference is growing, and it is then more easier to transfer thermal energy to it. At cooler air temperatures, our skin in contact with the air will reduce in temperature or "cool down" or "cool off" (ie., reduce in value) as the heat from it is more easily transferred to the cooler air and the it slightly increases its temperature before it rises upward into the more cooler air since warmer air is slightly less dense. Cooler skin will then absorb heat from the body, and this process will eventually cool the body to a lower temperature if there is no thermal or heat energy created by the body to keep it at a constant temperature.

If the air temperature is greater than 72°F, then the higher it is, the more difficult it is for the body to transfer energy to the warmer air since the temperature difference is less, and this is why it feels very warm or hot at say, beginning at about 85°F. Today as of about the year 2010, an electric air-conditioner to reduce a rooms air temperature and-or humidity is no longer an expensive luxury, but nearly an inexpensive necessity since it is often warmer indoors than outdoors during the summer months, and that is no way to live - stay cool, comfortable and be healthy indoors. Being hot often raises people "tempers" and discomfort, and things become more challenging to complete. During hot temperature, people can find shade to stay cooler, drink cool liquids, and-or wear a wide hat and- sunscreen ("sunblock") lotion if necessary.

object. The twisting or turning input and output forces or torques will be analyzed using simple direct, linear (non-twisting or rotating) forces. How can a smaller input weight or force lift a larger output weight or force? This will be mathematically shown now.

First consider that using weights (a force) of the same value and distance on each side of the fulcrum point of the lever, that the lever would be in balance and not moving downward or upward (ie., rotating). The effective output torque would equal the input torque, or in simple words, the effective output force would equal the input force:
Tout = Tin   and-or   Fout = Fin   : we will also use: F2 = Fout   and   F1 = Fin

If the weights or forces are different values, then their distances along the leverarms can be adjusted so as the system is kept in balance and not moving or rotating due to an unbalance of forces and resulting in some net force being applied to and then moving (here, rotating) the lever.

From the concepts of work (W), a measurement of work equals a constant force applied through a distance so as to do that work. For this lever system and analysis, we will let the distance (D) moved equal the height (h) moved:

W = (force)(distance) = (F)(D) = (mass)(acceleration)(distance) = mad   : = energy , joules

Win = Wout   this is equivalent to: energy in = energy out   and   power in = power out less any losses

(F1)(h1) = (F2)(h2)   from this we have the reverse ratios of: $\frac{F2}{F1} = \frac{h1}{h2}$ mathematically:

$F2 = \frac{(F1)(h1)}{(h2)}$:clearly as h1 increases, the output force F2 increases. The input force (F1) amplification value or factor is equal to the ratio of: (h1/h2). To increase (h1), the length of the leverarm (L1) on that side of the fulcrum must be increased. The longer (h1) and (L1) is, the larger the output force (F2).

In the figure shown, the angles on each side of the fulcrum are equivalent vertical angles, and the lines (h) dropped to the lever create similar right-triangles. The lever arms are effectively a (hypotenuse) side of these right-triangles created. Corresponding sides of right-triangles are proportional (same portions or fractional value of their respective triangles) when one is essentially just a similar triangle or magnified construction of the other:

$\frac{h1}{L1} = \frac{h2}{L2}$ mathematically:

$\frac{L2}{L1} = \frac{h2}{h1}$ or $\frac{L1}{L2} = \frac{h1}{h2} = \frac{F2}{F1}$

We see that the height ratio is equal to the corresponding leverarm ratio which can then be mathematically substituted in the above equation for F2:

$F2 = F1\frac{(L1)}{(L2)}$: * The gain in input force (F1) is equal to the ratio of the input and output lever (L) arms. The greater L1 is and-or the lower L2 is, the greater the input force amplification factor. From this we have the familiar torque (T) equation(s):

(F2)(L2) = (F1)(L1)   , which can be expressed as a ratio of torque or force values:

$\frac{F2}{F1} = \frac{L1}{L2}$: these are also "reverse ratios" since corresponding values are not in the same numerator and denominator.
Consider also for example, the less the output lever arm (L2), the greater the output force (F2)

Here is a related note about actual levers and their lever arms:

More About Matter, Mass And Mol

Chemistry is the study and measurement of matter (ie., chemicals composed of atoms and molecules) and its interactions with other matter and-or energy such as heat and-or electricity. The word "chemistry" is rooted in a description of Egypt as "Khemia" which is based on the dark soil material called "khem" about the fertile Nile river delta. The word "chemical" is based in the word of "khem". khem basically means substance(s), and chemistry basically means the study of substance (ie. matter). From a modern point of view, the ancient Egyptians had much knowledge about chemistry or matter and which then inspired the world in that field of science.   **Matter** is a real physical material or substance. The word and meaning of "matter" is based on the words of "mother" and "source". The word "material" is based on the word "matter". **Mass** is a measure of the amount of any type of substance or matter. The word "mass" is a form of the word "material" and which is being combined or assimilated (based on the word "similar") together so as to have a quantity of it. A **mol** is a unit for a defined quantity of similar objects or particles of fundamental units of matter such as for example protons, atoms, or molecules.

We can divide an objects weight (a force) by (g) to find its mass.

From: force = (mass)(acceleration) = weight = (mass)(g)   :g is the amount of local gravitational acceleration caused by the constantly applied gravitational force upon a mass, therefore producing the corresponding force or weight of that mass. (g) at Earth's surface is ~= 9.81 m/s^2.

Force or weight has units of Newtons = (kg m/s^2)   =   the result of an acceleration applied to a mass

mass = ____force____ = _force_ = _weight in Newtons_   , with units of kilograms
               acceleration   g   9.81 m/s^2

Most household scales are calibrated so as to automatically do the above calculation for mass using its weight, and then to display the result in grams or kilograms. Most scales are technically "mass scales" and not "weight or force scales".

Atoms are commonly understood as the building blocks of matter (ie., "mass"), and are the smallest part of a unique material or substance called an element (of matter). An atoms fundamental structure is composed of particles of material (ie,. matter that has a measurement of mass) called protons and neutrons located at the center or nucleus of an atom, and electrons which orbit that nucleus of the atom. The number of protons in an atom is formally and usually equal to the number of neutrons. The number of electrons is equal to the number of protons. Each element (ex., oxygen, iron, gold) has a unique atom or atomic structure because it has a different number of particles than that of an atom of another element. The unique structure of the atoms of a particular element gives that element its unique physical properties such as mass and its corresponding weight, density, color, hardness, bending or twisting strength, melting temperature, electrical conductivity, light energy sensitivity, etc. [FIG 218A]

and oxygen atoms. Liquid water compose of two or more of water molecules. A **mixture** is two or more substances (elements and-or compounds) mixed together. Note that a mol (a specific numerical amount) and a molecule (a structure of two or more atoms) are two different concepts. [FIG 218B]

MOLECULE STRUCTURE EXAMPLES

WATER

H_2O_1

Hydrogen and Oxygen

SALT

NaCl

Sodium and Chlorine

CARBON DIOXIDE

C_1O_2

Carbon and Oxygen
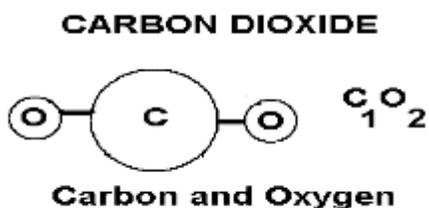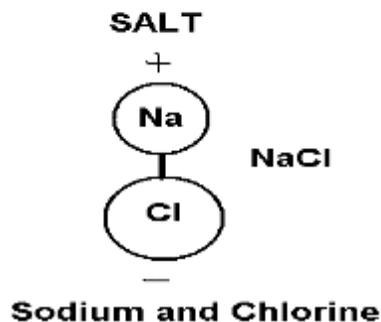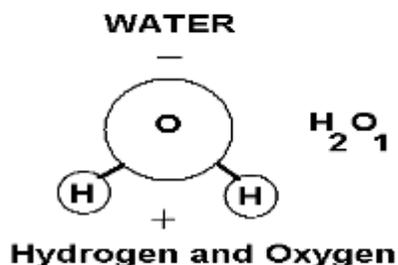
Also indicated in the above molecules is the "electric polarity" (if its substantial) of the molecule, and this is useful for some chemical and-or electo-chemical reactions. The main common factor of chemistry and electricity is the electron. For a simple example of how this can be is that in chemistry, an electron is shared by atoms when they combine together to form molecules, and in electricity, a moving electron is used to carry energy through a circuit.

In chemistry science, when substances and-or chemical processes and reactions are being made or considered, sometimes a specific (from an existing formula or recipe) amount or mol of the substance(s) are to be used. Having and using a specific number of mol of particles (usually atoms) ensures having enough particles of that material or substance to make another substance, and to reduce waste and prevent making a chemically wrong substance and-or mixture with too much or too less of something.   A chemist will easily weigh (rather than count the mass) each substance so as to have a particular number or mol of particles (atoms, electrons, protons, molecules) of it needed for the chemical process, etc.

Some fields that may include some chemistry knowledge are healthcare, pharmacy (medicines), agriculture (farming), nutrition, and research. In about the first half of the 1900's, the agricultural scientist, chemist, and teacher **George Washington Carver** (1864-1943), from America, made many significant discoveries in the field of agriculture, chemistry, and new products, with many being plant based, and so as to benefit both industry and mankind. Many of his new products were created from the peanut plant and of which is also said to be a "complete protein" in terms of nutrition. Carver created the concept of "plant rotation" by periodically planting a different crop(s) in the same location (ie., farm, garden), so as to help replenish (ie., fertilize, replace) surface soil nutrients, particularly the nitrogen compounds created by bacteria on plant roots so as to have "nitrogen fixing" and availability). Soil nutrients can be depleted and-or lost by plant "takeup" or absorption of nutrients, and mainly rain runoff (removal, erosion) from the top layer of soil. For a grown food to be nutritional, the soil it was grown in must first be nutritional. Carver, being an African-American, also promoted racial harmony and his scientific knowledge and expertise was desired by many others, and of which some new scientific discoveries are still based upon.

A **mol** (based on the concept and word of "molecule", and sometimes pronounced as "mole", and sometimes referenced as "molar") is a standardized (accepted) unit of measurement (reference, for comparison to) that is essentially just a

The atomic structure of a carbon-12 atom was found to contain 6 protons and 6 neutrons, hence it has 12 atomic mass units per atom = 12 amu / atom.   It will be shown in a calculation below that 1g of carbon-12 has 1 mol of amu, and-or that 12g of carbon-12 is 1 mol of atoms = 1 mol-atoms. 1g of any element will have 1 mol of amu, but not atoms, except for hydrogen which has only 1 proton in its nucleus..

Note, as mentioned above, each atom of carbon-12 also has 12 smaller particles of matter (atomic mass units = amu = protons and neutrons) with mass and with each having 1 atomic mass unit (amu). The atomic or molar mass (mass of a mol of it) of carbon-12 atoms is about 12 grams of weight per mol of atoms or= 12 grams per mol-atoms:

   12g carbon-12 = 1 mol of carbon-12 atoms   : or= 1 mol-atoms of C-12 ,   and since each C-12 atom has 12 amu:
   12g carbon-12 = 1 mol of carbon-12 atoms = (12 amu/atom)(1 mol atoms) = 12 mol-amu   : = 12 mol of atomic
                                                              mass units

    If 12g of carbon-12 is to 12 mol-amu,   then mathematically:

1g of weight (ie.,1g of mass being displayed on a common "corresponding weight to mass scale") of carbon-12 atoms is to 1 mol-amu = 1 gram of mass = 1 gram unit of mass or= 1 mass-gram

   As a reminder, most "weight scales" are actually calibrated to display the objects corresponding mass value associated to its measured corresponding force=weight=Newtons value. A common "weight scale" is more correctly a "mass scale" whose value was automatically calculated (in the scale) from the corresponding weight of that mass.

   If another element(s) or substance(s), other than carbon-12, has a weight of 1 gram, then it will have more or less than the number (1 mol) of atoms of carbon-12 in 1 gram of mass, but it will still have the same amount of mass and weight as that of 1 gram of carbon-12 = the mass and weight of 1 gram of atomic mass units (ie., protons)

If two substance, and regardless of their size or volume, have the same total mass, the weight of each substance is the same, and vice-versa.

1g of mass (matter, substance) placed on a common "weight scale" which is actually a "corresponding mass of that weight scale" will be displayed as 1 gram.

1 gram of mass = 1 gram of amu = mass of 1 mol of amu = mass of 1 mol-amu

   The actual weight or force of a gram or other mass value can be found from:
   F = weight = (m)(a) = (m)(g) newtons   : with units of kg m/s^2. Note, 1g = 0001kg , and   g = 9.8m/s^2

Ex. How many particles, objects, cells, atoms, protons, molecules, items or things is 5 mol?

   Since: 1 mol = (6.022)(10^23) things   if we multiply both sides by N, we have:
              N mol = N (6.022)(10^23) things   using substitution of the actual value in question, here 5:

             5 mol = 5 (6.022)(10^23) things
             5 mol = 30.11 (10^23) things = 3.011 (10^1)(10^23) things = 3.011 (10^24) things or items

**   Total Items = (N mol) (items per mol unit)   :Total Items or Total Quantity Of Items In N mol Units**

   If Given: 3.011 (10^24) things, how many mol units of things is this?

     Since: N mol = N ((6.022)(10^23)) things   using substitution of the actual value:
              N mol = N ((6.022)(10^23)) things = 3.011(10^24) things

         We can solve for N by dividing both sides by the number of items in each mol unit:

The reciprocal of this is: 0.083333 mol of carbon 12 atoms / gram of mass.
In short, **more massive, larger or "heavier" elements, have less atoms / gram, or less mol-atoms / gram , but also have more grams of mass / mol of atoms.**

The "atomic weight" listed on a table of elements, such as the common Periodic Table Of Elements, is actually the atomic mass units (amu) of each atom. 1 fundamental particle of matter has is 1 unit of mass = 1 atomic mass unit = 1amu. 1amu has a corresponding fundamental unit of weight (whose value depends on the locations gravity in the universe) which could be called an (fundamental) atomic weight unit. Therefore, in any location in a universe, the "atomic weight units" = "atomic mass units", or stated as: "atomic weight" = " atomic mass units".

The volume of 1cc = 1 cubic centimeter = (1/1000)Liter = 1mL = 1cm^3 of water at 4°, (ie., at the max. density of water, having the most mass of water per volume), is defined as having a mass of 1 gram. If an amount of material weighs the same as the weight of 1cc of water, and having any volume, it is said that it also contains the same amount of mass as that of 1 cc of water, and that amount of mass is 1 gram. In short, **if substances have the same weight, they have the same mass** since these two values always directly correspond to each other, and it is said that they are linear or proportional to each other. A 1cc volume of a denser substance and-or element will have more mass / volume and therefore have more weight / volume. For example, 1cc of iron has more mass than 1cc of water, and will weigh more by the same ratio or factor of two masses.

1 mol of atoms (ie., "mol-atoms" sized particle units) of carbon12 atoms will have (6.022)(10^23) atoms of carbon12. It actually takes 12 grams (mass) of carbon12 so as to have that many, 1 mol, of atoms. Since each gram of carbon12 element is composed of atoms with 12 amu particles each, the number of atoms in that gram is 12 times less than the number of its atomic particles (protons and neutrons) in its total mass such as 1 gram. This is why it was necessary to have 12 grams of the carbon12 element - so as to have 1 mol of complete carbon12 atoms. The **molar mass** (or mol-mass) of an element is the number of grams of it needed to have 1 mol of atom particles. The molar mass of carbon12 is 12 grams per mol = 12g / mol, or expressed as: 1 mol per 12 grams = 1mol / 12g.

1 atomic mass unit = 1amu, is the mass of a proton or neutron which are the two fundamental parts of the nucleus of an atom, and which has most of the mass of the atom. 1 mol of atomic mass particles is defined as having a mass of 1 gram. Since carbon-12 = C12 has 12 amu per atom, 1 amu can be defined as (1/12) the mass of a C12 atom. If 1 gram of C12 has 1 mol of amu, and to have 1 mol of entire or complete C12 atoms, it will take 12 times the amount of that mass. It will take 12 grams of C12 so as to have 1 mol of C12 atoms.

For C12, there is 12g / 1mole of atoms , dividing numerator and denominator by 12: we have:
1g / ((1/12)mol) atoms = 1g / ((6.022)(10^23) / 12) atoms = 1g / (0.502 x 10^23) atoms , and therefore, there is (5.02 x 10^22) C12 atoms in 1 gram of C12. We can multiply this value by so as 6 to find the total number of either protons, neutrons, or electrons in 1 gram of C12.

Ex. If 12g of C12 atoms is equal to 1 mol of C12 atoms, what is the mass of one-thousandth (0.001) of a mol of C12 atoms? [As a reminder, most modern scales are calibrated to display the equivalent mass value of a given weight].

Setting up a proportion type of equation:

(1mol-atoms of C12) / (12g of C12) = (0.001mol-atoms of C12) / (Xg of C12)   after solving for Xg of C12:
X = (0.001) 12g = 12g / 1000 = 0.012g = 12 (10^-3)g or= 12(0.001)g = 12 mg of C12 material

Ex. If 3g of C12 is added into a mixture, how many mol of C12 atoms did you add into that mixture?

First note that given 12g/mol, it can be also said there is: 1mol/12g. = 0.0833mol / 1g or 1g / 0.08333mol.
1g of C12 = 0.08333mol , and multiplying both sides by 3, we get: 3g = 0.25mol. This is also the result after multiplying both the numerator and denominator by 3 in the last previously expressed fraction above.

Or by setting up a proportion type of equation:

$\dfrac{12g}{1mol}$ = $\dfrac{1g}{0.083333333\ mol}$ or = $\dfrac{1g}{(6.022)(10^{\wedge}23)\ /\ 12)\ atoms}$ = $\dfrac{1g}{(\ 5.018)(10^{\wedge}22\ )\ atoms}$: for C12 only

Expressing a proportion equivalence or equation:

$\dfrac{Xgrams}{(1)(10^{\wedge}11)\ atoms}$ = $\dfrac{1g}{(\ 5.018)(10^{\wedge}22\ )\ atoms}$, after solving for X: Xgrams = 1g / (5.018)(10^11) and

X grams =~ (0.1993)(10^-11) grams =~ 0.2(10^-11) grams = 2 (10^-1)(10^-11)g = 2 (10^-12)g = 0.000,000,000,002g

= 2 picograms

From this we can also find that 1pg corresponds to 50 billion atoms of C12.

Ex. It was found previously that 1 gram of carbon12 element had about (5.02)(10^22) atoms. We can solve for the mass of just 1 carbon12 atom if we divide 1 gram of mass by (5.02)(10^22).

is to: $\dfrac{1\ gram\ of\ carbon12\ element}{(5.02)(10^{\wedge}22)\ atoms\ of\ carbon12}$ or: is to: $\dfrac{(5.02)(10^{\wedge}22)\ atoms\ of\ carbon12}{1\ gram\ of\ carbon12\ element}$

Dividing the numeric values in the numerator and denominator, we find:
1 atom of carbon12 element has a mass of about (1.993)(10^-23)g =~ (2)(10^-23)g ~= 2(10^-26)Kg

Since C12 has 12amu, the mass of one C12 atom can also be found by multiplying the mass of 1 amu by 12. **The mass of 1 amu or proton = 1.6726219 (10^-27) kilograms = 1.6726219 (10^-24)g** , and this will be derived here:

Since a carbon12 atom has 12 atomic mass units (6 protons and 6 neutrons), we can divide the mass of 1 carbon atom by by 12 so as to find the mass of 1atomic mass unit or particle:

2 (10^-23)g / 12 =~ 1.672621898 (10^-24)g :Mass of 1 atomic mass unit or particle such as a proton or neutron. (mass of 1 electron) < (proton mass / 1000). **MASS OF A PROTON. The diameter of a proton is about 0.833 femtometers = 0.833fm = 0.833 (10^-15) meters. The reciprocal of this value is how many protons could be set or lined up side by side in 1 meter. If you divide that value by 100, you can find the number of protons lined up in a centimeter. If you then cube (take the 3rd power) of that value, you can find the number of protons in a cubic centimeter.**

Since 1 gram unit is one-thousandth of a kilogram unit, 1g = 0.001kilogram, we can express this result in units of kilograms:

1.673 (10^-24)g =~ 1.673 (10^-24) (0.001)kg = 1.673 (10^-24) (1)(10^-3)kg = (1.673)(10^-27)kg

Ex. An estimated number of cells in the average human body is 30 trillion cells = (30)(10^12) cells. If each cell was to be given 1(more) atom of carbon12 element, the total mass of that amount of carbon12 element needed is:

(mass of 1 atom of carbon12 element) (30 trillion) = (2)(10^-23) (30)(10^12)g = 60 (10^-11)g =
6 (10^-10)g = six-hundred trillionths of a gram = 0.6 billionths of a gram = 0.000,000,000,6g =
= 600 pico-grams = 0.6nano-grams

Ex. How many mol of atoms is 30 trillion atoms? Expressing this as an equivalent proportions type of equation:

16/18 = 0.8888... about 89% of it being oxygen atom mass. This is so, even though the number of hydrogen atoms in a water molecule is greater at twice as much than the number of oxygen atoms, but the (atomic particle(s), amu) mass of a hydrogen atom is much less than that of an oxygen atom.

Note that the ratio of the atomic mass (ie., amu) of oxygen to the atomic mass of hydrogen is 16/1, and so if there are twice as many hydrogen atoms in a water molecule, the ratio of the atomic mass of oxygen to the atomic mass of hydrogen in water or a water molecule is: (atoms)(oxygen amu) / (atoms)(hydrogen amu) = (1 x 16) amu / (2 x 1) amu = 16/2 = 8. It could be said that the mass of water or a water molecule unit or particle is due to that there is 16 atomic mass units or particles of oxygen and 2 atomic mass units or particles hydrogen for each molecule of water. Therefore, any mass or amount of water has (16/2) = 8 times more of its mass (and-or weight) due to oxygen than its mass due to hydrogen.

Technically, a gram or any other specific amount of any matter such as atoms and-or molecules of an element or elements will have the same number of atomic mass units (amu) and will also weigh the same on a scale. A mol or mol count of different atoms and-or molecules will generally not have the same total mass (ie., grams) due to the different and total amounts of atomic mass units within those atoms or molecules. For example 1 atom of hydrogen has, contains or is composed of just 1 amu, and 1 atom of oxygen has, contains or is composed of 16 amu. 1 mol of hydrogen atoms will have (1 amu / atom)(1 mol atoms) = 1 mol amu. 1 mol of oxygen will have (16 amu / atom)(1 mol atoms) = 16 mol amu. Since 1 mol amu of hydrogen is defined as having a mass of 1g, 16 mols amu, here from oxygen, will have 16 times more mass, hence a mass of 16 grams.

A water molecule has a mass of 18 atomic mass particles or units. 16 are from oxygen, and 2 are from hydrogen.

Water molecule = H2 O1 = (2 hydrogen atoms)   + (1oxygen atom)
                                = (1amu/atom + 1 amu/atom) + (16 amu/atom)   combining the amu's:
                                = 18amu / water-molecule = mass of a water molecule   and:

$\underline{\text{ waters atomic mass due to atomic particles or mass units from oxygen }}$ = $\underline{\text{16 amu (1 atoms)}}$ = $\underline{\text{16 amu}}$ = 8
   waters atomic mass due to atomic particles or mass units from hydrogen   1 amu (2 atoms)   2 amu

Therefore, mathematically:

waters atomic mass due to oxygen = 8 (waters atomic mass due to hydrogen)   : 8 times more mass particles
waters atomic mass due to hydrogen = (waters atomic mass due to oxygen) / 8   : 8 times less mass particles

16 amu oxygen / 18 amu total = 0.8888... oxygen,   or   2 amu hydrogen / 18 amu total = 0.1111... hydrogen

Since 0.1111.. of waters mass is due to hydrogen , (1 - 0.1111...) = 0.8888... of the mass of water is due to the other substance(s) in that water molecule, and here it is just oxygen. It could be said that about 88% of the mass or weight of water is due to the mass or weight of the (condensed, liquefied) oxygen (gas) in it, and that about 11% of the mass or weight of water is due to the (condensed, liquefied) hydrogen (gas) in it.

0.8888.. oxygen / 0.1111 hydrogen... = 8   : a water molecules internal amu particle ratio, and **not** the
or= 16amu O / 2 amu H = 8   ratio of the number of atoms (here 3) or their ratios in a
                                                          water molecule. In short, the mass of an atom from different
                                                          elements is not the same value, and that an equivalent mass
                                                          of different elements will have a different number of atoms.
                                                          Both of these facts are due to the different number of atomic
                                                          mass units (amu) or particles (protons, neutrons) in atoms of
                                                          different elements.
Mathematically:

(0.1111...)(8) = 0.8888...   or~ =:   (11.1%)(8) = 88.8%

coulombs of charge was 5 times less, or the amount of current was five times more at 5A = 5Qc/1s , that time of 26.81 hours would be 5 times less at: 26.81h / 5 = 5.36h .   1 minute = 60s   and   1 minute as a fraction of an hour of is:
1min / 1h = 60s / (3600s/1h) = 0.01667h ,   or simply: 60s / 3600s = 0.01667 =~ 1.67%   and:
1 min / 0.01667h = x min / 0.36h   , after solving for xmin , we find xmin = 21.6min , hence 5.36h = 5h and 21.7 min

To split a mole of water molecules = 18g of water into hydrogen gas and oxygen gas, will require at least a total current of 2 mol of electrons since each water molecule requires 2 electrons to be removed or unbonded and pass through the external battery circuit. 1 mol is to 96500C as= 2 mol is to Xc: 1mol / 96500C = 2mol / Xc   , solving for Xc we find: Xc = (2)(96500C) = 193000C of charge. Note that if the water gets hot, some its volume be lost as steam, and in practice, more than 18g of water would actually be used - perhaps 500 to 1000 grams would be used, but only a maximum of 18g of that total volume of water would be converted to hydrogen and oxygen if only 2 mol of electrons flowed in this electrolysis circuit, and regardless of the time taken.

According to Einstein's famous equation: $E=mc^c$, the nucleus of atoms contains a very high amount of stored potential energy, and likewise it took at least that much energy to create that nucleus. The more protons a nucleus has, the greater the nuclear energy required to make that nucleus, and the greater the stored potential energy in that nucleus.   In terms of molecules of two or more atoms, they take a much smaller amount of energy to create since the atoms of the molecules share electrons that are relatively at a large distance from the nucleus, and are bound by the electric force rather than the strong nuclear force such as found at the nucleus and its particles. To split a mol (6.022×10^{23}) of water molecules into its hydrogen and oxygen parts will require about 237000 Joules = 237kJ, plus about 20% more lost, wasted or transferred as heat energy during the process. Water molecules can be split apart by using electrolysis, and where a voltage is applied to it. This will create an electric circuit and current through that water, and which will replace electrons when they combine with positively charged ions (atoms that are unbalanced or non-neutral in terms of electric charges).

A mole of water has a mass of about 18g. To start removing electrons from this molecule so as to split it apart will require a minimum voltage of about 1.23v potential difference across the water if it is warm, and slightly more voltage if the water is cool. Since the atoms of each water molecule share 2 electrons, also called its (molecular, atom) bonding electrons, it will take 2 times the total energy to free them all from their molecular bonds than it does just to free one electron from its molecular bond. When a single electron is influenced by the energy of a 1V (ie., 1J / 1C of electrons) potential difference, it is said to have gained 1 electron-volt of 1eV of energy. For water to split into hydrogen and oxygen atoms, it will require at least 1.23v potential difference because it will take about 1.23ev of energy to break an electron from its water molecular bond, and 2 electrons will then require twice as much or about 2.46ev of energy.

1V = (1J / 1C) = (1J / (6.24)(10^18) electrons)   , and after dividing the numerator by the denominator, we have:

1.602 (10^-19) Joules / 1 electron = **1ev**   , To split each water molecule of water will require:

2.46 ev / water molecule = (2.46 ev / water molecule) (1.602 (10^-19)J / 1 ev) = 3.941 (10^-19) J / water molecule

(3.941 (10^-19)J / water molecule) (1 mol of water molecules) =
(3.941 (10^-19)J / water molecule) (6.022 (10^23) water molecules) =
237327 J =~ 237 kJ to split 1 mole of water molecules = 18g of water molecules   : 1 tablespoon = 15mL = 15cc = 15g
                                                                                                      of water
From **V = J / C** , we have: **J = VC**   , and from : **1A = 1C / 1s**   , we have: **C = As** , and using substitition:
                    J = VAs = (Pw)(Ts)   , and if J=237000 , V = 1.23 , and A = 5 how long will it take, hence
                         solve for s:
                    s = J / VA   = 237000J / (1.23v)(5A) = 237000J / 6.15 watts = 38537s = 10.7h

The ratio of hydrogen atoms to oxygen atoms in a water molecule is: 2 hydrogen atoms / 1 oxygen atom = 2. The corresponding amu's of the numerator and denominator is: 2 amu / 16 amu = 8 . The amu due to oxygen is 8 times more than that of hydrogen. When a water molecule splits into its component hydrogen and oxygen gas atoms, there will be twice as much hydrogen atoms produced than oxygen atoms, but the mass and corresponding weight of those oxygen atoms is 8 times more than that of the hydrogen atoms. In water molecules: hydrogen mass / oxygen mass = 1 / 8 , or

More About Electricity And Electric Charges

Below are brief articles which contain selected key points of common knowledge, and consists of common brief facts, standards, some history and noted people, and so as to also not be excessive in detail and beyond the scope of this book and its practicality. The reader of this book may skip over these until another time needed and-or casually glance at the highlighted topics. They were included in this book so as the reader may have at least some practical familiarization and knowledge of these topics so as to be able to have a more robust, scientific understanding and use of how math can be applied in a practical manner, and to understand and-or possibly have some discussions with others. The science articles below are mostly about electricity, energy, flight, space, and measurement conversions. Each topic could fill a book by itself, and it is often not practical for all the knowledge, thoughts, examples and ideas to be presented in any one book, and you are therefore encouraged to seek more about any specific topic that you need to know more about. The way knowledge is presented, expressed or written, such as a derivation of a formula, also plays a role in how it is perceived as interesting, necessary, and-or easily remembered, such as being an important step for further knowledge.

Amber is a hard, petrified form of ancient tree sap, and was used to deliberately create some noticeable and interesting effects; usually showing the attraction and-or repulsion of small (electrically charged) objects after friction rubbing it with other objects. The words "electricity" and "electron" as we know of today is from the Roman word "electrum" that was used for the Greek word "amber". Today, we, now know that the friction (surface contact rubbing, or "drag" force) of the amber and-or other material(s) essentially applied enough energy to free some electrons, creating (highly mobile) electric charges, and charged objects (objects containing a net charge of either positive or negative).

William Gilbert (1544 - 1603), from England, made a study many materials, such as minerals, that can be electrically charged or affected by it, including a thin metal pointer which can turn direction in the presence of electricity rather than magnetism. In particular, he studied electrical attraction. He thought that the electric charge was a form of energy due to friction and the resulting heat increase of the object. He was the first to understand that the Earth had two magnetic poles, one called North, and the other called South due to their general directions on Earth. In the 1740's, **Benjamin Franklin**, from America, discovered that there are two distinct types of electric charges or "fluids": positive and negative. Current flow was then later conceived to be from the positive to the negative side of a battery, and this is called conventional current flow, and this actually helps theoretically and mathematically such as that a voltage or energy potential greater should have a mathematical positive sign. It was unknown during Franklin's time that electron particles usually carry energy and (are the) electric charge, and they actually flow from the negative side of a battery to the positive side. In most circuit analysis situations, this strict technical or scientific truth does not need to be considered.

Once a practical way was found to make and-or store electricity or charges, such as with Volta's battery in about the year 1700AD which made the study and use of electricity very practical, the revolution in electronic discoveries, understanding and applications grew exponentially where newer technology grew rapidly by using existing discoveries and technology so as to make further discoveries.

At about 500 B.C., ancient Greeks called the smallest bit of matter of a particular substance an atomos (the "uncut" part left after repeatedly cutting something into smaller and smaller pieces, and yet "unaltered" and still has the same physical properties). In the 1600's, the similar word of: **atom** was used for this smallest individual bit of a substance. The word "atomic" is in reference to atoms. Not long after Volta invented the battery in about 1800, in about 1803, **John Dalton** (1766-1844)**,** a chemist from England, called this smallest bit of a particular element as an "atom" of that element, and that each one is identical in a particular element. Some elements had atoms which behaved differently (such as observed in different gases) and must be somehow constructed differently and have a different mass because the same volumes of different elements weighed differently. Dalton was the first to consider that **compounds** (several elements that compose or create a unique substance) are composed of two or more elements bonded together so as to create the smallest fundamental particle of a compound which is called a **molecule** (since the word "mole" is an old word for mass, "molecule" means "mass part", "a part of a mass", "a small particle of mass") of that compound substance. For example, a water molecule is the smallest piece or particle of the substance called water, and each molecule of water is composed of hydrogen and oxygen atoms bonded or joined together - technically via electric forces and the "sharing of electrons" in a molecule. A molecule is significant particle in the field of chemistry science.   Dalton started his atomic quest after theorizing that gases must be made of tiny moving particles that have mass and kinetic energy so as to put pressure upon

Crookes tubes were also used during the discovery of X-ray [essentially, high frequency and energy, rf (radio frequency), penetrating particles] production as discovered in 1895 by the German professor **Wilhelm Rontgen** (1845 - 1923) after noticing that some of the mysterious, yet unknown (= X) rays (called Rontgen rays) from a Crookes tube were also passing through a thin layer of paper and-or metal near the outside wall of the tube. Today these rays are usually called **X-rays** and are often used in hospitals and dentistry so as to help have a reasonable internal (photographic, light and X-ray sensitive) image of internal things such as bone and metals which are not normally seen. X-ray radiation is created when a high velocity electron, such as in a Crookes tube, collides with certain materials. To create this high velocity, high kinetic energy electron(s), a high voltage is used. At about the same time, Nicola Tesla and others were also investigating X-rays. Rontgen noticed that these rays would easily pass through some materials, and also affect a photographic or light sensitive plate material as if it was a type of visible light . In 1895 Rontgen took the first medical X-Ray of his wife's hand, and too this day (year, 2022), X-rays's and-or X-ray imaging are widely used in the practice (ie., use of) and research of medical and scientific fields.

In 1953, a special type of X-ray image to study crystals helped discover the helical (twisting) of two main parallel rails and inner connecting rungs (ladder, step-like) of the DNA structure that is often described as the "code" or "blueprint" or "instructions" of all life. The field associated with the structure of crystals is called crystallography. The structure is similar to a twisted ladder in its lengthwise or long direction. **DNA (DeoxyriboNucleic Acid)**, a special and very long type of molecule tightly packed into every cell, particularly its nucleus or governing part, of the body and is responsible for all the chemical (chemistry) structures and processes in the body, and heredity from both of a persons biological parents. In 1882, structures called chromosomes were discovered by **Walther Flemming** from Germany, and later found by others to be responsible for "traits" or inheritance, and the next question was how is this actually done.   The word chromosome means "colorful bodies", as seen through powerful microscopes.   When a baby receives the chromosome structures from each parent, these mostly contain a copy of the DNA from each parent. The baby will then actually have a unique or combined DNA that is composed of both of its parents DNA. Each new cell of the growing baby will then need a replica (ie., a copy) of this unique DNA of the baby, and this is accomplished by a process called cell division. The DNA structure or code is composed of using 4 possible (chemical) molecules joined together as one rung or step of the long (many steps) DNA sequence, "chain" or ladder". Because of the preceding facts, certain sequences or lengths of DNA are said to contain the specific genetic data of a particular part or process of an organism or person, and these unique sequences or segments of DNA are therefore called genes.   The main researchers to find the previously proposed fundamental structure of life, now called DNA, were **James Watson**, **Francis Crick** and **Maurice Wilkins**, while in London, England. **Rosalind Franklin,** a chemist, is credited to actually taking the difficult to obtain, x-ray photograph(s) of DNA which showed its (atomic) molecular and (double) helical (twisting or spiral) structure, which is much like a twisted ladder.

In 1888, **Philipp Lenard** in Germany, previously noticed some of the effects seen by Rontgen.   Later, Hertz and Lenard are credited to discovering the photo-electric effect where surfaces, such as a cathode, would emit ("photo-induced") electrons or ions when struck by electro-magnetic radiation such as light and radio waves. This effect can also be seen when shining UV light onto the plates of a charged electroscope, and then the plates get closer together as the charge and-or electrostatic force is depleted. He also notice that the speed or (kinetic) energy of the electrons emitted depended on the wavelength (ie. frequency) of the incident light striking the surface, and not necessarily the intensity of the energy of the light. **Albert Einstein** would later suggest that this is due to that light or photons are composed of individual, discrete "packets" (ie., pulses) of energy, and this lead to the concept called quantum (energy) theory which is basically the study of individual, discrete, packets or steps (levels) associated to energy and matter such as electrons. Photons (ie., light particles) are essentially a form of a mass-less, electromagnetic radio wave   energy and vice-versa, that is, radio waves are a form light, but having a frequency so high or so low that we cannot see it, much like how we cannot hear high frequency audio waves called "ultrasonic" audio frequencies.

Max Planck (1858-1847), from Germany, is credited (for his scientific contributions from 1900 to 1918) to creating the concept of quantum (energy) theory which is proven by the fact that the energy of a photon (a light "particle") is directly related to the frequency of the light and which also determines the color of the light seen. When an electron gains kinetic energy, it will eventually release that gained energy as a photon (ie., photo, light, radio-frequency (rf), radiation, energy particle). Considering the rainbow or spectrum of colors, a particle or photon of red light has less energy than a photon of violet light. The color of a heated object such as a piece of metal can therefore be used to determine its temperature. When the mass conducts or radiates this energy away, it will reduce in temperature. Since wavelength is equal to the

highest power optical microscopes available, and the resolution is therefore much smaller so as to see (ie. "resolve") smaller objects and details. The electron microscope discovery and creation is rooted in a (1926) theory by the German physicist **Hans Busch** that a magnetic field(s) could be used as an electromagnetic focusing lens for a beam of electrons. Some of the already known concepts of the oscilloscope surely helped this effort. The first basic electron microscope was then created in 1931 by physicist **Ernst Ruska** and **Max Knoll** who was an electrical engineer who helped build it. Electron microscopes are more complicated than optical microscopes, and therefore, they are more expensive, and they usually require special preparation of the object being viewed, but it is still worth it, and even outdated and-or replaced electron microscopes are sill sought after.

Attempts at creating **television** (tele is a word prefix for "remote" or "at a distance", ex. telephone, TV = television = "remote vision", a "transmitted vision or image", electronically sending or transmitting images so as to be viewed at another location) has a long, varied and incredible history of electro-mechanical devices and people such as **Paul Nipkow** (1860-1940), from Germany, who created the Nipkow disk in 1884 that had a spiral array of holes to "scan" (horizontally, and vertically) or bathe an image with light, and which its reflection was then sensed with a (electronic) photo (ie., light) sensor.

The first electronic imaging (ie., camera) tube for television was called an iconoscope (ie., an image viewer, electrical device, an electronic camera) and was created by **Vladimir Zworykin** (1888-1982), from Russia, while working with RCA (Radio Corporation of America) in America. Some of its initial concepts were conceived in about 1923, but it took until about 1931 for it to be demonstrated as fast and practical. In 1929, **Philo Farnsworth** (1906-1971), from the United States of America, is credited to the first completely electronic, demonstrated, fast, non-mechanical television system. He created both an electronic camera or image sensor called an "image dissector (tube)" in 1927, and also the image viewing screen, for a complete and functional TV system. It could be considered that Crookes (ray, electron, "cathode ray", CRT) tubes is where TV (television) and many other modern scientific discoveries such as the electron atomic particle, were initially rooted in, discovered with, and-or based upon. When a thin beam of electrons strike the phosphor coating on the viewing screen, a visible dot will glow due to the energy it absorbed and is re-radiating it as light. The intensity of the electron beam will determine the intensity of the visible light seen, and this is the way that a black and white or "shades of grey (greyscale)" TV works

It could be said that both Farnsworth and Zworykin independently share the majority of credits to the initial invention of the (practical) electronic television system. The people and companies involved in the development of the first practical electronic TV system occasionally filed suit against each other. The iconoscope was eventually favored as being more practical than the dissector tube. The dissector tube was (independently) proposed as early as 1925 in Germany. Farnsworth also invented the "**electron multiplier**" during his TV research. With this vacuum tube device, each incoming electron can strike a charged plate that had a high voltage potential applied to it, and therefore having a high electrostatic force upon an incoming electron(s) and causing them to accelerate with more kinetic energy, and this will cause a few other electrons to be emitted from that plate when it collides with it. These plates are usually placed at angles and in a series arrangement so as each electron, perhaps nearly undetectable or impractical to, can be easily sensed when there is many, or a multiple, of them. Some devices which use an electron multiplier are some spectrometers which can determine what element(s) a substance is by the light and wavelengths emitted form it, and the electron multiplier has found use in some night-vision, low light sensitive, cameras.

A **movie** (ie., moving or motion pictures) is a recorded series of consecutive images or pictures of a subject in motion. A good quality movie recording process will recording enough needed images, pictures, "stills" or "frames" per second so as to have a seemingly continuous viewed subject having natural motion, and without any apparent "missing image gaps" or distortion. A movie recording when viewed or "played back" should also appear as consecutive images having a nearly or seemingly smooth, unbroken, natural looking motion or sequence of the subject that was recorded or "filmed" (ie., onto photographic film reels [containing many consecutive individual, small photos or "stills"], but modernly as digital or electronically recorded, stored and viewed images). The movie concept most likely got inspired from what is known as a "flip book" series of drawings or photographs, and it was only a matter of time until the technology allowed movies to be made and played back for viewing in a practical and pleasing manner, and for wide distribution for many uses.   Before TV or video recorders, many of the first TV transmissions were of movies shown on a screen for the TV camera to record and broadcast.

> 35mm : Before electronic or digital photography, 35mm film, still image cameras were the most popular type of camera. "Studio, professional quality", "portrait film" and-or older "plate or large format" photographs used negatives that were large, perhaps 4" =~ 10cm for each dimension. Compare this to a modern (2022) "smart phone's" digital camera with an image sensor that is perhaps 2mm x 2mm and produces good quality, inexpensive images.

TV images can be stored on magnetic ("reel") tape recorder images (called "video") were developed in the early 1950's by the Ampex Co,, and this made it possible to store (record, save) electronic images from an electronic ("TV") camera, and later broadcast (transmit to a wide area and-or population) and-or rebroadcast the recorded images as a common TV radio-wave or electro-magnetic signal. TV signals can be transmitted through the air like a radio-signal, and-or through direct wire connections such as for "cable TV". This direct connection usually results in a received TV signal with much less electronic and atmospheric noise in it, therefore producing a more pleasant viewing image and-or experience for the audience.   Today many of us take these technologies for granted as part of everyday life and not even having a basic knowledge of how they came to be and-or how they work.

The first light to be deliberately or artificially produced by man, besides light from a fire, was produced from the friction of two fast moving dry sticks (wood from trees, having a high carbon element and low moisture content) producing red glowing embers and-or (yellow colored) fire (hot gas particles), and sometimes as visible bright sparks created by two rocks colliding. Later, candles were created that used oils and waxes as the fuel, and the produced a soft yellow-white light. Maybe a few thousand of years later, man began tinkering with the mysteries of electricity - first with various methods of friction induced static electricity and charges, and sometimes producing both visible "sparks" (quick pulses of light) and-or felt as (electric) shocks. With the invention of the electric battery which can accumulate (slowly increased its stored amount) electric charge, particularly electrons, it was noticed that a bright spark of light could be produced that was very similar to that of lightning from the sky, and that lightning must also be due to the same cause, that is, lighting must be electric in nature, and contains electric charges. Lightning can be considered as a big, long and powerful electric spark. People noticed that with a higher battery voltage (ie., electromagnetic force or energy, emf) applied, a longer spark could be created across a greater distance, and surely lightning between the ground and clouds must be due to a very high voltage. But what was producing the actual spark of light? If the air was taken away, would a spark still be seen? It was found out later that no spark would be produced or seen when air is not present such as in a glass vacuum tube, but still, the electricity or current would flow through a vacuum if the voltage or electric field strength was high enough for it to happen. With air or some other gas, and a moderate voltage applied to it, a glowing (faint light) gas could be created such as seen in the Geissler Tubes which are discussed next. Surely, electricity was doing something to the air or other gas so as to cause it to give off or radiate some light.   It was discovered later that when an electron gains kinetic (ie., motion, speed) energy, such as due to a nearby electromotive (electricity causing motion) or electric force (ie,, in a nearby region or "field" of influence) caused by other charges (charged particles with their electric force), it can then release that gained energy as electromagnetic energy (such as light) when it collides with another particle and-or settles (ie., "grounds") into an orbit about an atom and electrically neutralizes that atoms net electric charge from being positive. The electron releases its gained or excess kinetic energy as light or radio frequency (rf) energy, but it could also include heat or thermal energy. For an analogy, consider how when a weight is dropped and it eventually looses its kinetic energy when it lands or "grounds" itself, but its gained energy (potential energy, or potential ability energy, from working and lifting it to a certain height and storing that energy in its location, and this is somewhat similar to how an expanded spring stores potential energy due to its configuration]) is transferred elsewhere such as radiated sound energy, and friction and heat, etc. during the impact.

Crookes tubes are based upon the glowing novelty (curiosity, fascination and amusement), neon-like gas tubes previously invented in 1857 by **Heinrich Geissler** (1814-1879) from Germany.   These tubes (see [FIG 219] for some very basic drawings of some electronic devices), and its operating principle are sometimes called "gas discharge" tubes or lamps, which basically means an electrically charged gas lamp that emits light. Geissler tubes could also be used to detect a high voltage nearby and-or connected to one or two of its protruding leads (leading into and-or out of the device), electrodes or terminal wires. At both ends of the sealed glass tube, a high voltage was externally applied (ie., connected) to the internal metal electrodes placed within a low pressure gas that could be ionized (produce free electrons that broke away from their atomic (force) bond with the atom) by the applied voltage electric field and create many movable charges so as to help the conduction of current and-or through it. The excited (energized, higher energy) electrons created would

nearby location) air pressure can also drop (ie., decline, reduce, decrease) in value due to an increase in the air or wind velocity. Today, we know that mercury is considered as a toxic substance and should be avoided, and therefore, it has become difficult to purchase without good cause and legal permission.

Not too long after the invention of the barometer to measure air pressure was invented, **Jean-Felix Picard**, (1620-1682), from France, noticed in 1675 that there would sometimes be a visible "glow (like a fog") of light in the (near) vacuum region in the top of the glass tube of the barometer when the mercury in it was agitated (ie., shook, stirred), and then by touching the glass with some static electricity which has a relatively high voltage value of a few hundred volts. This was the first publicly known instance of what would later be called a **gas discharge tube, bulb or lamp**. Electric charges were exciting the trace of mercury vapor ions in the top of that tube, and electrons settling into an orbit around atoms would release their excess, gained energy as some visible light This simple and accidental discovery, curiosity and novelty would eventually inspire and lead all the way to the inventions of Geissler Tubes, Crookes Tubes, and later, vacuum tube amplifiers several hundred years later and which inspired modern electronics, This discovery was therefore a very significant and important discovery, much like how Edison discovered the (electronic) diode action in a light bulb experiment and didn't have much use of it at that time, but it did lead to the vacuum tube amplifier years later.   Picard also created many more advanced devices so as to obtain more precise (ie., precision, more accurate, "finer", greater resolution) measurements for astronomy and various geometries (or "geodesy") of the Earth, land features, and making making (cartography), and   these were also the forerunners of the more modern devices which have slightly better accuracy. He worked with other scientists and astronomers who made several important geological and astronomical measurements. Picard is credited to a calculating a newer and more precise calculation of Earth's radius.

Otto Von Guericke (1602-1686) from Germany invent the first practical **vacuum pump** in 1650, and this allowed experiments and devices that needed a substance-less volume or space, which is called a vacuum (vacated, evacuated, empty) region, and so as to produce a high quality experiment results, and to be practical. This pump consisted of a piston (much like a rod and-or "plunger") inside a cylinder (here, a rigid or solid tube shape) and having one-way directional openings or "valves" controlling the flow (input and-or output) of air so that it is removed from the vacuum region being created when the plunger is pulled outward, and then discarded into the air when the plunger is pushed inward. This pump could make high quality vacuums in rigid (ie., strong, non-flexible) containers. He then discovered that when the air (a gas) is removed from a container, that its internal pressure is also removed, and then the outside air pressure will still exert a great force (14.7psi) everywhere upon the surface of that container, and directing it to the much lower pressure region inside, and it therefore tries to compress (ie., "squash") it and may collapse it. This fact also makes it more difficult to separate the parts of some containers of which a vacuum is within. Otto also made an electric charge generator out of an iron rod which held a rotating sulfur sphere of which friction was applied, and so as to give that sphere some charge which could then attract or repel nearby small objects like amber would.

In 1705, **Francis Hauksbee** (1660-1713) from England, and who was Isaac Newton's assistant, would use a vacuum pump to do some electric experiments. He made a powered rotating glass sphere that had a vacuum within it, and with a trace (very small amount) of mercury in it, and would see a (mercury gas vapor or atoms) glow when it was agitated and particularly when the surface of the rotating sphere was touched, creating some static electricity, or touched by another (statically) charged object, and this is very similar to that of the historical, barometer glow effect discovered by **Jean-Felix Picard**, and was mentioned previously in this book. These discoveries and technologies would later lead to **Geissler tubes** in 1857 that also created a gas discharge or glow when a high voltage and low current is passed through a gas in a vacuum, rather than using (high voltage) static electricity to excite or energize the gas. Hauksbee's glowing sphere would later lead to the invention of electric **fluorescent tube lamps** ("lights"), where much of the invisible radiated (light) energy from the glowing mercury gas is converted to a visible (or "fluoresced") white light by using a thin, phosphorous (a natural element), paint-like coating on the inside of the glass tube.

In 1729, **Steven Gray** (1866-1733), from England, found and demonstrated that static electricity (ie., static = non-moving, stored charge) can essentially move or "flow" from point to point along a (static) electricity (ie., electric) conductor such as various types of (non-metallic) materials including cloth and rope, and to then attract or repel other lite-weight objects near the end of a length it. What he also discovered is the first form of distant or remote communication with the aid of electricity, and before the much longer distance, (non-static, but current flow operation) single-wire, electric telegraph communication system developed over 100 years later in 1837 by **Samuel Morse**. This was just a few years after the

battery. Callan was influenced by Michael Faraday's discoveries. These coils produced a high voltage and spark across a short distance called a "spark-gap", and would later be used by others to study and transmit radio waves, and even be used to produce the high voltages necessary to produce X-rays for hospitals and dentists. Callan's interest in electricity was inspired by **William Sturgeon** (1783-1850), from England, who invented the first electromagnet in 1824 which was made from an insulated coil of wire wrapped around a piece of iron, and then connecting the ends of the coil to a battery so as to have electric current flow through the windings ("turns" or "loops") of the coil so as to produce an electromagnetic field about it. The more turns of wire or loops in the coil, the greater the accumulated magnetic field from the smaller magnetic field of each length segment and turn of wire in that coil. This essentially concentrates the magnetic field of the entire length of a long wire into a smaller volume. The intensity or strength of the magnetic field produced is directly related to the number of turns of the coil and the current through it. Sturgeon also made the first practical electric motor. Note that an electromagnet does not actually need an iron core, but the core improves its performance since the magnetism does not have to travel through the air which does not pass magnetic fields as easy as the metallic iron. Air is then like a resistance to magnetism, and iron is a conductor of magnetism.   Some common examples that use an electromagnet is in an electro-mechanical buzzer or sounder, audio speakers, a crane to pick up scrap iron metal pieces, and a relay that is essentially an electro-mechanical switch. Note that although a DC electric current can cause and keep a steady valued magnetic field about an electromagnet, a DC electric current cannot be used with transformers when there is a need to transfer an AC signal from the input coil to the output coil, and this is because the input current must also be AC or changing in value and not DC which has a steady, unchanging value of current and this will not induce any voltage and current in the output coil. In short, besides for a brief, temporary or quick magnetic pulse during current "inrush" or application creating the magnetic field pulse or rise, a steady magnetic field strength of any value does not induce a voltage or electric current, but it is rather a changing magnetic field strength that does.

The Electric Motor

To help understand how a common electric motor works, consider two magnets on a flat surface with the same magnetic poles or magnetic polarity facing each other. If magnet (1) is pushed closer to the other magnet (2), magnet (2) will be repelled by the magnetic force from magnet (1), and then it will move away from that magnet (1). This is an example of a linear or non-rotating motor (such as a solenoid) and its motion. In a rotating motor, magnet (2) would be affixed to a rotating shaft called the rotor which rotates. Magnet (2) would then move in a circular direction or motion rather than in a linear (line-like, straight) motion. Here, one or both magnets may be electro-magnets rather than "permanent magnets" (non-electric, no current needed) made of iron metal and-or ceramic material.

Michael Faraday (1791-1867), from England, and who first discovered in 1831 that even a single wire with electricity (ie., electrons, current flow) going through it would produce a magnetic field around it, and after noticing that it would make a compass pointer (as a magnetic field and direction sensor) change direction. The more current, the greater the magnetic field produced and the greater the compass pointer deflection - hence the compass become a current sensor and meter. When pulsing current through this wire, Faraday noticed that another nearby wire circuit would then also have a pulse (temporary) of current flowing through it.. Faraday called this phenomenon of creating or inducing a (changing in value) voltage and current due to a (changing in value) magnetic field, particularly its strength, as **electro-magnetic induction**.

Faraday also invented the first simple (but not yet practical) electric motor in 1821 and which produced some mechanical motion from the applied electricity. **Moritz Jacoby** would improved upon all the previous motors and made the first useful electro-magnetic motor in 1834. Before the electro-magnetic motor, there were the scientifically interesting, but not of much practical use at the time, electro-static motors created by people such as Andrew Gordon (Scotland), and Benjamin Franklin (United States of America, U.S.A.).   These electric motors used the acceleration and velocity caused by accumulated electric charge which has an accumulation of electric or coulomb forces which can affect and apply energy to nearby electrically charged objects.

Anyos Jedlik, (1800-1895), from Hungary, invented an electrical connection called a **commutator** of an electric motor in 1827 and it was a significant for improving the direct current (DC) electric motor, and which rotates in only one direction. His invention allowed motors to be powered by DC (direct current, single direction, steady voltage, such as from a battery) power. He also made a small "toy car" with his motor which surely inspired cars of the future.

discharging cycles possible with a battery, hence improving the "battery life" (ie., the total amount of the rated usability [supplying a rated voltage and current], and before it drops below its ratings) with a relatively low cost improvement to it.

While on this subject of transportation, the **Studebaker Co**. in U.S.A. made the first practical electric car (automobile) for the American general public in 1902. This was at a time when gasoline was a limited produced item, relatively expensive, and not yet available in many locations, and most public transportation roads were still dirt covered and unpaved with a hard surface. The batteries in these cars was a lead design and was therefore heavy and reduced the efficiency of the car. A few years afterwards, in 1904, Henry Ford began making the Model-A automobile. In 1913, the (Henry) Ford Motor Co. became a factory for mass producing relatively inexpensive gasoline powered automobiles, and this greatly diminished the public's desire for an electric car or horse-and-carriage for personal transportation. The "Model T" car was a very popular and relatively affordable design, and it was the initial basis to many future designs in the entire automobile industry. Many dirt roads soon became paved roads for better travel and efficiency. Trucks could now transport many and heavy items to many miles away and with greater speed than a horse (an animal engine) powered vehicle. As of about the year 2000, with advances in lite-weight electric batteries to power all types of devices, the electric car is once again on the wish-list of many people. During the same year as Ford's Model-A car was available, the Wright brothers created and flew the first practical plane in 1904, and which would revolutionized long distance transportation and made it practical.

Heinrich Emil Lenz, from Estonia-Russia, in 1834, created "Lenz's Law" which describes the voltage polarity and direction of the induced currents. The "Right-Hand Rule" considers conventional positive (+) to negative (-) current flow, and with your right hand thumb going in the direction of the wire and current, and with your other fingers of your right hand then wrapped around the wire is the (clockwise) direction of the magnetic field about around that wire. The symbol L, in honor of Lenz, was first used as the unit of measure of inductance, and others may use H (Henry's), instead of L, as the unit. Still, many inductors are still noted using an L symbol, but measured in henry units.

In 1841, **Michael Faraday** discovered (Faraday's Law of) electromagnetic induction by applying a voltage (and resulting current) to two coils placed closely together; essentially a transformer, and which are said to have "mutual inductance" in that they actually affect each other with their magnetic fields. As mentioned, Faraday also discovered that a moving magnetic field (and-or wire, relative to each other) such as from a magnet moving near a wire or coil of wire, will produce (ie., induce or generate) a voltage across that coil's terminals (end points, connections or leads) and create a resulting current through that coil and circuit connected to its ends. Faraday's discoveries about magnetism and electricity (magneto-electricity or electromagnetism) are crucial for generating electricity and using electric motors, both solenoids [non rotating, but for linear (push and-or pull) motors providing non-rotating, but rather straight motion], and rotational or spinning motors. The unit of measurement of charge storage or capacitance is now called Farads (F), and was given that name in honor of Faraday.

In 1865, **James Clerk Maxwell** (1831-1879), from Scotland in the U.K (The United Kingdom of Britain) was greatly influenced by the work and discoveries of Michael Faraday, and he took electro-magnetic (electricity and magnetism) science to new heights and future possibilities with his four electromagnetic "Maxwell equations" of which were later simplified and credited to: **Maxwell**, **Faraday**, and **Oliver Heaviside**.   These equations showed the mathematical relationships between electricity. magnetism and light, and even considered the transmission of electro-magnetic energy (ie., radiowaves) when they are considered as a type of light and-or vice-versa. Heaviside invented the coaxial cable in 1880 for short or long distance, efficient, and noiseless (low electromagnetic interference) transmission of signals through this special (outer "shielded", inner cable) "wire", "cable" or conductor. Maxwell's work influenced Albert Einstein with his discoveries. One equation by Maxwell relates the amount of created magnetic flux (ie., magnetic field strength) of a coil to the amount of input current (Amps) and the number of turns (windings, loops) of that coil. Maxwell also deduced that light is an electromagnetic (ie., a radio) wave (ie., has a frequency like a wave or oscillation). In 1861, Maxwell made the first, but crude, color photographic method, and where three color (green, blue, and red, or RGB) filtered images of the subject were made onto three separate, (transparent) glass plate, black and white photographs. The amount of greyscale (or sometimes spelled as grayscale) of the image on each photograph would effectively indicate and record the intensity and-or color of the light of the subject.   If bright light was then passed through each photograph, and then that light being passed through the same color filter that was used to created it, the three separate color images can then be combined, by "overlaying" (to be at the same position or location) the project images to coincide, be combined with, or "on on top of the other", into one multi-colored image of the subject. Besides Heaviside's work on Maxwell's equations, he also

charge, will attract each other like the two opposite poles of a magnet, and so as to naturally and eventually balance (settle, rest, equalize) the charges and have a net charge of 0, such as in a non-ionized or neutral atom with a balance of (ie., equal) electric charges in it when it has the same number of electrons and protons. Each element has a unique number of protons, neutrons and electrons. Generally for a given element, both the number of neutrons and both the number of electrons will equal the number of protons. An electrons attraction to a proton and-or the repulsion by other electrons, will cause (ie., induce, force) a "free" or unbound (from its atom) electron to move or flow towards a positive charge, such as when it is moving in an electric (electrically conductive path such as a metal wire) circuit to the positive voltage polarity and terminal side of a battery. This direction of current flow is for the true, physical and actual flow, but "conventional" or "common" flow of electricity, or electrical energy and-or power, is considered from positive terminal to the negative or "ground" (considered at 0 remaining potential) terminal of the voltage source, and this does aid with the math of it. For most common circuits, there is not problem using conventional flow and-or analysis. For things like Crookes and other vacuum tubes, the actual direction of electron flow matters more than having a (generalized) current flow and-or direction.

1911, Rutherford, who discovered the electron particle of an atom, theorized that the proton and electron composed a larger particle (called a nucleus of the atom) with a balanced or neutral charge state and called this unknown particle the **neutron** particle, and that an atom must have a large center or "**nucleus**" of mass. Rutherford and his team of scientists then experimented by sending fast moving alpha (protons, positive charged) particles at a very thin layer of gold foil, and because most of the alpha particles went through the foil, he theorized that the atoms of the foil had mostly empty (no matter) space between them. Since alpha particles have a positive charge, they should also be attracted to, and combine with the electrons in the gold foil, but it was noted that this alpha particle was sometimes changing or swerving away from its expected straight direction after leaving the foil, hence it must be colliding with and-or deflecting (changing) its direction due to a positive charged particle at the center of the atom. He theorized that electrons must be very small, and must compose some space or volume of an atom, and that they must then be attracted to, and orbit the nucleus.

Neutrons are very similar (but actually just slightly greater) in mass to that of protons, however, they don't have any electric charge like a proton or electron. They are not influenced by an electric field, and are only slightly influenced by a magnetic field, but are rather influenced only by kinetic force or collision. Later, in 1932 **James Chadwick** (1891-1974), from England, discovered a particle that did not have any electric charge, hence neutral in charge, and therefore called it the **neutron**. Neutrons do not attract or repel each other, and it is thought that they therefore do not have a "rotational spin" on its axis, and that they are attracted to the proton by a mysterious "strong atomic [nuclear] force". In the 1960's it was theorized and discovered that protons and neutrons are composed of even smaller particles called quarks. Each proton or neutron is composed of 3 quark particles. In 1979, even smaller particles called "gluons" were discovered and are thought to hold ("glue") the 3 quarks together. The concepts of quarks and gluons is generally not needed and-or used in the practice of electricity and electronics. If a nucleus has an unbalance (excess or lack) of neutrons, the atom is called an (unstable) **isotope** of the stable or balanced form of the element, and it can then split or decay into other lower mass atoms while releasing or radiating energy in the form of radioactivity, and then becoming (more) stable. Nearly all of the mass (real physical material, or substance) of an atom is due to the total mass of its protons and neutrons.

A proton is said to have and be the 1 fundamental unit or particle of positive (+) electric charge and force, and an electron particle is said to have and be the 1 fundamental unit of negative (-) electric charge and force. Neutrons have no charge. Protons will repel each other, and electrons will repel each other. Electrons and protons will attract each other due to their opposite electrical charge. Since the electric charge of a proton is equivalent to that of an electron (e), the electric charge of a proton may be expressed as (+1e = e), and the electric charge of an electron may be expressed as (-1e = -e). Note that the mass of an electron is much smaller than the mass of a proton, and yet an electron has an equal, but opposite, electric charge as that of a proton's electric charge.

Just like magnets having a magnetic (force) field (area of presence and influence), or field of force about them which can induce, attract or repel other magnets and (magnetic, iron) particles, charged particles also have an electric (force) field about them which can induce, attract or repel other charged particles. Because of an object just having a non-zero electric force, these electrostatic particles and their charges can also cause (induce, force) other particles and their charges (particularly electrons) to move or flow (attracted to, or be repulsed away) without any electrical (ie., electron) path or conductor between them such as a wire. The electric field strength value is much like that of a magnetic field

(sometimes) recorded, collected and then decoded or reconstructed back into letters of words and written down by the telegraph operator so as to create a written "telegram message" (or simply a "telegram") to be delivered as mail to the receiver of it. This was a significant step in fast, low cost communication, and it helped society advance more efficiently and rapidly. The telegraph did not replace the mail system, for it was, and still is needed for the many and long messages and packages, whereas a telegram was often limited in the "length" of it in terms of the total number of characters and-or lines permitted by the station, and therefore, the messages were often short and "to the point" or issue, much like that on a postcard (mail).   The Morse code is given an article in the "Late Entry's" section in this book.

Rubbing (friction) can "charge up" some objects. Rubbing two objects together can free some electrons from their atomic orbit or bond so as to produce ("free", movable) charges or "electricity". Larger, more massive atoms of elements have a weaker bond or force applied to their more distant outer electrons, and they are more easily freed using less force such as electric force. An object that has more negative charges (free electrons) than positive charges (positive ions, positive ionized or charged atoms) is said to be negatively charged.

In 1832, and not long after Michael Faraday discovered electro-magnetic induction, **Hippolyte Pixii** (1808-1835), from France, and inspired by Faraday's crude non-coil design of an electricity generator, created a hand cranked and powered by arm force, rotating machine which can be turned or rotated so as to generate (AC, Alternating [in two directions] Current) electricity. This machine converted or transformed input mechanical energy and-or power to electrical energy and-or power. His machine rotated a magnet near a coil of wire. With this, he invented a practical A.C. **electric generator** machine or electricity generator. This simple concept and device was actually monumental in terms of the electricity advancements it would inspire and be used to power. **Andre Ampere** gave Pixii the idea of using a mechanical device (now called a [electric] commutator) so as the current would be caused to flow in just one direction, somewhat like DC does, but here, with AC involved and always changing in value, the generated DC will also vary in value, magnitude or amplitude, but not in direction like AC does.

A **dynamo** is an advanced type of electric generator created by Werner Seimens in 1866, and which used electro-magnetics instead of permanent magnets. These electrified coils (electro-magnets) have a higher magnetic strength or power, and therefore, the output electric power was (theoretically) greater if the generator's rotor (ie., its rotating or rotational part) was spinning as fast as that without electrified coils. It should be of note that generators, for either mechanical or electrical output power, can contain mechanical input and-or output gears so as to alter the rotational speed and torque (force) needed for the rotor and a stable (ie., a constant value of voltage and current, or power), regulated electricity production or output.

How electric current is generated in a generator.

A changing magnetic field, such as through the coils in a spinning electric (electricity) generator, can be used to cause ("induce" or generate) electrons to be free (from their atoms) in a wire (or coil of wire) so as to then have an electric (electron) current (flow, movement) in a circuit. The magnetic field force will give kinetic energy (ie., J/C, joules of energy per coulomb of electrons, and this is defined as Voltage) to the electrons in the wire of the spinning coil (or sometimes the magnets are rotated, rather than the coil), and an external electric circuit will allow those energized electrons to flow. The (alternating, and in a sinusoidal wave) voltages produced from a generator and-or output from a transformer can be very high and dangerous, and can easily force a relatively low current (due to high resistance) through long distance wires to the destinations of use. At the destination(s) for public use, the voltage available at the electric power transmission lines is reduced ("stepped down") by a device called a **transformer.** An electric (power = voltage and current) transformer is basically two electrically separated, but magnetically coupled (magnetically influenced) coils that will convert a high voltage, low current input, to a much safer and lower voltage (about 220v, or 120v) output and with a higher current ability than that normally found in the long transmission wires. Long transmission wires have a certain resistance per meter or mile and therefore have an additive or accumulative resistance and would waste (as heat) much of the power generated and available for use. When the resistance is higher, the current is reduced, and the power loss becomes more significant because power is related to the square of the available current value. The voltage on a long distance transmission wire may be about 10kv to 20kv and is extremely dangerous, lethal, and can easily cause electric arcing (ie., like lightning, electrical discharges), shocks, damage and-or fires.

phase of electricity from the local area, (voltage) step-down or reduction transformer. Dobrovolsky is also credited to the "squirrel cage" AC motor design that uses segmented metal pieces instead of larger pieces, and so as to reduce the unwanted (magnetic field caused) "eddy" (ie., side or residual) energy storage and resulting wasteful (back emf caused) current reduction and resulting reduction, and this improves the efficiency of the motor.

A contemporary to Nicola Tesla, named Karl Steinmetz (aka., **Charles Steinmetz**, while in the USA), (1865-1923), from the Prussia area that is now located in Northern Germany and Poland, made many practical discoveries about AC electricity, and made some of the equations for it to be more practical and learnable. He gave us the understanding of the phenomenon of (magnetic) hysteresis in electro-magnetic motors which contained magnets.   Hysteresis is when an electro-magnet briefly retains some of its same magnetic polarity after the current has switched to the opposite direction such as what commonly happens with AC (Alternating, changing direction, Current) electricity.   If hysteresis can be reduced, it will improve the efficiency of motors.   He is honored by the IEEE (**I**nstitute of **E**lectrical and **E**lectronics **E**ngineers).   Charles was born with a form of dwarfism and a genetic [DNA]) physical handicap and-or (bone, osteo) disease. After his education and several initial discoveries, he went to the United States of America in 1889 and worked as an engineer for the (Rudolph) Eickemeyer Company that specialized making (AC) transformers, and then worked with the General Electric Company so as to further develop both the science and application of electricity.

Edith Clarke (1883 - 1959), from the USA, was one of the first women to be recognized and (formally) certified as an electrical engineer by the IEEE, and is honored for her practical work, calculation charts, and significant educational writings about AC electricity, circuits, and AC power transmission so as to help make them more understandable and practical for many to study and comprehend.

RADIO

Radio is a word derived from the radial-like (radius from the center) direction of the outward transmission or radiation of electromagnetic energy from a central source such as from a metal wire antenna with electric energy applied to it. Visible light is one form of electromagnetic radiation and is of the (radio or electromagnetic) frequencies that are visible to the human eye as colors.   Most radio frequency (RF) waves are invisible radio waves such as used for the electronic transmission of audio (sound) such as for a radio receiver and its audio speaker, and-or for visual images as for a TV receiver and its viewing screen. The telegraph and-or phone system requires a direct electrical connection (with a metal wire) to the people communicating, but radio communication (ie., "wireless" communication) does not require a direct electrical connection, and is therefore available and cost effective for many more people, and-or for all at the same time such as with standard TV (television). This discussion is mainly an introduction to radio, and some of the technical or specifics of it are discussed further ahead in this book.

Ever since electric sparks were created, invisible and unknown radio wave energy was also automatically created and transmitted, and it was also not known at the time that these waves could pass though non-metallic things such as a wooden box or wall that is not too thick and-or made of metal. It was also not then known of how to sense or receive this invisible radiated (ie. radio) energy.   A high voltage spark of light through the air is visible, but was there also some invisible (radio) waves also being transmitted at the same time? In 1888, **Heinrich Hertz** (1857-1894), from Germany, detected the first invisible (or radio) wave, by using a receiver (ie., antenna) that was composed of just a single loop of wire having a very small spark-gap and (initially) placed just a few feet of distance away from the electric spark-gap The transmission spark (which created the (invisible) radio wave energy) used was made using a high voltage induction coil, which is a type of transformer that can also raise the voltage of the input signal, and so as produce a high voltage output signal, but actually at a lower maximum amount of current since the energy output cannot be greater than the energy input. That is, (energy out) = (energy in), and-or (work out) = (work in) or (power out) = (power in) if there are no losses of power along the way. Modern radio transmitters and-or receivers do not use a spark-gap, but rather use much more modern and sensitive radio wave circuits. The electric spark and spark-gap in this type of application became known as the (electric) spark-gap (electricity, energy, radio-wave) transmitter.

Hertz probably got his idea while considering Faraday's discoveries and that an electric current pulsed wire and-or coil could transmit a magnetic field, of which each (transmitted) pulse can be detected, sensed or received by a (nearby) magnetically sensitive compass. Would a spark be an indication of some type of charge or energy being radiated? Hertz

Sound Recording

Thomas Edison (1847-1931), from the United States of America (USA), and his company made many significant advances in technology, such as for a practical electric light in 1879, and a completely mechanical audio (ie., sound) recording and playback machine in 1877 called a **phonograph.** A phonograph could both record and play back sound vibrations. These mechanical machines were hand-cranked powered, and to hear the recorded audio at its proper frequency, the recording needed to be played back the same amount of revolutions per minute (rpm) as it was recorded with, and it was often about 120 rpm. A recording standard was later recommended and used, and it was 160 rpm which improved the higher frequency range up to about 5000 hz., making it more suitable for accurately recording high frequency musical instruments. At those times, due to the lack of electronic amplification for both recording and playback, a sound needed to be relatively loud to record it, and sound collectors shaped like funnels were used to concentrate sound so to increase its intensity or pressure for the recordings.

In the times of Edison, the concept of "motion or moving pictures" to photographically record and display objects in motion was around for many years, but he made the first practical **motion picture camera,** and of which began to set the standards for motion pictures or "movies"**.** He also invented a greatly improved **microphone** (converts an input sound energy or signal into a corresponding or simulated electric energy signal) in 1876 which was then used to significantly improve the existing, but relatively crude telephone system created by Alexander Graham Bell. Edison is generally credited to the invention of the carbon microphone in 1877 which also allowed larger (ie., in amplitude, louder) audio signals to be used with the telephone and other devices. Noted credit to the invention of the carbon microphone at about the same time as Edison is also given to David Hughes from England, who also invented a telegraph system that printed telegraph messages onto paper tape, and to **Emile Berliner** (1851-1929), from Germany-USA, who worked with Edison in 1876 and created a better carbon microphone in 1878 that Alexander Graham Bell would use to significantly improve his electric telephone system. Berliner would later, in 1887, invent the ("Gramophone" phonograph) flat or disk phonograph recordings (commonly called as just "records"), and he also invented a significant record playback reproduction method for mass distribution of audio recordings or records. This technology was a vast improvement over the relatively soft and degrading wax recordings made previously by both Edison and Bell. Berliner created a process to record the audio into a thin wax coating on a metal disk, and then electroplated it and this metal disk would then last much longer than (flimsy) wax cylinder recordings and any possible (degraded) re-recordings of it. This metal disk was not actually played back, but was used as a type of mechanical die or imprint that could then be used as the source or mold to stamp (ie., press, imprint) exact (but reversed) copies of itself onto a heat-softened plastic that would soon harden firm when cooled. These recordings would last much longer than wax would when a vibration sensor called a "pickup needle" was physically dragged through its groove (ie., a long indentation) as the recording was played back so as to hear it. These disks were to be heard or played back using a rotational speed of 78 rpm (revolutions per minute), and have a recording that had a maximum of about 5 minutes of time duration or length. These disks originally were 7 inches in diameter, but later, and mostly, the 10 inch and 12 inch diameters were used. As of 1948, and created by the **Columbia Record company**, similar 12 inch diameter types of records having a longer recording and playback (ie., "listening") time and having improved sound or "audio fidelity" quality were called LP (Long Play) records. These records were to be played back at 33**.**3 rpm using a sensitive electromagnetic needle (physical-mechanical vibration "pickup", sensor and transducer to an electric signal) so as to then be heard louder by using an electronic sound (signal) amplifier. Each side of these records had a recording and-or playback time of about 30 minutes maximum, and therefore these records could contain a total of: (30 min / side)(2 sides) = 60 minutes of audio.

In 1948 the **RCA Victor Co.** invented the 45 rmp, 7 inch diameter plastic record, and which was often used to sell "single" (ie., one song per side) recordings of which were often the most desired or popular songs found on the more expensive LP records. These relatively small disks were also used with coin-operated (ie, pay first) playback machines, often called a "Juke Box", that had many available songs on these small disk records inside it, and for a coin fee, it could play a desired song using its internal record player and audio amplification. As of the year 2022, a few of these original machines could still be found working, and usually inside a restaurant, bar or night-club. They are quite collectible, especially if they contain old recordings. A popular and-or song being promoted was usually placed on what is called the A-side of the record, and the B-side of the record was generally a song that still needed to be heard, but it often was not promoted and -or it did not have much of a chance of being a popular or "hit" record, but there are some exceptions to this.

As of about the year 2000, music and-or other (computer or digital) data can be recorded, stored or saved onto non-mechanical memory IC's (ie., integrated circuits, or "chips") devices, and one common method and-or portable device is called **USB** (Universal Serial Bus having a standardized (bit by bit) serial data connection so as to store data to or from a computer) memory card, a "thumb drive" or "memory stick".   A particular modern USB memory stick can hold from 1 GigaBytes (GB, = 1 billion bytes) to perhaps 128 GigaBytes of binary and-or digital data. If this device gets lost or stolen, much valuable data, perhaps proprietary, company, employee, family, friends and personal data is then at a security risk, and you may wish to use your computers software (ie., perhaps a Windows program(s)) so as to secure folders of data on your device(s), and to also make and store "backup" copies of your data. If possible, include a readable, plain text file in the data on the device, and which includes your phone number so as the device may be returned to you, and-or write or engrave your number on the device.

A dynamic (or "magnet and coil", electro-magnetic) microphone was invented in 1877 by **Ernst von Siemens** in Germany and it is very similar to a small loudspeaker, both electrically, physically and mechanically with its suspended, movable coil of wire around a fixed in position, permanent magnet.

Today (as of the year 2021), **condenser microphones** with great audio input sensitivity have mostly replaced the use of the more rugged carbon and dynamic microphones. This microphone consists of a capacitor with two plates, and with one plate that has low mass and weight and is very sensitive to sound (audio) waves and could easily vibrate or move in unison to sound. Because a plate of this capacitor can move, its capacitance value will vary and which will then cause its output electric signal to vary at the same frequency and amplitude as the received audio. The condenser microphone was first invented by **Edward Wente**, from America, in 1916. As of about the year 1995, condenser microphones are in great use as for computers and phones. There are also other types of microphones available, such as the (thin, metalic) ribbon microphone.

Before the invention of the first audio tube amplifier, some principles of the carbon microphone were sometimes used to make audio (low frequency) amplifiers before the invention of tube (or "audio" or "[electronic] valve") amplifiers. The concepts of the change in resistance and-or conductance of the microphone so as to amplify sound would also later be found as part of the understanding and functioning of the transistor amplifier. Before the transistor, hearing aid sound amplifiers used a small **carbon amplifier**.

In about 1875 while Edison was experimenting with his **electric light**, he notice an effect that the electricity (ie., current, later to be called as electrons or electron flow) from the hot filament would flow to a (voltage applied) charged metal plate inside the tube or light-bulb, and also having a much higher current when the plate had one voltage or charge polarity than the other. After applying a reverse polarity voltage connection, this resulted in no current flow, hence there was a very high resistance to the flow of current in one direction, and a low resistance to current in the other direction or polarity. Edison therefore essentially created the first man-made electronic "one-way" (ie., direction of flow) valve or diode for electric current, and he did patent it, but he had no immediate use for its application since he was more interested in perfecting the electric light. This "**Edison Effect**" (ie., technically a diode with the **diode effect**) would later be improved and used in a radio receiver as the first practical and reliable (vacuum tube, "tube") diode or [electronic] "valve" that was formally and practically created in 1904 by **John Ambrose Fleming** (1849-1945), from England, and while working for the Marconi Company**.** Before this time, Fleming once worked for the Edison Company in the United States of America and was well aware of the discovery and basic knowledge of the Edison Effect.

Before Edison's incandescent electric light, many already knew that electricity through a material or that a high voltage arc could produce light, but Edison's electric light filament (a hot, light radiator or emitter) was placed in a vacuum so as to not burn (slow combustion) quickly. His filaments of choice was a carbonized wood fiber made from bamboo, and a very thin platinum wire.   In 1875, **Joseph Swan** (1828-1914), from England, independently created a practical incandescent light bulb at about the same time as Edison, and his filament was a piece of carbonized plant fiber thread, and then later a drawn cellulose thread. Edison and Swan actually created a company to manufacture electric incandescent lamps (ie., lightbulbs, light-bulbs, light bulbs) which consisted of the best research, manufacturing technology, cost, power required, and the lifespan (average time of use before failure) of the filament and lightbulb. Swan would also invent the use of nitrocellulose plates for creating photographic images, and this eventually lead to movie film reels made from a thin (ie., a

In 1912, **Edwin Armstrong** (1890-1954) from the USA, while experimenting with a radio receiver with a triode tube amplifier, he invented what is called positive (ie. enhancement, additive) feedback or "regeneration" (ie., reusing), and where a small amount of the output signal of the rf tuner and amplifier circuit is electrically recombined or joined with the input signal at the control grid (or signal input grid) of the triode amplifier. With this feedback, the total gain of that amplifier was many times more than than what the triode could otherwise produce without it. Now, instead of using earphones to hear the sound, large diameter, high power "sound reproducers" or "loud-speakers" could be used. Today these are simply called as "speakers". If the feedback was strong enough, a radio wave frequency, electronic **oscillator** or "signal generator" could then also be created which is often a self-starting oscillator and circuit without the need for any input signal to start the process of making a (tuned, specific frequency) waveform, and this is called an **Armstrong oscillator circuit and-or concept.** This greatly improved the field of radio communication, for before this time, continuous-wave (sine waves) were limited to a relatively low maximum frequency due to that they had to be mechanically made using high speed alternators, or by relatively crude, low frequency electro-mechanical oscillators which worked well for musical organs (electric piano's).

Armstrong was one of the contemporaries of engineers who invented the concepts of superheterodyne for radio receivers. Heterodyning (ie., to create a difference by force) was previously developed by Reginald Fessenden and is a type of advanced tuning concept that is more stable, efficient, but it requires more circuitry such as to combine (ie., "mix") the received AM carrier signal with a fixed high rf frequency sine wave so as to produce a fixed or constant "intermediate frequency (IF)" signal that is the difference (ie., the "beat frequency") of those two combined signals and which is then demodulated (ie., to remove) so as to have just the desired audio signal. **Heterodyning** allows the radio receiver to be more sensitive to the (weak) received radio waves, and more "selective" by offering a more precise (ie., a limited or narrow frequency bandwidth or range) tuned signal with little interference from other radio stations that might be using a transmission frequency just a little higher or a lower than the one desired or selected. Armstrong also found a way to stop the high frequency oscillations in a regeneration receiver, hence this allowed even more amplification, and therefore, he invented the super-regeneration concept and receiver.

In 1933, **Edwin Armstrong** made the first practical "wideband (bandwidth, frequency range)" **FM** (Frequency Modulated radio) transmitter and receiver which has nearly no audible noise, such as from "static electricity" interference caused by lightning which is commonly heard as noise during AM (Amplitude Modulated radio) reception which is based on the amplitude or strength of the received radio wave energy, and lighting will greatly affect that strength, both increasing and decreasing it. FM (Frequency [changing slightly] Modulated radio waves) is often used for "high fidelity" (low noise, "high quality") radio communication. Frequency modulation is when the input information or audio signal slightly can vary the carrier or main radio frequency to be slightly higher or lower in value, and which corresponds to the input signal's amplitude. In an FM receiver, the frequency changes in the tuned or desired ("channel") frequency are converted back into changes in audio amplitude. In general, most FM transmitters are designed to be of lower transmission power than **AM** transmitters, and therefore have a shorter range or distance of radio transmission. This also prevents possible reception interference from other distant (FM, wideband, frequency range for each channel) stations that may be assigned to and using the same frequencies. The standards set by the **FCC** (Federal Communications Commission of the USA) allow AM radio signals to have a maximum allocated frequency bandwidth (ie., total change in frequency) of 10khz and with 5khz below the main carrier, center or "channel" frequency, and 5khz above it. FM radio signals are allocated a 200khz frequency bandwidth and with 75khz below, and 75khz higher than the main assigned carrier frequency. Since FM broadcast stations and their signals and are allocated a wider frequency bandwidth than actually needed, the full audio spectrum of 0hz to 20000hz is easily included into its rf signal, but this also greatly limits the total number of (nearby) stations allocated to the complete, but limited, FM frequency bandwidth communication allocation.

Some "primitive, solid-state-like" or semiconductor discoveries and devices were made years before the first transistor amplifier was discovered, and some of these are discussed next.

In 1874, in Germany, Professor **Karl Braun** first noticed an electrical diode action in certain (semiconductor) crystal materials. A diode is the name given to any device that only allows current in one direction, and is the electric equivalent or analogy of a special (mechanical, force sensitive and operated) valve that is used to permit only one-direction of a gas or fluid flow in a pipe (conduit). This type of mechanical valve is often a "ball-valve" due to its internal construction with a ball that will essentially block the pipe and fluid flow if that fluid goes in the opposite direction for some reason, such as

The **transistor** amplifier device is based on the concepts of electrostatic forces (such as mainly used in the **FET transistor** which has just two semiconductor layers), solid state (metal) diodes (such as used in the **BJT transistor**), and charge carriers (such as electron and hole [absence of electrons; net positive] flow of current). The transistor electronic device was invented in 1947 at Bell Labs in the state of New Jersey, USA. The first working transistor amplifier is credited to **William Shockley, John Bardeen and Walter Brattain.** A bipolar junction transistor (BJT) is a small, three layer (NPN type or construction, or PNP type or construction) metal semiconductor(s) material device that functions as an electronic signal controller (ie., an on-off switch), or an AC signal amplifier, and which has replaced most vacuum tube amplifier devices since it is much smaller, much cheaper, and much more efficient since it wastes far less power as heat or thermal energy. Modern computers have many transistors.

The two main types of transistors are the FET, (electric field, electrostatic) Field Effect Transistors, and the (BJT), Bipolar Junction Transistors which can be described as a two-layer (solid state) diode of identical semiconductor material and with a thin (control, or "base") layer made of opposite (electric charge, and carriers) polarity semiconductor material between them. A transistor can be tested as being good if the two diode sections are tested as two good diodes that will conduct in one direction and not in the other direction. Essentially, transistors control the flow of current through it with the control layer called the "gate" as for FET's, and the "base" as for BJT's.   In general, FET's require higher load voltages to operate than BJT's, and this is due to the (Source to Drain) channel material and its resistance in the FET.

An FET is basically controlled by its input voltage, and a BJT is basically controlled by its input current. It is of note that a single FET generally does not come close to amplifying as much as a single BJT could, but the FET has the advantage of having a very high input resistance or impedance, and therefore it requires almost no input signal current to function, and therefore nearly electrically isolates that transistor's output circuit from that transistor's input circuit. An FET is usually functions faster than a BJT, and therefore a FET is often chosen to be used in high frequency circuits such as for radio frequency (rf) amplification of the weak radio waves received by a radio antenna. Due to its high input impedance an FET is an ideal choice for the output signal from a tuning or oscillation circuit, whereas, if a lower impedance device was used it would actually reduce the performance of that tuner and or oscillator by effectively lowering its (tuned) impedance and the outputs signal's maximum voltage.   In 1948 **John Bardeen** is credited to inventing the Insulated Gate Field Effect Transistor (IGFET), and which technology inspired the more practical Metallic Oxide Semiconductor Field Effect transistor (**MOSFET**) invented by **Mohamed Atalla** (from Egypt) and **Dawon Kahng** (from Korea) working for the Bell Labs company in 1957. MOSFETS eventually became widely used in both low power, inexpensive (cost per transistor), "high density" or transistor count, per unit area, integrated circuits, and for high power transistor applications such as for switching and amplification.   It is of note that the BJT was created in 1947, and before the theoretically "more simple" (yet more difficult to produce, and consistent device reliability) FET was created and made available between 1957 and 1959. The Junction Field Effect Transistor (**JFET** which is often called as the common FET) manufacture was being studied and developed by several people and companies between 1950 and 1953. Compared to BJT's, JFET production and reliance have been much more of an issue at first, and it took until many years later to gain a high device reliance and expectation from each JFET created, such as a consistent source (S) to drain (D) resistance of a specific (ie., manufactured, registered number, type) JFET of which a circuit was initially utilized in its design, and for replacement if needed.

Besides being a signal (ac wave) amplifier, the transistor is also used as a simple switch, a fast electronic on-off switch or (solid state, fast, no coil) relay (ie., an electronic equivalent of a water on-off valve), for logic and signal control, and for data storage in computer or digital memory devices. When used as a switch, the control signal such as the base current in a BJT is at a high enough level so as the collector to emitter resistance becomes the lowest value it can be, much like the low resistance of a closed switch. The transistor is said to be "full on" or completely on and conducting as much as possible. When the control signal or current is 0A, the transistor is not conducting and is said to be "off" and its effective resistance is very high, much like that of an open circuit or switch. In terms of (on-off switching) speed, as of the year 2022, BJT's are still capable at operating at a higher frequencies than MOSFET's, and in both cases, the smaller the device, the more faster it can be due to less internal capacitance, and the smaller the distance that the charge carriers have to travel through the device during the full on (conduction) to full off (no conduction,high resistance) states of it. JFET's can operate at high frequencies than MOSFET's, and are becoming close the (switching and frequency handling) speed of many BJT's as of the year 2022.

In digital ("1's and 0's") electronic circuits, transistors are usually used as very fast and simple electronic ("on and off", or "1 and 0") switches for logical (truth) decisions and results, performing mathematical operations using binary values, and-or for storing the bits (ie., digits) of data and-or binary numbers to be used for mathematical calculations and their results.

As previously mentioned, some preliminary solid-state or semiconductor electronics before the invention of the transistor actually goes back to the first effective discovery of effectively a **light emitting diode** (LED) by the British scientist **J.H. Round** in 1907. He noticed some "electro-luminescence" (electrically caused light) being created when a voltage (and a resulting current) was applied to certain minerals (such as silicon carbide) crystals composed of certain elements. Many of these (solid state, metals) minerals were often used for a "cat whisker" (a thin hair-like metal wire for point of [surface] contact, electrical connector) part of a radio wave "detector" device. A radio wave detector used in the first radio receivers was effective as an (ac) radio wave (audio signal) extractor or demodulator, and has a one-direction (dc) diode-like action, hence a basic (ac) to (dc) signal converter. A theory of a special boundary or "junction" (depletion) layer (such as an oxide layer) between each element in the crystal and-or the cat whisker wire, was mostly likely started here, and which eventually lead to further understanding and man-made production of the diode and further development of modern high speed and-or power diodes, the three-layer (NPN, or PNP) BJT transistor amplifier - diode-like. layerd construction, and the modern and efficient light emitting diodes.

In 1894, **Dr. Jagadish Chandra (JC) Bose** (1858-1937), from India, was the first to discover and use (what was to be eventually called semiconductor) metals (here, naturally formed "metallic crystals") as a radio wave detector, particularly with **microwaves** which have a very high radio wave frequency and with a very short wavelength. Unlike the first coherer, radio energy or wave detector, this device was very fast and with a with high frequency ability, and it did not need to be physically reset after each pulse of energy. In 1901 he patented the first crystal **(**modernly called "solid-state", semiconductor, and here with **galena** [a lead and sulfide (sulfur) compound] metal**)** radio detector, and which had a thin metal wire as the other metal necessary for the detector (ie., diode or rectification) action. Today, we understand more thoroughly that a (necessary) junction and-or a (electric field caused, current) barrier region between the two metals is created. Perhaps Bose was considering the mystery of the two different types of metals used in the construction of a single voltaic cell of a battery. Two years before Marconi's radio-telegraph communication system, in 1895 and inspired by Hertz's previous experiments with both creating radio waves and the detection of, Bose was the first pioneer in microwave (have a short wavelength) radio-wave creation, transmission and detection (via a solid-state metallic detector that he invented). Bose made microwaves using a specially designed electric spark-gap, transmission and detection system. He was the first to create what can be called as radio signaling, and which can be thought of as (electronic) remote-control.. Like Hertz, he associated and studied invisible radio waves as being similar to that of visible light waves which can be directed, reflected, but can also pass through non-metalic materials like walls between the rooms of a house. Though this device (fundamentally called a radio-wave or microwave detector) worked, the physical properties and concepts of semiconductors and diodes were generally unknown at that time, very limited in use, and that it just happened to work so as to be useful. This mysterious, magical, and very useful device inspired others to research these type of devices (ie., diodes) and materials so as to improve upon radio wave detection, and then later to be used for voice demodulation or audio detection (particularly with isolating it from the received radio signal [electric wave energy, voltage, current, frequency] so as it could be properly heard and not effectively lost due to signal cancellation) such as for AM (Amplitude Modulated) radio wave communication and for many other electronic circuits. After many years of research and technology, high frequency signals such as microwaves are used in RADAR (to locate distant planes via radio reflection), communications, and to heat food in a microwave-oven.

Greenleaf Whittier (G.W.) Pickard (1877-1956) from the United States of America is credited to the first study of identifying and using many types of semiconductor metals that when deliberately joined together, they and their **junction** (surface area connection) have a diode action necessary for radio wave receivers that are often called "crystal radios" so as to distinguish them from the vacuum-tube (electronic amplifier) radios available. He also patented the first **silicon crystal** as a radio detector in 1906. This book contains a basic and common crystal radio circuit of which you can also research, experiment and improved upon. You may also research both long-wire and (portable) loop (directional, coiled) antennas and their circuits so as to improve the amount of radio energy collected so as to be able to hear weak and-or distant stations. AM radio reception can often be improved simply by turning the radio and-or antenna to a direction having the best audio performance.

type of **Schottky diode** which is a metal and semiconductor junction diode, and has a low "turn on" (forward, conduction or barrier) voltage of about 0.35v (when the current is kept low, otherwise it will increase) and it has fast turn on (conduction) and turn off (non-conduction) times and can therefore be used for very high frequency circuits. It is a very practical and efficient diode to use with weak radio signals such as that received by many radio receivers.

Russell Ohl (1898-1987), from the USA, while working for AT&T Bell Labs in 1939 is credited to purifying silicon to over 99% purity, and so as to eventually make the first practical and commercial "(PN) solid-state" diode. He is also credited with the modern understanding and concepts of the diode barrier-junction (PN) region, and the concepts of solar cells to create electricity (as of 1941) using the light sensitive properties of silicon and the barrier region between the P and N joined pieces. Light striking the barrier or junction region would create a voltage (ie., electronic energy so as to move electrons (ie., current) with electric forces). These silicon **solar-cells** created by some engineers at **Bell Laboratories** ([Alexander Graham] "Bell Labs") in 1954 had a relatively low efficiency (about 5%) of converting solar light energy into electric energy (moving or flowing [like a water current] electrons with kinetic energy), but it was still very promising and also useful for electronic light sensors and small electrical devices and batteries. A solar cell is sometimes described as a large PN junction diode. Solar-cells were used to power most of the man-made satellites sent to outer-space starting in 1958. The PN solid-state discoveries would lead Bell Labs to begin research for a solid state electronic signal amplifier based on the practical solid-state diode, its manufacturing process, and the basic concepts of the vacuum tube amplifier having a current control grid in the middle. This "solid state" (semiconductor) amplifier device invented a few years later in 1948 would be called a transistor. Many other similar (germanium or silicon, PN junction(s)) electrical devices, including the first practical and commercial LED (Light Emitting Diode) light soon followed these (seemingly more advanced) discoveries.   Below is a crude, but helpful basic drawing of some of these electronic devices. [FIG 219]
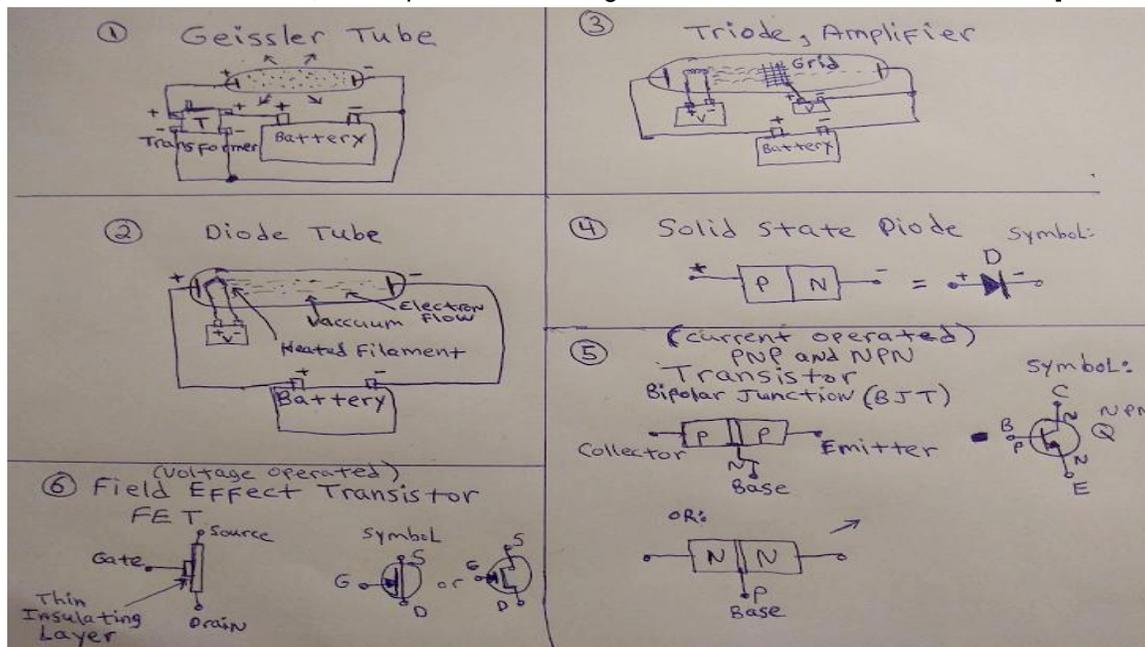
The first practical **LED** (light emitting diode) light was invented by **Nick Holonyak** (born in 1928) at a General Electric Co. laboratory in 1962. The light emitted was red in color. These LED's required about 2 volts and 10mA of current to function reasonably. If the current is increased slightly (up to a few milliamps (mA) more, maximum), the light intensity will also increase due to the increase in power sent to it. The power (voltage x current) to an LED can be controlled or limited by using a resistor with an appropriate, calculated ohms value of resistance. The price of these LED's was initially very high, and as of the year 2020, a bag of 100 red, yellow or green LED's can be had for about $5USD. 7 or 8 segment LED numeric displays are composed of 7 or 8 individual, long LED's of which each can be individually turned on or off by a circuit and so as to display a number such as for calculators, watches, meters and other various counters. The general construction of an LED is very similar to that of a typical PN diode, except that the LED is also designed to produce and radiate much more light. White light emitting LED's were always desired and were finally produced in 1996 and became

image transmitted via the telegraph system. Gray is also credited to making what could be called the first practical electronic musical instrument and-or system in 1874. It was called an electro-harmonic or music telegraph. Since speech could not be transmitted over the telegraph system for various reasons such as the technology was not yet available or impractical, Grey considered transmitting musical tone signals (particularly their frequency's) using via their equivalent or representative electronic signals. With this invention, Grey invented a form of (mechanical) vibration microphone or "sound pickup", and a form of a loudspeaker which has an electromagnet to make the sound vibrations from the input electric signal. This instrument was basically an electric organ, and where each note and its corresponding current variations or "fluctuations" at a particular frequency were automatically determined by a corresponding, thin metal reed that could vibrate at a particular natural resonant frequency. Each acoustic and-or electric representational tone of it was therefore produced by what could be called a "(metal) reed oscillator", and which was powered by an electromagnet, and much like how an electric buzzer is operated as a type of (electro-mechanical) oscillator. What was Grey initially thinking? Grey most likely wanted to replace the "dot" and "dash" electronic Morse code pulses with two different and discernible audio tones at the receiver, thus improving the telegraph system for some people to decode the pulses more accurately.

Alexander Graham Bell was born and raised in the United Kingdom but immigrated to Canada, and then to Boston city in the United States of America (USA) just to the south of Canada. Bell often worked with educating the deaf and helping their community. Bell made an improvement in sound recording and playback by creating a modified version of Edison's phonograph system. Bell also invented the photo-phone where sound could be transmitted using sound modified ("modulating", varied, here the intensity) light waves transmitted through the air to an electronic sensor or receiver placed at a short distance away. This was one of the first significant uses of the newly discovered (by **Willoughby Smith** (1828-1891), from England, in about 1873) light or photo-sensitive, photo conductive electrical materials. The light sensitive or reactive material used was selenium which is a metallic element that can vary its electrical (electricity) resistance to the flow of electric current, and directly (a direct relationship) to the light intensity upon its surface. Selenium is said to be a semiconductor material that does not conduct electricity very well, but still conducts better than a resistor. Semiconductors are usually sensitive to light, heat, etc., which will vary their electrical properties such as its electrical resistance. It is said that the "band-gap" or amount of energy (joules or "electron volts") needed for a semiconductor atoms to release (from their orbit and force of electric attraction) or give off electrons and conduct electricity is slightly greater than that of a standard electric conductor or metal. The temperature level of something very hot can be sensed by using a probe that has temperature sensor (ie., a heat to electricity "transducer" or converter) made of a semiconductor material.

Charles Fritts (1850-1903), from America, essentially created the first solid-state electrical generator or **solarcell** in 1883. He created this device when he coated selenium (a semiconductor metal) with a very thin, nearly clear layer of gold on the light receiving side, and it had a metal plate on the other side so as to make a second electrical connection for it to function in an electrical and-or conduction circuit. It had a low efficiency (less than 2%) of converting sunlight energy to electrical energy. Surly this device must have eventually inspired some of the later solid state research, construction, manufacturing and technology such as with the diodes, transistors and other metals used for modern solarcells. His PV (photo-voltaic [pv], photo or light to electricity) solar-cell fabrication would be called today as a form of a Shottky (semiconductor to metal) type of solar cell where one terminal or solarcell layer was not a semiconductor metal, but rather a conductive metal. In theory, an area of a square meter of his pv-cells could provide about $(1000w/m^2)(1.5\%$ efficiency) = 15W (ie., 15 joules of energy per second) of power, hence it is enough to charge a battery over a period of time, and which can then deliver a much higher amount of power (energy / second), in particular a high current for a shorter amount of time when needed.

Before the PV device invented by Charles Fritts, **Alexandre Edmond Becquerel** (1820-1891), from France and who was the father of Henri Becquerel who made discoveries about radiation energy, previously made the initial discovery of the **photo-voltaic (PV) or "Becquerel Effect"** in 1839. With this effect, a generated voltage (emf) is created due to light energy, and therefore, a current through a circuit is also produced. His light sensor and-or (energy) converter used two platinum wires in a silver compound solution. It must be mentioned that at those times, the technical understanding of how these (semi-conductor, layer, junction) devices worked was mostly unclear until the research by Albert Einstein years later, and then by Bell Labs who created the first transistor. The basic process of the photo-voltaic effect is that if a photon of light has enough energy, it can collide with and knock an electron out from its orbit about an atom of certain materials and-or electrons with a low enough (energy) "band-gap" that is equal to or less than the energy of that photon of light.

be replaced in about 1963 with "push button" or "tone" phones which sent fast and unique electric-audio tones to the phone station, and which when electronically decoded, it represented the phone number being requested or called. Many modern phone systems also use the special number of **911** to remember and quickly call for emergency assistance. Most phones today as of the year 2022 are wireless or radio telephones called as "**cell phones**", **"mobile phones"** and-or "**smart phones**" due to all their abilities such as having access to the Internet communication system. Older phones that used a direct wire connection to the phone system and company are now called as "land-line" (ie., using wires, "electric lines") phones.

currently popular, 18650 lithium-ion rechargeable battery has an amp-hour (aH) rating of about 2000 maH = 2 aH. A "protected lithium-ion" battery has a small internal circuit to prevent overcharging and discharging, and these will cost a little more, but it is worth it. Some lithium battery charger devices might include some battery protection circuitry, but do not assume an electrical device, being powered by these (unprotected) batteries, as having any battery protection circuitry within it.

It is possible to make a homemade, portable, and-or emergency power supply for an (electric) **arc-welder** by connecting about three 12v large, high current batteries, such as automobile batteries, in series with thick wires that can transmit high amounts of current due to its low resistance. Technically, 1 to 4, 12v batteries will work, but lower voltages produce a lower amounts of current and-the corresponding generated heat to melt metal with. Higher voltages above 48v are generally not used for arc-welding. Common "jumper cables" (wires) for a typical automobile batteries are rated at about 400A, such as to power the starter motor (ie., "cranking" or [piston] "turn-over" motor) during a brief amount of time, however to charge an automobile battery, usually much less current is used and for a longer period of time (ie., "slow charging").

On the amount of power used from a battery:

A battery may be rated by the recommended, maximum amount of power that can be safely drawn for one hour of time so as to keep it from getting hot. This value does not consider any high power drains of energy during just a brief amount of time, perhaps a minute or less.

Since 1 hour = 3600s, 1wH = 1 watt of power applied for a total of 3600s = 1 joule of energy used per second, and for a total of 3600 seconds, which may be intermittent and actually occur during a much longer period of time, and sometimes with a high amount of energy or joules being used, and sometimes with a low amount of energy or joules being used.
1 wH = (1J/s)(1H) = (1J/s)(3600s) = 3600 joules of energy.   :1 watt-hour = 1wH of energy

To help rationalize this value, consider the energy needed and-or stored by lifting a mass of 1Kg a height or distance of 1 meter: E = (force in Newtons)(distance in meters) = (ma)(meters) = (1Kg)(9.8m/s^2)(1m) = 9.8J =~ 10J. (ie. 10 Joules per kilogram, and per meter = ((10J/1Kg) / m) = 10J/1Kg-m). If the distance was 1000 times more, hence 1Km, the energy needed and-or stored would be 1000 times more at: (1000) 10J = 10000J.

Likewise **1 kwH** = 1000 watts of power applied for a total of 1h = 3600 seconds. 1kWH = (1000J/s)(3600s) =
1 kwH = 3600000 joules of energy = 3.6 million joules of energy = 3.6 MJ of energy = 3.6 mega-joules of energy = 3.6 (10^6) Joules .

Watts = energy / second = Joules / second = J/s . V = J / C , J = VC and A = C/s , C = As , using substitution:
J = VAs = (power)(s) = (watts)(s)   , also note above that: s = C / A = (total charge) / (rate or amt. of charge flow)

1 watt-hour = 1 wH = (1 watt)(1 hour) = (1 watt)(3600s) = (1 watt = VI of energy or power) for (1 hour or 3600s of time)
Note that 1 watt-hour of energy can be obtained by many other ways such as:
1wH = (0.5w)(2) = (0.5J/s)(7200s) = 3600J

1 kwH = 1000 wH = (1000 watts)(1h) or= (1000 watts)(3600s) = (1000J/s)(3600s) = 3600000J = 3.6MJ of energy

A lithium-ion battery rated at 3.6v, and 2000 maH = 2aH of current = 2 amps of max. rated current for 1 hour of time, will have a power rating of: P = (V)(I) = (3.6v)(2aH) (3.6v)(2a) for (1H of time) (3.6v)(2a)(1H) = 7.2wH = 7.2 watt-hours. Since 1 hour = 3600s, 7.2wH = can be thought of as: 7.2 watts applied for a total of 3600 seconds = (7.2w)(3600s) = 25920 watt-seconds = 25920 (J/s)(s/1) = 25920 joules in total. Some simple examples of using this rated power value would be using 1 watt of power for 7.2 hours, or 7.2 watts of power for 1 hour.

On the (output) current and-or power rating of a battery:

A battery may be rated as how long it can supply an amount of current an at the battery's rated voltage. This does not

the sulfuric acid electrolyte) and-or it may then have problems recharging from the electricity generator in the automobile, and it is best to slowly recharge that battery using a low current if this is the case, and so that there is more time to effectively reverse the problem. A damaged battery may have a reduced current output, and may not be able to charge up to its rated voltage, hence its total power output will be reduce.

The liquid electrolyte between the plates of each cell in a lead-acid battery helps the creation of free electrons and current conduction through the battery without shorting out the plates of the cell with a direct connection. Automobile batteries are designed with large lead plates and so as to provide low resistance and to release a large amount of power (energy, joules, Power = (V)(I)), and particularly in the form of a large amount of amperes of current in a brief amount of time so as a high torque, electric "**starter motor"** can "turn" (move) the cylinders in the engine and cause the initial combustion to get the engine functioning without any further need for the high starting power. Once the car's engine is functioning or "running", the battery and a high voltage step-up **ignition coil** is still needed to produce a high voltage spark in the combustion cylinders when a piston in the cylinder routinely compresses the gas-air mixture near the spark-plug terminal. The spark will ignite the compressed fuel and cause it to combust, and the hot kinetic gases will push the piston downward, and this motion and force will eventually be transferred to the wheels of the car, and causing them to rotate which move the car.

A battery will be recharged as needed via a (diode(s)) rectifier and voltage regulator circuit, and while the automobile engine is working and the generator (or "AC alternator") is recharging that battery. When an automobile is not in use, it is recommended to shut the power off to all unnecessary devices being powered by its battery so as to prevent the battery from being drained of its stored energy. If it gets drained, it is possible to "jump start" or recharge that battery by using high power "**jumper wires"** for the connection cables that are usually red and black in color and made of thick wire that connect that drained battery to the same polarity terminals of the battery of a running automobile or battery charger. These cables are relatively inexpensive and should be stored in the trunk of the automobile so as to have them available when in need. Portable and rechargeable "battery packs" can also be purchased for emergency use for when a battery is drained. Automobile battery chargers that use "household" 120Vac power can be purchased and can charge an automobile battery up to 14.4v at reasonably high currents if there is no overheating of the battery. If enough time is available, relatively low currents of just a few amps, say 2A to 5A, is recommended to charge a drained battery. The recharging current to an automobile battery will also decrease as the battery voltage increases, and this is due to that the difference in voltage potential between the charger and the automobile battery is getting less, and there will be less effective voltage available to charge the battery. This is generally why it takes so long to charge a battery. Some lead-acid batteries use water as part of their liquid (acid) electrolyte and it could become low in volume over time and use, and it then needs more water, and rather than use "tap water" with various minerals (ie., elements) in it that can reduce the function of the battery, try to use pure water only, and this can be purchased as distilled or "evaporated" water that is available at many stores.

As of the year 2021 there have been many recent developments in battery materials, construction and better storage ability or capacity. It is promising for the future when people and businesses are becoming more focused on "green (renewable, non-fossil fuels)" energy and energy security so as to have enough electric power available when needed, and this will require more localized power storage and-or generation stations, and of which will also include homes generating electricity from solarpanels on their roofs.

Capacitors store electric charges (electrons on one plate, and positive ions on the other plate) and their energy on its two closely spaced, thin metal plates (ie., surfaces, with area for the charges to fit) with the aid of the electrostatic (or electric field) force between them. Since the charges no longer have kinetic energy are not moving, there is no magnetic field created or involved in the storage of static or stationary electrons. The kinetic energy of the electrons has been transformed into electrostatic (field) energy which is a form of potential energy. Since the charges on a plate are similar, they are in repulsion to each other after being forced onto the plate with an electromagnetic force of repulsion by the same polarity, and attracted to the plate by the electric field of opposite polarity on the other plate. These stored charges can then be thought of as like a spring that has been either compressed or stretched and having stored or potential energy to do things. Capacitors were also mentioned previously in this book. For comparison, an **inductor** (ie., a coil of wire) will temporarily store the kinetic energy from electrons into a generated magnetic (field) energy around that coil.

in a parallel connection with the voltage source or supply, and that capacitor will charge and will be the same voltage as the voltage source, and then the current flow will stop because there is no difference in voltage potential (emf) for any current to be flowing. For example, if the supply voltage is +5v and the capacitor charges up to +5 volts and effectively in the opposite or reverse polarity direction, there is a net voltage difference of 0v, and which will cause no flow of current. A resistor could be placed between the power supply and the capacitor so as to reduce the charging current, perhaps to a safe level, and this will also cause an increase in the time it takes to charge that capacitor up to the supply voltage. The limited amount of current due to the resistor will also decrease even further due to the increasing "back-emf" or (reverse polarity) voltage (emf) of the capacitor.   A formula for the voltage or current of a charging or discharging capacitor is shown ahead in this book. [FIG 220]

CAPACITOR
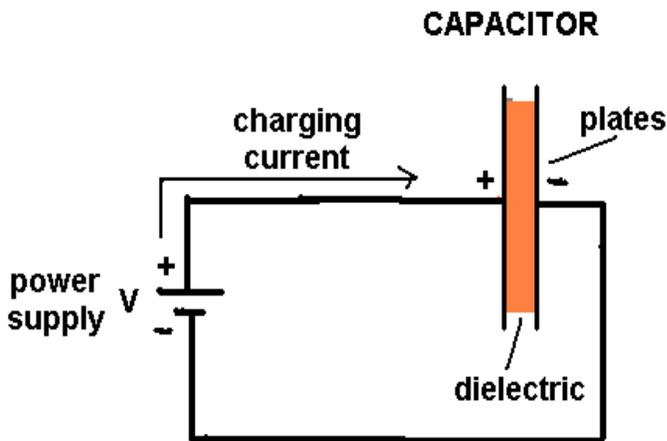
Considering the common conventional current flow analysis from the positive terminal of a voltage to the negative terminal of a voltage or potential difference, and which could be thought of as positive or real energy flow analysis, the positive terminal of the battery will repel and force like charges which then then will gain kinetic energy and also be forced and-or be repelled onto the positive plate of the capacitor as potential or stored energy. Likewise, opposite polarity charge will be stored on the opposite side of the capacitor. The unlike charges across the capacitors dielectric material will also attract each other due to their electric or coulomb forces, and the capacitor will store those charges, even if the input voltage is removed.

To create practical **radio waves** at a specific or tuned frequency for transmission and reception, a stable (ie., non-fluctuation in frequency) **electric oscillator** circuit is required. To receive those same radio waves being transmitted at a specific frequency (such as that being transmitted from a certain radio broadcasting "station" and its transmitting antenna), an electric circuit called a "tuner" in a radio receiver is "tuned" or adjusted to accept just that particular frequency and reject, block or remove all the unwanted frequencies, particularly those higher and lower in the desired frequency and-or a limited band, bandwidth or range of nearby frequencies necessary to contain and convey the source information. A tuner circuit is actually a type of (frequency) filter circuit that allows frequency rejection and-or the selection of a particular output signal and frequency.

An oscillator circuit is designed to (electrically) resonate at, produce or generate a specific AC signal or frequency, and which usually has a (natural) sine waveform. A simple tuner circuit contains just an inductor (L) and capacitor (C) placed in a parallel electrical connection, and which they becomes both a filter and-or resonant (ie., oscillator) tuner circuit. This tuner circuit will allow (pass) a certain radio frequency and electronically reject all other undesired frequencies by sending them to the ground or lower (ie., 0V) voltage potential via a lower impedance or reactance path than the rest of the circuit. The frequency filtering will remove unwanted frequencies from being heard and-or interfering with the radio station and information broadcast that you want to hear. The parallel LC tuner actually has its highest electrical impedance (ie., a (frequency dependent) reactance or effective resistance to the AC signals) when the desired frequency of the input signal is equal to the specific natural oscillation and-or "resonant" frequency of that tuner circuit. The result is that the voltage of the desired frequency will be at a maximum value at the tuners output, and then it will be available to the rest of the circuit

A **pendulum** is a mechanical (force powered, non-electric) oscillator. It contains a mass, usually called a bob, that is held in position by a length of a string or or wire. The mass at the end of the pendulum can freely swing (ie., move) side to side and has a (desired) constant time or period of movement during each swing cycle or oscillation. This time period is determined by the length of the string. The longer the string, the longer the arc length or distance of motion of the pendulum mass object (or "bob") and the greater the distance of travel and time of travel, and therefore, the slower the oscillations and longer period time, and less frequency of those cycles per unit of time. The pendulum will cycle through converting its gravitational potential energy (after a person raises it a height) to kinetic energy as it swings. It is interesting to note that the mass and-or weight of the pendulum does not determine its period of oscillation, and the reason for this is that masses of different amounts of weight will fall at the same speed due to being influenced by the same gravitational acceleration (g = 9.8m/s^2 for Earth). Different sized masses and-or weights will also reach the ground or lowest part of the pendulum swing cycle at the same time when raised to and dropped from the same height. A larger pendulum mass will contain or store more kinetic energy ((mv^2)/2) and have more momentum (p=mv) so as to keep swinging, but a heavier weight may also cause more friction due.

Pendulum Swing Period = Ts = (2)(pi) (square-root of (length / g)) : theory, and accurate for small angles of swing. Credited to **Huygens** in about 1673.

In the above formula, length is measured with units of meters form the pivot to the center of mass and-or weight, and time is measured with units of seconds. Note that: frequency = (1 / Period) = (1 / Ts) hz or cycles per second. For a practical example, a pendulum with a string that is 1m long will have a (total swing, cycle) period of 2s to reach the same location it started from, and-or 1 second to swing from say left to right. A pendulum with a string that is 0.2485m = 24.85cm = 24cm + 8.5mm, (nearly 25cm), will have a (swing and-or oscillation) period of 1s. Note that 25cm = (1/4)m. If the string length increases by 4, the period increased by 2, or doubles, and this is the square root of 4. Because (g) = gravitational acceleration is in the period equation of a pendulum, a pendulum can also be used to measure the local gravity in an area. A (very near, metric) standard of 1s unit of time, and-or 1m unit of length can also be verified by a pendulum.

Although it is fairly easy to find the local (north, south) latitude by observing and measuring the angle that the North star (Polaris) is above the local horizon line, it is not so easy determining your local (east, west) longitude. It would take an accurate clock set to Greenwich local time in England. At you local longitude, when the Sun is the highest in the sky, it is local noon or 12 P.M. If the accurate clock shows 1P.M., this is still the local time in Greenwich, and you are 1 hour behind that time, and therefore, you are 15° west of Greenwich.

Some oscillators are composed of just transistors, resistors and capacitors, however some radios use crystal oscillators for the frequency transmission and reception. Transistorized oscillators are mainly based on the fact that a capacitor takes a certain amount of time to charge up when a resistor is in series with it. The first type of oscillators used for tuning (or selecting) a specific frequency radio waves are called "(LC) tank circuits" or "(LC) tuned circuits". This circuit is simply a capacitor and inductor electrically (with a wire) connected in parallel to each other. When energy is applied to this circuit, it will resonate or oscillate current (electrons) "back and forth" from the inductor to the capacitor, and then back to the inductor and to repeat this process "over and over" periodically at a constant amount of time and frequency called the (natural or electrical) resonant frequency of that circuit. Smaller valued capacitors and inductors will have a lower time constant (RC and-or RL) for charging and discharging, and the natural frequency of that oscillator will be higher.

For a periodic signal and its waveform from an oscillator or repetitive circuit: time = 1/frequency , and frequency = 1/time, hence the time and frequency values are mathematically reciprocal in value to each other. Time is the number of seconds to complete one complete cycle or repetition of the wave, and frequency is the amount of cycles per second of that wave. This will happen for as many cycles as possible and before the energy losses in the resistance of the metal connection wires essentially deplete the energy left or available, and the output voltage and current of the oscillator gets lower and lower in strength (ie.,amplitude or peak) until it is practically useless to the rest of the circuit. To keep the oscillations going and at a strong amplitude, a resupply of (external) energy to that oscillator circuit is needed. A transistor can be pulsed on for a short duration so as to supply some energy to an oscillator so as to keep it going. Due to the electric current or flow of electrons with their electric charge and kinetic energy, the inductor (ie., coil) stores its input energy in a

vibrating string are anchored to or connected to the body of the musical instrument. Vibrations and-or the energy of the string will be transmitted or conducted to that musical instrument's vibration "soundboard" or (energy, frequency, vibration) resonator surface, and these vibrations will then be converted into (compressed, pressurized, more dense) air vibrations that are radiated or transmitted outward to the less dense air about it. These compressed and vibrating air waves will radiate or travel outwards from that source, and can then be heard as audio (sound waves) and having the same frequency(s) as that of the vibrating string. [FIG 221]
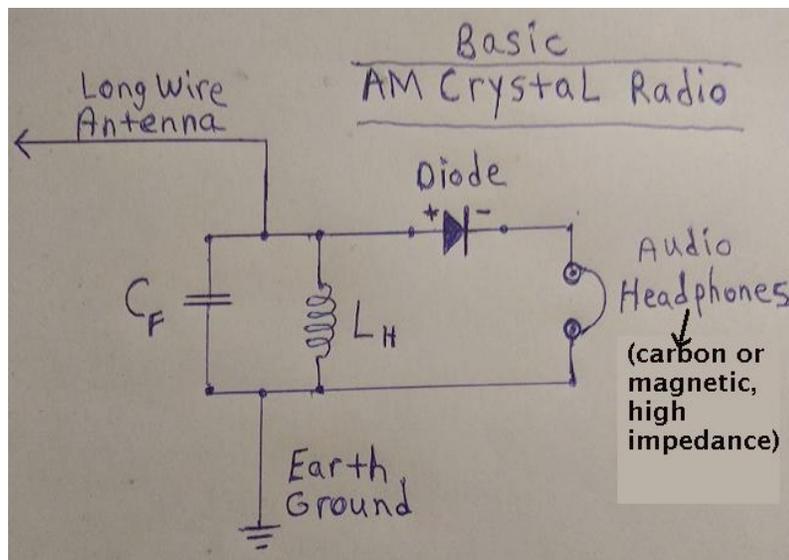
For experimenting with an AM, crystal radio circuit as shown in the above drawing or schematic, it can be pre-made, homemade and-or purchased as a kit to build. The following discussion is of the typical common parts, and in particular, of the typical capacitor and inductor values used for the tank or tuner circuit so as to be tuned approximately (via the above calculation) to a common 1Mhz AM radio frequency. Due to this receivers low tuning selectivity (ie., rejecting other close frequencies) issues, you will probably hear some stations "overlap" or interfere with each other, nonetheless, you should be able to at least hear one nearby, "strong" (high power and-or close distance) station so as to verify it is working and-or to help your circuit and reception experiments continue.   There are many nice books about understanding and-or making and-or improving crystal radios. Some radio receiver circuit kits include all the necessary parts, including the high impedance piezo-crystal earphone. This type of earphone is a very voltage sensitive, miniature speaker that can flex and-or vibrate when a small voltage is applied, and the amount of current needed is very low, almost 0A. A piezo-crystal has a very high impedance since it is nearly a non-conductor material, and a piezo speaker or buzzer works via the supply voltage creating electro-static forces which will then physically bend the crystal material, hence causing a electro-mechanical vibration of which can then be transmitted to and through the air as sound.

A , the **A**ntenna, 10 ft to 100 ft of insulated "hookup" (connection, or speaker) wire strung off the ground and not
   during any possible high power (voltage and current) lightning, of which the antenna will essentially become a lightning (directing) rod. During a storm or potential storm, disconnect the outside portion of the antenna several feet from your dwelling, and having several feet of unconnected wire and-or gap (space).   Do not connect the antenna or any other other part of the radio circuit to an electric power or telephone line which can then cause injury and-or fire. Because of the preceding facts, it may be best to place the antenna indoors, but if it is long, it will have to be bent so as to make it fit, and-or raised up near the ceiling so as to not cause a potential problem for the movement of people and pets.

   In theory, the antenna length should be equal to the wavelength of frequency of the radio wave, and be perpendicular to its direction of travel, so as the antenna has maximum interception, reception or signal collection and efficiency of converting the received (rf) energy to electricity energy as the antenna, and much like a musical string, can electrically resonates at the same frequency as that of the desired radio frequency.

   Modern portable radios have a relatively small antenna that is only a few feet long and-or is part of the tuning coil

cannot vibrate much with high frequency, (rf) electric signals, and so as to efficiently converted them to sound. These high frequency sounds can not even heard by the human ear anyway. It needs to be noted that the diode has a fairly high signal impedance (ie., resistance) to the low low voltage (and current) rf-input input signal from the antenna and the output from the LC "tank" circuit, frequency selector, filter or tuner.   A Schottky diode is recommended since it has a low voltage drop when used with low currents. The received (rf) signals are often very low in voltage, and therefore, a very low current is being generated and flowing in the circuit. Having a lower diode impedance allows more of the received or input (rf) signal's voltage to be applied to the headphones and produce a louder sound. Perhaps the most popular diode for a crystal radio is the germanium IN34, however nearly any other type of diode can be used, but some diodes may require a forward bias to set the diode near its conduction voltage, and since the voltage available from the received radio signal is too low to even allow the diode to function. Reducing the available voltage will reduce the volume of the sound output from the high impedance earphones. A simple method to bias a diode is to use a 1.5v battery cell and an adjustable resistor, say 50K to 100K ohms so as to limit the current to a very small value.

Headphones , Headphones are essentially small speakers that have only a relatively small amount of power output and are therefore placed close to your ears. Headphones, like a loud-speaker, are an electrical energy to audio or sound energy transducer or converter, and can therefore be considered as a type of (energy) transformer. **WARNING**: If the sound is too loud, and-or for too long, hearing damage will usually occur.

For crystal radios, use high impedance carbon, or (fine wire) magnetic earphones which contain a high impedance to a low impedance transformer (2 coils). A relatively inexpensive "crystal (piezo)" earphone can also be used.

You can also use a high input impedance audio amplifier with a speaker level signal for its output , and this will also amplify the weak, input audio signal to be used for an 8 ohm impedance, loud-speaker or headphones output. The output signal of the high impedance amplifier or circuit may even be sent ("ie., "connected") to a computer's "line-level [a certain std. max. amplitude]" signal input. A "line-level" input typically has about a 10k ohm input impedance, and about 1mv max. peak to peak, maximum input signal before (amplitude) "clipping" or limiting of the signals waveform takes place and the signal gets distorted. The audio signal from a (non-crystal) radio can also be sent to a computer amplified ("electret") microphone input (~1k ohm input impedance, and ~100mv max. peak to peak input signal). The computers audio circuitry can sometimes amplify these input signals and-or send an output signal to an amplified speaker system.

For the crystal radio's tuner circuit to remain at high impedance to the input signal, and for maximum circuit efficiency, anything connected to that tuner or LC circuit needs to be a very high impedance since it is essentially connected in parallel to it and will reduce the impedance of the tuner like how two parallel resistors effectively creating a lower resistance. Then for maximum power transfer to the headphones, those headphones will also have to be a high impedance type so as to match that of the high impedance tuner and diode.   For maximum power transfer from one impedance to another, those impedance's should "match" or be the same value, and this concept of "impedance matching" is used for the concept of maximum power transfer and vice-versa. Some advanced crystal-radios may use a high impedance to low impedance transformer to match the circuit impedance before and after the transformer such as to the headphones, and so as to have greater efficiency and some more audio volume.   Using common 8 ohm to 32 ohm headphones will not work for this circuit since the output power of it is too low to physically move or vibrate these speakers so as to make sound (air pressure waves), and there is also an impedance mismatch which will reduce the electric power transferred to the earphones.

G , **G**round, a wire going deep into the ground, such as being connected to a metal radiator and-or water pipe.

"**Hookup wires**" are quick connection wires, and are therefore helpful for quickly building, testing and experimenting with all kinds of electronic circuits. These conductive wires have a small spring powered clamp (or "alligator clip") at each end so as to easily connect (or disconnect) and hold the wires and-or components electrically together without using soldering

A BASIC TRANSISTOR AMPLIFIER CIRCUIT [FIG 222]

It = total current

Ib dc

small AC input signal

Physical Construction Q

N P N

E   B   C

V +

Rb

C1

Rin

common or circuit ground

Ib →   B

Rc

C

Q

E

Ic = (gain)(Ib)

C2

large AC output signal

R load

Packaging Typ. Bottom View

Pins

C
B
E

Q
NPN

R load = perhaps an 8ohm , AC impedance audio speaker

Q = transistor , the arrow indicates an NPN type
B = base , C = collector, E = emitter
Rb = base resistor , Rc = collector resistor
C1 = input capacitor , C2 = output capacitor

Ib = base current
Ic = collector current
Ie = emitter current = Ib + Ic

Rin = input resistor, fixed or variable to set input level, this can also be a potentiometer (adjustable, variable voltage potential setter, usually via an internal resistance and voltage divider) with one terminal connected to the ground, and so as to divide the voltage of the input signal so as to have the desired level to the transistor base from the center terminal of that potentiometer. A safety resistor can also be put before the potentiometer, and provide some transistor safety in case the input signal has too much current and the potentiometer is set to a 0 resistance.

An electronic amplifier will amplify (increase, gain, magnify) the power of an input signal such as an electronic, AC waveform equivalent, representation or facsimile of a sound waveform. Power = P = VI = (voltage)(current). The increase in power can come from an increase in output voltage and-or current of the input signal. Since the input signal controls the transistor amplifier, the output signal of it will be of the same waveform shape and frequency (cycles per second) as that of the input signal. This transistor amplifier circuit and-or configuration is called a "common emitter" since the emitter is common to both the input circuit and output circuit:

Gain (A) in voltage = Av = Vout / Vin , Ex. If Av = 10 and Vin = 0.1v = 100mv,   Vo = Av (Vin) = (10)(0.1v) = 1v
Gain (A) in current = Ai = Iin / Iout
Gain (A) in power = Ap = Po / Pi = (Av)(Ai)

The values of the components in the circuit are determined by many factors such as the input and output impedance needed so as to have maximum power transfer. An FET transistor has a high input impedance, and rather just amplifies the input voltage. The transistor (Q) in the above circuit is a NPN polarity, BJT (Bipolar Junction Transistor) and is mainly

How does this signal inversion happen in a single stage or transistor amplifier? Signal amplification by a transistor happens since when the input signal voltage increases or rises, it puts more current (ie., electrons that can carry charge) into the base or central region of the transistor and it then can conducts more, and therefore, the transistors internal resistance (R) decreases. The voltage drop across the transistor will decrease since V = IR. Since the output signal is taken from or across the collector and ground, its voltage will also decrease.

The capacitors (C) shown in the circuit above are used as "(AC signal) bypass" and-or "(AC signal) pass through" capacitors which will block a steady DC current, and also keep the bias settings of the transistor constant by isolating it from any external (ie., such as input and-or output) circuit DC power. The capacitors also allow AC signals to flow into and out of that amplifier circuit. Capacitors and-or inductors (such as a "choke") can also be used as signal filters that can block (not allow) and-or pass (allow) certain unwanted (such as noise) frequencies, and for example the filtering depends on the (Farad, charge storage) value of that capacitor used and the frequencies of the AC or varying signals. In general, smaller capacitors tend to block low frequency signals and pass high frequency signals.

A "load" resistor or device can receive or "draw" some of the available signal (v, I, power) from the transistor amplifier circuit. This "load" is also considered as a "load" (quantity or value) of current, hence a "current load" or quantity. A common load device for an electronic audio signal amplifier is an 8ohm speaker and-or coil in it. An 8ohm speaker is a common or standard value that a circuit can be designed for. Technically, 8ohms is the impedance or effective AC resistance of the coil when a 1000hz frequency, electric and-or audio signal is applied to it. This is due to the magnetic field and "back-emf" (in reverse polarity and-or opposing that of the source) voltage created in the coil when there is a change in current such as during the applied AC signal's waveform. The DC resistance, or the "wire resistance" of this same speaker coil is about 6 ohms, and the impedance of that coil will rise as the frequency of the input signal increases.

Since a class-A amplifier is biased to be always ON (functioning) with current flowing thorough it, a current on-off switch should be used so as to not waste any battery energy or power when the amplifier is not in use.

The above circuit for a transistor mostly considers the necessary DC (steady, fixed, V, I, R) biasing or operational settings. What happens to the AC signal is somewhat of a different nature. We want to consider both the positive and negative halves of the AC signal, and not just the positive half of the AC wave (ex., a sine wave). A simple description of what happens is that the incoming AC signal to this amplifier will electronically "sees" or "encounters" the battery with its low internal resistance, and which can then be considered as 0 ohms in ideal conditions. This effectively shorts the positive and negative "rails" or conductors together for the AC signal. For example Rb will then effectively be connected across the input of the AC signal, and Rb will also essentially be across or in parallel to the transistor. Rb will then also be part of the AC signal's encountered input impedance of that amplifier, hence it will also be a part of the AC biasing of that amplifier. Consider if the DC biased transistor had a very low internal "on" impedance or resistance, then Rc is also essentially connected across that transistor in terms of the AC signal.   For a class-A amplifier circuit as shown above, and for the most signal output, the value of Rc could very well be set to the average value of the transistors internal (collector to emitter) resistance. Rc will also help determine that amplifiers, AC signal's output impedance seen or available for any circuit connected to it, and of which should be matched for best power transfer, to or by any connected device for best power and -or signal transfer. A typical value for Rc may be about 1000 ohms = 1k, but it may be as low as 10ohms, or as high as 50k ohms. A topical value for Rb may be about 10K ohms, and since the DC bias base current to a transistor is relatively a small value, say about only 1 milli-amp (mA) or less.

More Advanced Uses Of Transistors And Their Manufacturing Technology

Soon after the discovery of the transistor and having affordable, stable and practical versions of it, the transistor began to be used in many circuits and many with new designs, and where a vacuum tube is no longer needed which made many circuits price prohibitive to even consider for manufacturing for the general public. The first transistor and "pocket (size)" radio was the **Regency TR-1** and it was available for purchase in 1954. The first transistor based computer was made in 1955 and was a closet sized, "mainframe" type of computer, but it was still much smaller than the room-sized vacuum tube versions available at the time. It was also much faster.

Some rare discoveries indicate that ancient Greeks had some primitive magnifying lenses. These are called **magnifier lenses** or glasses. Their rarity is mostly likely due to that clear glass manufacturing was rare and-or that larger clear crystals were difficult to find and-or shape. It is not too difficult to imagine that many were broken after several years of time by accident. In nature, a drop of water is similar to a magnifier or vice-versa, and its curved surface surely indicated what shape of clear lens is necessary so as to make something appear bigger, and which eventually lead to both the telescope and microscope image magnifiers, particularly with two or more lenses.

Sir **Issac Newton** began to study light and optics in the late 1660's and was one of the first people to scientifically study and publicly demonstrate (in 1672) that a beam of white light can be separated into a rainbow of 7 main colors by using a specially (triangular) shaped piece of clear glass called a (light or optical) **prism**. A rainbow or spectrum of colors of white light can sometimes be seen being emitted (ie., refracting and-or reflected) by some crystals or gems such as quartz and diamonds when in the bright sunlight, and sometimes when sunlight is reflecting from ice (frozen water) or snow (small frozen ice crystals, crystallized water). He then showed that these separate colors of light can then be combined back into a beam of white light by using another prism. Newton proposed that light is made of particles instead of waves. Today, as of the year 2020 with the popular "wave-particle duality" theory of light, it is not too difficult to imagine light as being a wave of light particles, much like a sound pressure wave that is a wave of air particles having a varying density (due to varying force or pressure upon the air from the source of the sound) or amounts, and of the same wave frequency as the source of that sound. A color is due to that it is a certain frequency of electromagnetic and-or radio frequency (rf) energy, and therefore, that energy has a periodic waveform and can be called a wave.

A magnifier and-or telescope lens that was ground and polished so as to have a high quality and without image distortion will have no apparent "chromatic aberration" (ie. color and image distortion) which is a prism-like effect of separating the received light from the image and into a visible rainbow of colors, and particularly due to the outer edge or rim area of the lens where the incoming light has a greater distance to travel and separate into its component colors.

A simple magnifier lens can be used to project an inverted (ie., "upside-down) image of an object onto a white surface such as a piece of paper, and this is the essence of a camera device that is used to make photographic images. In a camera, an image is made when the light coming from an object or subject passes through the lens and to be focused onto a light sensitive material (film, or a photo sensor array) and recorded or stored as an image of that subject. The first cameras often had a pinhole for the lens, and which only allowed a small amount of light into the camera, and was therefore a slow camera system, but it provided a sharp or well defined image, and also allowed a large distance or depth of field that was in focus. Many simple lens systems, including a the pinhole opening and even many large telescopes initially will show the viewer an inverted or image of the subject. This has to do with the way light rays from a subject will pass through a lens and are diffracted (ie., bent by some angle).by that glass and shape of the lens.   If need be, another lens can be used to upright an inverted image.

Today, as of 2020, the costly light sensitive paper or film material has been mostly been replaced by a light sensitive electronic image sensor in an (electric) digital camera. Here, digital means that the camera image senor saves the light or pixel (picture element, a small part of) data of the image as binary (1's and 0's) numeric data in the cameras memory and-or a file. Digital cameras have made photography very inexpensive, quick and practical for the average user. Most modern "smart" phones (ie., telephones) have a built-in camera(s) and internal computer circuitry so as to access the (electronic) internet communication and information, and-or run useful programs (ie., "apps", applications) such as a photo editor, perhaps to increase the brightness and-or contrast of the photo so as to make it more presentable and-or printable.

As mentioned previously, the light energy not reflected off an object is usually converted directly into heat energy in the atoms of the object. If a photon or "particle of light" has enough energy, it can collide with an electron and free it from the electric force of an atom by increasing the kinetic energy of that electron, and this creates electrical energy and this is the basis of how **photo-voltaic** devices such as solarcells function. Current in a circuit is electrons that are free to move and have kinetic energy, and can be influenced or directed to flow via an electric force connected to that circuit.

A photon or particle of light energy striking a **solarcell's** atoms can have enough energy or force (the application of energy) to free an electron from the atom it is bonded or electrically attracted to. Solarcell is similar to and associated with the words: Photo-voltaic (light energy to voltage, electrical energy or power), Solar-voltaic and cell such in reverence to

them in parallel to one another, and with all their positive terminals connected together, and all their negative terminals connected together. If the solarcells are (and should be) the same size, type and have the same electrical characteristics (ie., "ratings", output voltage and current), the output current of parallel connected solarcells will be the sum of current from each solarcell, but the output voltage will still be 0.5v for parallel solarcells. As indicated, when connecting solarcells together, either in parallel and-or series, to help avoid potential solar-cell damage, such as over-heating, solarcell damage and safety problems, it is recommended to use (identical) solar cells of the same type, ratings and size. The wires carrying the output current from a solarpanel (series and parallel connected solar cells, a "battery" or array of solarcells) and-or battery should have a maximum current rating that is greater than or equal to the maximum amount of current that will be flowing through it.

Because sunlight energy has a maximum amount of energy per square meter, a solarcell collecting that energy is limited to receiving a certain maximum amount of energy, and therefore its output electrical energy will be limited to a certain maximum amount of watts which is determined by the energy conversion efficiency of that particular solarcell and-or solarpanel. An 18% to 20% energy conversion efficiency is a typical value as of the year 2020. For example, if an 18% efficient solar panel receives 100 watts of sunlight power, then (100w)(0.18) = 18watts of it will be converted to electrical energy and power (energy/time, voltage and current), and most (100% - 18% = 72%) of the rest will be converted (and usually wasted) to heat energy, and some will be reradiated or reflected. If heat energy is desired in the form of warm air, a **solar air heater** can heat air for free after it is either purchased or made and installed, and with the sunlight and energy shining on its black, heat-absorbent collector surface inside an insulated rectangular box-like shape, and located a few inches beneath a layer of tempered glass. **Solar air heaters are very efficient, often 80% to 90% efficient at converting solar light energy directly into heat energy.** A solar air-heater is also less expensive (perhaps, half the price) than a solar electric panel of the same size. Due to the high conversion efficiency of a solar air heater, it is not recommended to use a photovoltaic solar panel to make heat (warm air, such as to heat a room or house) with an electric air heater of which requires high current and will quickly drain your energy storage batteries. Clear glass or transparent plastic will allow the lower frequency, near-infrared solar radiation to pass through it, and this is ideal for warm greenhouses where plants are grown in cooler climates and-or weather and seasons of the year. Glass will block much of the higher frequency light and-or energy such as ultra-violet light (UV).

If a solarcell is rated at 0.5Vdc and can produce 4A of output current, the theoretical or effective internal resistance of that energy source can be found using Ohm's Law: $I = V / R$, therefore, $R = V / I = 0.5v / 4A = 0.125$ ohms = (1/8) ohms = "an eighth of an Ohm". If the solarcell is producing less than 4A due to that there is less sunlight on its surface, its internal resistance will be higher. Here is how to calculate the output power: The maximum amount of output power of this example solarcell energy source is: $P = (V)(I) = (0.5v)(4A) = 2watts$ = 2 Joules of energy per second.

A **solar panel** (or **solarpanel**) is simply a convenient array of electrically connected solar cells so as to increase (ie., multiply) the available output power which is many times more than that of a single solarcell. Many modern silicon solar cells are in the range of about 18% to 24% efficient, and are often priced according to their efficiency, and the 18% type is very common as of the year 2021. That is, they will convert 18% of the light energy into electric energy, and therefore waste (100% -18% =) 72% of it when its converted to heat (molecular vibration) energy on the surface of the solar cell and-or having a low (light to electric) energy conversion inefficiency to begin with and is therefore incapable of a higher efficiency. In terms of efficiency, they would make a better solar heater than an electric generator, but the usefulness of electricity in our modern world of electronic technology (devices, machines, lights and phones, that need and use the power of electricity to function) is important enough to tolerate some energy losses or waste. Energy availability or security is important, and reducing wasted energy or losses by creating more efficient solarcells and devices is an ongoing challenge.

The available and-or maximum amount of solar energy per square meter is about 1000W/m^2. Because of cloudy skies, sunlight incidence (application, input) angle, fixed positioning as opposed to more efficient solar (Sun) tracking or aiming, internal resistance, wire and battery resistance, diode(s), voltage regulators and converters, etc., the actual power available from a solar panel, especially the current, will be reduced below the maximum possible value and-or the solarpanel's rated output value. To help overcome these types of problems, and have a more consistent or stable amount of expected power, the solar panel should be designed and created so as to produce a slightly higher voltage or output power, say 30% more than what is actually needed due to an estimated 30% energy loss or waste in entire solar electric

Connecting solar cells in parallel means more total surface area and more current generation without the associated power losses due to solar cells connected in just a series configuration where the total resistance of that circuit increases for each solar cell used. Connecting solarcells in series will not increase the current because the effective or total resistance of each solarcell in series is then a multiple of the resistance of one solarcell, and this high resistance will limit the current to just that from one solarcell. Placing solar cells or panels in parallel is also problematic since one may be at a different output voltage and-or resistance than the others and they will electrically conflict to some degree. In that situation, it is best to always use an output power diode on each solar panel so as no current can then flow back through it in the reverse direction. The preferred type of diode to use with a solar energy system is of the Schottky diode variety which has a low forward or "turn-on" voltage and power loss, but any high current (perhaps 6A to 10A max.) power diode should work reasonably for relatively low power systems. You will have to use a diode capable of safely handling or passing the expected maximum current, and power loss in it - $Pd = (Vd)(Id)$, for a solar electric generator system (SEGS).

Solar cells and-or solar panels can be compared by various factors such as the advertised or rated solar energy to electricity conversion efficiency which is typically about 18% for the (lower cost, blue color) high quality amorphous ("multi-crystal mix") silicon solar cells, and 22% for the more modern (year 2020) (black color) "mono-crystalline", single or solid crystal silicon. Solar cells and-or solar panels can also be rated and compared in terms of maximum power, voltage, and current output, or possibly even amps (current) or power output per square centimeter of surface area. Solar cells and-or panels can also be rated and compared in terms of price per output watt. On a clear day, the sunlight energy per square meter is about 1000 watts per square meter of Earths surface area when directly illuminated by the overhead Sun and its sunlight energy. Since there is (1m x 1m) = (100cm x 100cm) = 10000 square centimeters per square meter, that is about (1000 watts / 10000 square centimeters) = 0.1 watt / cm^2 = 100 milliwatts per square centimeter. If the efficiency of the solar cell or panel is 20%, then the wattage per square centimeter is reduced to just: (0.1watt)(0.20) = 0.02 watts = 20 milliwatts (mW). The (max.) current generated per square centimeter is therefore: $I = Pw / Vv = 0.02 / 0.5 = 0.040A = 40$ mA per square centimeter in ideal conditions.

As of the year 2020, the total cost of a pre-made solar panel with an aluminum frame and a tempered safety glass protective layer is as low as $1USD a watt. Some 100W lightweight, more portable solarpanels do not have the aluminum frame and heavy glass, but still have a protective plastic layer coating are as low as $50USD, hence 50 cents per watt. These somewhat fragile solarpanels can be "stiffened" or "firmed up" and protected with a lightweight wooden square frame on the back and sides, and-or attach a "tilt stand" so it can be aimed more directly at the Sun. You can search websites such as the Amazon.com and-or Ebay.com for solarpanels and other supplies for a solar electric generator system (SEGS), but locally in your area there may be a store such as Harbor Freight, Lows, Walmart, and Home Depot, etc. selling these products and having experienced workers there who may help you choose and-or install a solar-electric system. Many stores also have an associated internet website where you can browse many items both shown and not shown at the store location(s), and to then make purchases "online" via the Internet, and possibly have the items delivered to our house. Sometimes shipping is "free" or relatively inexpensive. A purchase helps both you, your community, the store's business and the product's manufacturing company so as the supply and demand of beneficial things can continue.

To make a complete and functionable basic solar electric generator system (SEGS), this is what is often recommended:

1. **Solar Panel** A solar panel (solarpanel, or solar-panel) is an energy collector, and specifically, it is a solar or light energy collector. On a more technical level, it is also an energy converter or transducer since the energy of the light (ie., the energy of the photons) is converted to another form which is electrical energy.

    This usually includes a blocking diode so as the battery or other electronics does not send power back through that solar panel and causing the battery to drain power through it and-or preventing damage to the solar panel. This diode will have about a 0.7v drop or loss through it, and therefore a power loss in it that is equal to Pd = (voltage across the diode)(current through the diode). Ex. (0.7v)(5A) = 3.5W

needed for running high power devices such as a motors and other devices, and especially at night when the Sun is not shining and delivering light energy to the solar panel(s).

Charge a battery at a recommended amount of current, say 10% to 20% of its amp-hour rating. A battery will get warm while charging because it has an internal resistance to the current flowing through it. As the battery increases in voltage, it will effectively become a "back emf" (ie., reverse voltage) and where the net effective sum of voltages in series creates a less effective supply voltage and-or voltage difference to be applied the charging battery, and this will also reduce the amount of charging current to the battery and increase the charging time. Still, this is a safe way to charge a battery. To then get more current into that charging battery, the charging resistance to it can be reduced, but do not increase the voltage supply of the charger as an attempt to get more current into a battery since this may lead to overcharging the battery, and it will then have a higher voltage than the safe rated maximum voltage that it can be charged up to.

[FIG 222B]
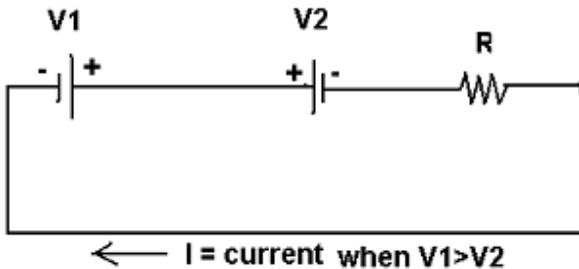
V1 = supply voltage , emf

V2 = charging battery or capacitor, emf, but since its polarity is reversed than that of V1, it is considered a reverse and-or back emf.

When V2 = V1, Vt or the effective charging v or emf = = V1 - V2 = 0, and I = 0V / Rohms = 0A

R is current limiter, for a safe amount of current. $I = \dfrac{\text{Effective charging V}}{R} = \dfrac{Vt}{R} = \dfrac{(V1 + (-V2))}{R} = \dfrac{V1 - V2}{R}$

4. **Inverter**. This is also called a power inverter. It will take the 12vdc to18vdc output of the solar panel and convert it to common household power of about 120vac, as used in the USA. Like the energy storage batteries, inverters come in all manner of sizes, maximum rated output power (watts), and corresponding prices. The trade off of increasing the voltage using a transformer-like voltage step-up system like an inverter, is a decrease in the available current, but the available power, less some small internal losses from the inverter, will be close in value to the input power. If the voltage boost or increase is 10, the current output will decrease by the same factor, hence it will be a tenth (0.1) of that at its input. An advantage of this is that cheaper, thinner output wires from the inverter can be used. With just one large capacity 12v battery, an inverted can be used to temporarily run high power devices such as tools (drills, saws, etc.) that contain high power motors which require a high input power to function properly. Two 12v solar panels connected in series and two 12v batteries connected in series can create a 24v solar electrical generator system (SEGS) of which the current, from the battery to the inverter, needed for a certain amount of power will be half as that of a 12v system, and this will reduce the cost of the inverter (especially if its a true or pure sine wave output type having thick wire coils for higher currents and power output ability) and have less (here, half) the battery drain because the voltage is now an increased factor of the power used, and therefore the current can be less, and at a safer and cheaper value with thinner wires needed to deliver that amount of power.

If a DC to AC 120v inverter is rated at 600 watts, the maximum output current is: From: P = V I , I = P / V = 600w / 120v = 5 amps. Since the inverter essentially boosts the input voltage of 12v to 120vac, a factor of 10, the output current will be reduced by a factor of 10, hence to use 5A of current at the output, then (5A)(10) = 50A from the battery will need to be input to the inverter. The cable to do this will need to be rated at 50A or higher, and particularly if it is long, rather than just a short distance of say 3 feet. The battery will also be needed to be rated as being able to safely deliver 50A max, hence use a 50aH rated battery. Pin = (12v) (50A) = 600w = Pout = (120v)(5A)

=0.125mm = 125um, about the thickness of an index or playing card, and are very brittle due to being a rigid glass-like crystal structure and will easily crack if they are bent and-or too much pressure is placed on their surface. For comparison, it is slightly thicker than a human hair which is about 0.1mm = 100um = 0.0001m thick. To cut solar cells is difficult, and special lasers or saws and methods must be used. Silicon dust is **toxic** to breath and therefore a mask and eye protection must be used when cutting them and cleaning up any solarcell dust. The best cutting methods seem to be with high power lasers. For most people, the best and modern low-cost (actually often cheaper than building a solarpanel) option and recommendation is to buy a completely made solarpanel(s) and other parts for a solar electric system of the size and power needed. Some full sized solarpanels are heavy and are not very portable for a person to carry, while others are small and inexpensive, and can even store energy in an internal battery bank until it is needed.

A solar cell area that is cut into a smaller area will should have an output current of:

$$\frac{A\ cutcell}{A\ uncutcell} = \frac{I\ cutcell}{I\ uncutcell}$$

: same ratio , I cutcell is the current from a solarcell that was cut to a smaller area.
If a solar cell is cut into (n) equal areas:

$$\frac{A\ uncutcell}{n} = \frac{I\ uncutcell}{n} = I\ cutcell$$

How big should a solarcell be cut for, or purchased to charge a given battery? Since it is usually recommended to charge a lithium type of battery at a maximum of 10% of its rated amp-hour (aH) rating, a solarcell or solar electric system only needs to to have a maximum output of 10% of this aH rating. It a battery has a 1 amp-hour rating, the charging current at 10% of this value is therefore, 1A (10%) =   1A (0.10) = 0.1A = 100mA. Use the above formula to solve for the corresponding area of the solarcell(s). Don't forget that a recommended diode will also drop the available voltage of the solarpanel by about 0.7V, and therefore, reduce the available current to your battery and-or device. To overcome this, design the solarpanel so as to be 0.7v higher than needed, and probably a bit more at 1v since two solar cells will produce 1v. Another option is to use a higher voltage solarpanel and a voltage regulator and current limiting and-or safety resistor to charge a battery, and this method also eliminates the need and expertise to cut solarcells.

Some new solar panel designs use "half cells" (half-size of usually a full 6" x 6" solar cell) so as to get a higher voltage from a smaller solar panel that has many more solar cells in it. A theory is that this will also reduce the power loss in each cell a little since the internal resistance will be less, and therefore making the solar panel a small percentage more efficient. Just remember that output current is charges (electrons), and that is what is actually needed and used to charge (accumulated, to be stored) a battery so as to store power. If the charging current is low, it will take that battery longer to charge.

If the output current is lower than expected in full sunlight, the solarcell might have what is called micro-cracks. You can determine if a solarcell has micro-cracks by shining a bright light through it from the back side, and then observing for them.

Some newer solar panels may include some solarcell **bypass diodes** that will conduct current around any (temporarily) shaded cells so as to prevent those high resistance, shaded cells from getting hot, creating a power loss and possibly being destroyed, and then ruining the function of the entire solar panel. If your solar panel does not have these, then prevent any shading on the solarcells in a solar panel, such as not letting people stand too long between the solar panel and the sunlight.

Here is a photograph of a basic solarcell kit that can be purchased so as to make a homemade solarpanel by an experienced hobbyist. The photo shows a stack of 3in by 6in solar cells, the dark solar cell surface with many thin, lengthwise conduction wires on it, and the back of a solarcell with its electrical connections. Also shown is a 40ft spool of thin "tabbing wire" so as to connect each solar cell in series, and a 4ft spool of wider "bus wire" which can carry more current safely. A large 6 amp diode is shown next to a U.S. penny coin (0.75in =~ 19mm wide, diameter) as a practical reference size and measurement. Generally, though this seems like the right and economical thing to do, these kits are intended for knowledgeable and-or experienced builders, experiments and-or for some type of custom solarpanel designed for a certain size and-or output voltage, current and-or power level. **For most people without the desire and-**

COULOMB'S LAW (of electric force between electrically charged particles and-or objects)

This topic and article is about the physical force of attraction or repulsion of electric charges and-or of charged objects containing those (net or total effective) electrical charges. An equation was created by **Charles Coulomb** of France in 1785 so as to express the value of the electric force and in relationship to those charges (Q1 and Q2) and the distance (d) between their effective center points of attraction or repulsion. In short, the electric force of attraction or repulsion is measured as a being a common physical force with units of Newtons (N). From this equation, an amount of charge can also be determined by the force and distance between those two charges. The charges for this equation are also considered as static, resting, non moving, so as to avoid the magnetic field and forces created due to moving current.

$$F = \frac{k(Q1)(Q2)}{d^2}$$

: Q=charge , Charges Q1 and Q2 are measured in units of Coulombs (c) of electric charge.
k (or Ke) is the Coulomb's Law Constant = 1/4(pi)Eo = 1/(12.566371)(8.854(10^-12) C^2/Nm^2)
Eo is the permittivity of a vacuum, and is also called the "electric constant"
with units of Farads / meter when used in terms of the capacitance of a capacitor.
Permittivity is a measure of the ability of the substance to allow the electric field.
k =~ (8.98755179)(10^9) C^2/Nm^2 =~ (9)(10^9) C^2/Nm^2 for air.
The unit for this value for k is: (Nm^2/C^2) , k and then F will decrease in denser mediums or
substances between the charges. **k** is sometimes called the **electro-static force constant**.
F = Fe = the electric (not gravitational) force of attraction or repulsion in units of Newtons = N
(d) or r = radius from the centers of the net charge, and has units of meters (m).

On the atomic scale, the distance between the nucleus and electron of a hydrogen atom is about 52.9pm =~ 53 (10^-12)m = 53 picometers, according to the (1913, Niels) Bohr popular atom model. This value is also equal to that electrons orbital radius about the nucleus (center). This distance is also considered as the minimum distance that an electron can be about an atom, but for other elements which have other electrons in "higher" or farther radius orbits, their distance from the nucleus will be more than 53 pm. Electrons in more distant orbits have a weaker (Coulomb) attractive force applied to them from the nucleus which contains protons which have an opposite and positive electric charge, and of which is even lessened or weaker from all the other electrons present in that atom and having some repulsive force. Loosely bound (by force) electrons will require less energy to be free and then have kinetic energy, and this is what electric current is composed of.

Coulomb found and verified these mathematical relationships while experimenting with charged metal covered balls (and possibly, sometimes using metal plates) near each other and then measuring and calculating the force of their attraction or repulsion with a sensitive torsion (twisting force, here, using a thin hanging string) balance with a thin "pointer" (ie., a needle, indicator) or rod on it to indicate the amount of twisting force. A coulomb (C) is a practical unit of measure for a specific quantity of charged particles, such as electrons, and-or their net charge. These particles are difficult to see, count and-or numerically quantify with a number, and the Coulomb unit is very helpful because it is practical. Others have also studied the electric force using capacitors with closely spaced, large metal plates of which can accumulate and store charges (ie. electrons, or positive ions (atoms)), and this would actually be somewhat like a type of (electric charge sensing) galvanometer with larger plates that move in the presence of an net electric force due to the net charges.

If two uncharged (or de-charged) objects such as hollow metal spheres are identical, and if one is given a net charge, and due to like charges repelling (via electric forces), half of its charge can be transferred to the second metal sphere upon electrical contact which allows the transfer or conduction of the charged particles and-or their charge. The result is that each of these objects have the same amount of charge, and without actually knowing exactly how much other than that they are now both equal in charge. It could also be said that the original sphere held twice the amount of charge,and this could be expressed as: (100% charge amount) / 2 = 1/2 = 50% = 0.5 charge amount on each sphere. The sum of charges on each sphere is: 0.5 + 0.5 = (1/2) + (1/2) = 1.

If an amount of electric charge is divided in half by Coulomb's reasoning and method, this smaller amount can also be divided in half with another uncharged sphere, and the resulting charge on the last sphere is (1/2) / 2 = (1/4) = 0.25 of the

Coulomb was rather measuring charge as he knew it as something that caused a force and was involved with electricity, rather than as being charged microscopic particles know today as electron particles that each has an electric charge and are therefore called charged particles or simply as equated to being charges. Coulomb did not know the number of these charged particles a unit or charge such as the Coulomb, but he knew that an amount of charge can be repeatedly divided. into smaller charges which have smaller forces associated with them.

Ampere created a unit to measure the flow of electric charge, which is called a current of charge or simply current:

1 Ampere = 1 Amp. = 1A = the amount of current that will flow through a resistance of 1 ohm when a 1 volt potential
difference or emf (electromotive force) is applied to it. And by Ohms law:
current = I = voltage / resistance = V / R

current = $\dfrac{1 \text{ volt}}{1 \text{ ohm}}$, we see that a unit for current is needed, and this is what is used:

Since an amount of (flowing) current should also be measured as a flow rate of electric charges, A unit of 1A of current was defined as 1C of charge past a point per second:

1A = $\dfrac{1C}{1s}$ = $\dfrac{1 \text{ Vv}}{1 \text{ Rohm}}$ also: 1C = 1A-s = the total amount charges in a current of 1A of current for 1 second. 1C = 1Amp-second

An electron (e) is the smallest charged particle or mass. A net amount of total charge on an object is due to the (algebraic) sum of these individual charges: positive charges such as (net) positive ions (atoms that lost an electron(s)), will have a positive sign (+), and negative charges such as electrons will have a negative sign (-):

Net charge of an object = positive charges + negative charges

1 C = 6.24 (10^18) electrons. **: 1 coulomb (number) of charge(s) and-or its carriers such as electrons**
A coulomb of electrons or charges, and:
This is derived below. For consideration: 1C / second = 1 Ampere . By this definition, an Ampere (A) is a rate of flow of (an specific amount of) charge past a point per second, and-or the quantity of charges past a point per second.
Amps = Coulombs of flowing charge / second . 1A =1Qc / 1s = flowing charge / time

Dividing both sides by 6.24 (10^18) we can find the amount of charge or the number of coulombs that 1 electron is:

The charge on 1 electron is : **1 electron charge = 1.6025641 (10^-19) C**

Total charge, Q = (number of charges)(charge of each charge) = (number of electrons)(charge of each electron) ,

Total charge = (number of charge carriers)(charge of each carrier) now consider:

If we assign 1C as the value of this total charge: 1C = (number of charge carriers)(charge of each carrier)

If each charge carrier is an electron particle, and is given its individual charge amount the same unit as C:

Total charge = Qc = (number of charge carriers)(1.602)(10^-19)C which can be expressed as
1Qc = 1C = (number of charge carriers)(1.602)(10^-19)C mathematically:

number of charge carriers = $\dfrac{1C}{1.602(10^{-19})C}$ = **6.24 (10^18) charges or electrons in a coulomb= (1C) of charge or electrons**

We can assign 1C of charge carriers = 6.24(10^18) elementary charges and-or electrons = 1C of electric charge

1 ampere unit of current = 1A.   Amperes = Coulombs per second

  1A = 1C / 1s   : 1 amp is or has a flow rate of 1 coulomb of charge (ie., electrons) per each second of time elapsed

If electrons carry or have the charge, 1C of charge is also said as being the amount of charge in 1C of electrons.

  total charge = (rate of charge)(total time) = (charge rate)(total time) = (coulombs/second)(seconds)

  1C = 1A x 1s   : 1 coulomb of charge or electrons will pass by a point in a circuit that has 1 amp of current
                   applied for 1 second of time.   In general: **C = A s   and   A = C / s**

  Ex. If the current was 2A applied for 10 seconds, the total number of electrons or charges that flowed was:

      (2A)(10s) = 20 amp-seconds = (2C / s)(10s) = 20C = 20 coulombs of electrons and-or their charges

  1C = 1 coulomb = 1 amp-second = "an amp for one second" = "an amp of current flowing for one second"
              = "a current of 1A for 1s of time will be a total of 1 coulomb of charge and-or electron charges"

From: A = C / s , we have **s = C / A**   so as to determine how long it will take an amount of charge to move past a point
when the flow rate of it is A. , A = C / s is essentially a flow of charge (time) rate = the amount of charge per second

Here are a few related equations in terms of energy:

  $1W = \dfrac{1J}{1s} = 1V\,1A = 1V\,\dfrac{1C}{s} = I^2R = \dfrac{V^2}{R}$: 1 watt of power = 1J/s is delivered and-or used up =
                                                      1 volt moved 1 amp of current.   From this, we can derive:
                                                      Note mathematically: **J = VAs = (watts)(s)**

  $1V = \dfrac{1W}{1A} = \dfrac{1J/s}{1C/s}$ Simplifying this fraction, we have:

  $1V = \dfrac{1W\,s}{1A\,s} = \dfrac{1J}{1C} = \dfrac{energy}{coulomb}$ and   1C = 1J / 1V ,   and   (1C)(1V) = 1J = W-s

  1J = (1C)(1V) can be thought of as:

      1J = (1V)(1C) = energy = work = (force)(distance) = Nm = W-s

      V = J / C = energy / coulomb   : **voltage is shown here to be the energy per coulomb of charge**

      Electromotive force (emf) applied to charges will give that charge a change in its kinetic energy which is its
      movement energy, or to give the charge potential energy which is stored energy. Potential energy means the
      potential ability to use that energy so as to do work. Energy stored in a battery is potential energy.

      If a coulomb of charge has 5 Joules of total energy, its potential energy is said to be 5V. 5V can be thought of as
      5J/C. If that coulomb of charge used up or lost 2 Joules of energy in a resistor, it now has 3 Joules of energy left
      for that coulomb of charges, hence 3V, and it is said to have changed its potential energy by 2V. 2V can be
      thought of as 2J/c. This difference or change in energy and-or the voltage is often called a potential energy
      difference or simply the potential difference. The potential energy of stored or static charges can also give
      nearby charges (ie., electrons) some motion (kinetic energy) due to how charges repel or attract.

      For charges to flow in a circuit between two points, there must be a potential energy difference between those two

available energy. Though a resistance is thought of as energy or power wasting thing, it can limit the current or voltage, and therefore power to a safe value where needed, particularly if the supply voltage is higher than needed.

If the resistance (R) doubles then the current through it will be half. Current (A) and resistance are inversely related:

     nR when _A_   : mathematical relationship between R and A when there is a change in either
               n   and when the voltage is the same.   Here, (n) is a multiplying factor.

    Ex. In terms of the factors involved: $2R = \frac{A}{2}$ or= 0.5 A = (1/n) A   OR $2A = \frac{R}{2A}$ or= A = $\frac{0.5R}{A}$ = (1/n) R
      (here, n=2)

    From also V = I R or= A R , and given a certain or fixed voltage, we see that the relationship between
    R and A is an inverse relationship, and they are not necessarily reciprocals, and that if one changes by a
    factor of (n), the other changes by a factor equal to the reciprocal of that same factor, hence by (1/n).

More force will be needed to push a certain amount of current through more resistance, hence a higher amount of energy is needed, therefore a larger voltage potential: V = J/C wil be needed to force the same amount of current through a larger resistance. In short, the electrons will need a higher kinetic energy. If the resistance (R) increases by a factor of (n), then the voltage needed for a given amount of current must be increased by that same factor. If the voltage is not increased, but left the same value, then the current will be reduced by that same factor.

    $V = (A)(R) = A R (1) = A R \frac{(n)}{(n)} = \frac{(A)}{(n)} (n)R = (n)A \frac{R}{(n)} = (1/n) R$: here A = I = current

If the voltage (V) or voltage force applied across that same resistance is doubled then twice the current (A) will flow. Voltage and current are directly related.

    nV leads to nA
    nV = n(A)(R) ,   when R is constant: nV = (nA)R   or when A is constant: nV = (A)(nR)

If 2 volts is across a resistor, it fundamentally means V = 2J / C of energy per coulomb is lost through that resistor when forcing each 1C of charge through it.   If the current or charge rate is 3A = 3C/s , then the total energy lost through that resistor per second can be found using a proportion:

    $\frac{2J}{1C} = \frac{xJ}{3C}$ and $x = \frac{(2J)(3C)}{(1C)} = 6J$: hence the energy wasted in 1 second is (V)(A) = J/s = watts

As for the specific value of this resistance (R) that passes 3A of current per second through it with a voltage of 2V applied across it, this can be thought of as 3C of charge that looses 2J/C of energy per second:

    $\frac{2J}{1C}$ x $\frac{3C}{s}$ = $\frac{6J}{s}$ = six joules will be lost each second = 6J/s = 6 watts   : P = (V)(A)

    $R \text{ ohms} = \frac{V}{A} = \frac{\frac{J}{C}}{\frac{C}{s}} = \frac{J\text{-}s}{C^2} = \frac{(6J)(1s)}{(3C)^2} = \frac{6 \text{ J-s}}{9 \text{ C}^2} = \frac{2V}{3A} = 0.667 \text{ ohms}$
                                                                                  This can be interpreted as:
                                                                                  0.667 volts per 1 amp of current
                                                                                  are needed to force that current
                                                                                  through that value of resistance.

    Js = Joule-seconds   : this can be thought of as an amount of energy applied for a number of seconds, hence

average value for the charge of 1 electron to be about:

Charge of an electron   = 1 electron charge   = -1e = -e   = Qe =   - 1.602176634 (10^-19)C

Charge of an electron or proton. The electric charge of a proton atomic particle is the same as that of an electron, therefore having an equal value of a fundamental charge unit, but opposite in sign with a positive sign (+).

The total charge of a mol of protons divided by the charge of a single proton is equal to Avogadro's Number = (Na)

The total charge of a mol of electrons divided by the charge of a single electron is equal to Avogadro's Number = (Na)
 Mathematically:
(charge of an electron)(Na) = (charge of a mol of electrons) = (96485C of charge)   : with Faraday's Constant
 Hence there are about: (96500 C / mol of electric charge) = (96500C / mol of electrons)

Extra: A mol of Atomic Mass Units = 1 mol of amu = is defined as having a total amount mass of 1 gram. Protons and neutrons are equal in number in each atom, and they are also equal in mass.   1 gram of atom matter has 0.5 grams of protons and 0.5 grams of neutrons. 0.5 grams of matter corresponds to half a mol = (mol / 2) of protons, neutrons or amu. The amount of mass form the electrons is negligible. Because of the above concepts, and that hydrogen atoms do not have neutrons, 1 gram of hydrogen atoms will have 1 mol of protons.

Charges tend to concentrate on sharp points, angled edges, and corners of metal, and therefore, there is a higher chance of arcing ("sparks", "lightening") there when the emf of the total charge exceeds the breakdown voltage of air and distance to another object, and then lets a current to flow from those spots. Sometimes electrical objects and-or insulators are deliberately made round in shape so as to help distribute the charges more and therefore reduce the chance of arcing. A **lightning rod** to protect structures (houses, building, towers, etc.) from lightening has a pointed or relatively narrow piece of metal high in the air above a structure so as to divert a possible lightening strike away from the structure and into the ground around the structure. The lightning's energy will be diverted into the ground via a high conduction wire attached to a reasonably deep (several feet, maybe 1 meter), metal rod(s) and-or metal plate(s) in the ground.

Not too long after Volta created the battery in about the year 1800, and which is a relatively easy and reliable way to make usable amounts electric power more available and practical. **Georg Ohm,** in 1827, discovered by observation and measurement the fundamental mathematical and physical relationship of voltage, current, and resistance of an electric circuit or path that the current is passing through and delivering energy. This relationship is known as Ohm's Law. It is nearly intuitive, easy to remember, and is the most useful formula about electricity: **I = V / R**   : Current (I = A, or Ia) is directly proportional to the applied voltage (V) or (electromotive) force to move that current, and is inversely related to the amount of resistance (R) in the circuit or path that impedes the flow of current. The units for current (I) are amperes (A, named after Ampere). The units for the voltage is volts (V, named after Volta). The units for resistance is Ohms (ohms, named after George Ohm):

$$I = \frac{V}{R} \quad , \quad \text{therefore mathematically:} \quad V = I\,R \quad \text{and} \quad R = \frac{V}{I} \quad : \text{Ohm's Law}$$

A common analogy to the flow of current (moving electrons, or "flow of electricity") and Ohm's Law is with water flowing through a pipe. The pressure (ie., force) due to the weight of the water at the start of the pipe corresponds to voltage (ie., emf, electro-motive force). A force can apply energy to things such as cause water and-or electrons to move after transferring some kinetic energy to them. A larger amount of force can move a larger amount of water or current. The moving current of water corresponds to electric current. If the pipe is wide (ie., its diameter) enough, a large volume or amount of water can flow per second through that pipe, and this corresponds to the amount of current per second or amperage (amps). The pipe itself corresponds to a wire (ie. electrical conductor, conduit, or "path"). If the pipe somehow gets narrower, it is like increasing the reverse force, pressure or resistance in the opposite direction to the flow of a current, and the rate of water flow will be reduced.   The flow rate = amount per time = (gallons or pounds) / second , correspond to: (current = electrons or coulombs of electrons) / second = I = A = C/s. The height and weight of the of the

NATURAL RADIOACTIVE ELEMENTS

It is thought that heavy elements which have a high amount of mass were created at "the beginning of time", energy and matter such as during the "Big-Bang", and-or then also created from the high pressures created during exploding stars of which may also have previously concentrated some heavier elements at its highly pressurized and dense core location. Earth too is thought to have some heavy elements that have sunk to its core and are then of much higher concentration, and that some of it can eventually reach to the surface via volcanic activity or plate tectonics (ie., surface, land movements). Radioactive atoms on the surface of Earth contribute to the natural background radiation we experience. Heavy element concentration near Earth's surface, either in mineral ore and-or atom form. is relatively low, and the amount of radiation from these that are radioactive is correspondingly relatively low and without issue.

In 1896, **Antoine Henri Becquerel** (1852-1908), from France, is credited to the formal discovery that some elements such as uranium, and without external stimulus or energy applied, produced "spontaneous", high-energy, penetrating rays of invisible light called radiation. A heater or heat radiator is a metal structure designed to make and radiate heated air, such as to heat a room. The word radiation means an outward release, emittance, or transfer of energy, and which extends "radially" in 360° or spherical-like pattern from the source of that radiation. Heat energy is generally invisible to the human eye, but special cameras can convert this invisible form of (light and-or rf) radiation it receives into a visible image on a screen.

The radiation Becquerel discovered was invisible (like heat is) and traveling outward like invisible light rays from its source at the center of that radiation. This "invisible light" was discovered using a light sensitive photographic plate, and is somewhat similar to how X-rays are produced from a Crookes tube were discovered a year earlier by Rontgen. His contemporaries at the same time and study were **Pierre and Marie Curie** (1859-1906 and 1867-1934 respectively), from France and Poland respectively, who discovered many things about radiation while working in France, such as the glowing light (ie., a form of radiation) emitting and highly radioactive unknown element they eventually purified (essentially filtered away from other unneeded substances) and-or concentrated it together into a larger amount and called it radium. This radium was refined - concentrated (unwanted debris removed) and isolated from the hard, black mineral called Uraninite ("pitchblende"), which contains the element uranium. The isotope, radium-236 has a half life of 1600 years and is considered as the decayed form of uranium of which the isotope form uranium-238 has a half life of 4.5 billion years. Unstable atoms which have an excess of energy and-or particles, can decay or change into another atom after releasing some energy and-or particles. Radium will decay into the element lead. Uranium, radium and lead are metals. She also discovered the element polonium in the pitchblende. Becquerel was the first to formally discover a radioactive element or material, although a few people have previously noticed some effects that uranium caused on photographic plates. Becquerel essentially discovered both nuclear or atomic energy and radiation. The SI unit (Bq) of measuring the rate or intensity of radiation is named after Becquerel. 1 Becquerel or Bg is the decay of or within one nucleus of an atom per second, hence it is also a rate of decay. **1 Bq = 1 decay / 1s**. This decay will cause the release of a pulse of energy which is the emitted radiation or radioactivity.

1 **Curie (Ci)** is 37 (10^9) = 37 billion Becquerels per second, such as emitted from 1 gram of radium-226. How much radiation life forms have absorbed per unit of time generally has other defined units such as **grays** , **ergs**, and **rads** (a word from the word "radiation"), and this and other related topics of radiation is beyond the scope of this book, and you may research these.

1 gray = 1Joule of radiation energy absorbed by 1 Kilogram of matter = 1J/k. :by Louis Gray (1905 - 1965), from England
1 gray = 1 **sievert** = 100 rads). : by Rolf Sievert (1896 - 1966), from Sweden

A combined or total amount of radiation of 1 gray = 1 sievert = 1000 milli-sieverts = 1000 mSv = 100 rads = 100 rem during 24 hours = 1 day of time could cause radiation sickness. A cumulative total of about 100 mSv of radiation per year , such as for nuclear energy workers, and hopefully even less such as the natural background radiation level (about 3 mSv total / year, or 0.333 rem / year) is generally considered as tolerable to the body and will generally not cause long lasting damage to the body. A cumulative total of about 1 sievert / year = 1000 mSv / year is considered dangerous and can cause cancer in some people, especially if that dose or amount occurred in less than a year and-or was concentrated in one region of the body.

observing hydrogen ions which is just a single proton of a hydrogen atom and without an electron. Note that a hydrogen atom does not even have a neutron counterpart of its proton, hence a hydrogen atom without an electron is a positive ion and that it is composed of just a single proton. He also classified and named the alpha and beta types of radiation, and conceived the half-life formula for radiation decay and intensity. Rutherford also discovered the gas and natural element of radon after experimenting with radium. Rutherford found that a radioactive element can change and-or decay into another element such as radon gas produced from radium.

Just as with common light or radio energy radiation, the farther a person is from the source of nuclear radiation, the less energy received per unit of area, and-or, the less intense it gets according to the inverse square law of electromagnetic transmission and reception. The intensity of radiation at a certain distance from the source can be indirectly measured by the intensity of the ionized radiation or particles produced by that radiation, such as what happens within the tube (ie., the radiation sensor) of a Geiger (radiation) meter or counter. A received (ionizing) pulse or instance of radiation will then produce or create some free electrons which is technically a small current of charge or electricity, and which can then be greatly amplified to produce a more useful amount of current or pulse of electricity through a meters and-or a signaling light or audio speaker.

Directed energy beams such as a laser bean do not follow the inverse square law concept of energy radiation, and the intensity of the beam does not diminish or reduce by a great extent as it travels further from its source. This is very helpful for outer-space communications where the signal power is often relatively small and the distances are great. "Pulsed" laser or radio wave beams is a way to transmit a very brief and circuit safe amount of (intense or concentrated) energy, and of which can travel and be received even much further than a standard laser beam. As a simple example, consider: Power = energy / time , with units of watts. 1 watt = 1 Joule / 1s , but if that same amount of available energy can be released in a quicker (ie., shorter, less of a duration) or faster amount of time, perhaps in one-thousandth of a second = 1s / 1000 = 0.001s, the pulsed output would be equivalent to a much more powerful: 1 Joule / 0.001s = 1000w = 1000J / 1s for a brief, but effective moment of time. High power lasers, with many used for cutting metals, use this concept of having a high available and useful amount of power for a brief (ie., "pulses") amounts of time.

Some long distance spacecraft, such as the one that went to Pluto, used a radioactive substance as part of an electricity generator, and since being far from the Sun, there isn't much sunlight energy to power all the equipment on it such as the camera, computer, and radio-data communication system by using just typical sized solar panels.

The following discussions are optional to the reader of this book. These discussions contain common knowledge about flight and modern technology so that the reader may have a basic awareness and understanding of them, and examples of how having some math and science knowledge is very useful in many fields of study and-or work.

How Balloons And Planes Can Fly, And The History Of Flight

This book topic is mainly used as a practical example of the progressive advances in science and technology - usually aided with some amount of math along the way. The reader may skip over this till a later time since its not needed for further use in this book. If a person has interest in these or other similar topics, it soon becomes clear that also knowing some math will be useful. The selected discussions are relatively brief and to the point so as to keep the style of this book, and if you are further interested, you can research each topic elsewhere, such as by visiting your nearby bookstores for specific books about each topic(s).

Since ancient times, it was well known that warm and-or hot air rose upward or higher in the air and sky. For example, this can be felt as warm air rising up and away from a candle. The warmest air of a house would often be located in an upper room and especially near the room's ceiling if it was not vented or released outward. Warm and-or hot water or liquid also rises upward and resides on the surface of the cooler liquid below it. A practical question would be how does this actually happen, and the following discussion(s) will help answer this question. A related topic and question is why does oil always rise upward and-or float on water, and even when they are both at the same temperature. The quick and short answer to much of these questions has to do with the density of the substance(s) or object, and the pressure or forces acting upon it.

First, a brief discussion on the topic of gasses and pressure so as to help bring some understanding to the technical issues of flight.

Joseph Priestley (1733-1804), from England,, discovered and published in 1774 the findings of a substance we know today as **oxygen**. Just a few years before this, **Carl Scheel** (1742-1786), from Sweeden, made a similar, but published later, discovery of a "fire gas" or some substance in air (a mixture of gases that we breath) that helps combustion (ie., fire, burning) occur or happen. Scheel is also generally given some credit for the discovery of some other elements such as barium, chlorine, manganese, molybdenum, and tungsten. Priestley's friend and contemporary **Antoine Lavoisier,** (1743-1794), from France, determined this gas to be a unique (chemical and fundamental) element and called it oxygen which is a word based on the Greek words for "acid (strong or sharp)" and "gene" (that which creates or produces, "genesis", "generator", etc). Later discoveries by others showed that the element hydrogen (first discovered by **Henry Cavendish** (1731-1810), from England in 1766 and named by Lavoisier) rather than oxygen, is actually the main substance or element for creating acids. Lavoisier then also made advances in the knowledge of oxidation and combustion. Priestley also discovered some other gas compounds, particularly some nitrogen gas compounds. Nitrogen gas was discovered in 1772 by **Daniel Rutherford**, (1749-1819), from Scotland. Nitrogen is an "inert gas" meaning it does not generally cause any chemical reactions with other elements and-or molecules, and is non-flammable and non-combustible.

Henry Cavendish determined that water is a compound of hydrogen gas and oxygen gas, after combusting the mixture and noticing water drops. Shortly after Alessandro Volta created the first battery in 1799-1800 which permitted a portable, reliable, and steady (direct) current flow and rate, and at a desired voltage, the concept of **electrolysis** (using electricity to cause chemical reactions by using electricity to reduce or break apart matter, and-or bind matter such as for electroplating one metal onto another) or electro-chemistry was discovered in 1800. Electrolysis will separate a water molecule (ie. a compound of two or more atoms) into its atom parts and therefore create hydrogen gas (at the cathode or negative terminal or electrode), and oxygen gas (at the anode or positive terminal or electrode) from it when a voltage (about 1.23V minimum, and any excess power may be wasteful if not designed correctly) is applied in and across the water with two metal electrode surfaces. Plain or pure water does not conduct electric current (ie., electrons and-or net positive atoms called ions) very well, and therefore some "electrolyte" (such as a small amount of salt, an acid or base) substance or particles are added into the water so as to create a (ion, solution, positive particles able to move) solution that can conduct electric current much better than pure or plain water. The electrolyte solution should also not conduct very well

The elements discovered by Humphry Davy using electrolysis are: Sodium, potassium, calcium, magnesium, boron, barium, strontium, and he also verified chlorine (a gas) as an element.

type of structure called a Chinese Lantern was invented and which was made out of lite-weight paper material. These devices were sometimes used in the night as a signal for an event. Because they use a small flame for the energy or power source to make heated, less dense air, they are a potential fire hazard.

Matter that is said as being "lighter" is due to that it is, on average, less dense per unit volume of that matter or mass (the measure of matter). Less dense means it has less matter per unit volume, and less matter means it will also have less weight (force due to gravity acting on or applied that matter). Denser things have more mass per unit volume and therefore will correspondingly (and proportionally) weigh more per unit volume, and are said as being heavy or heavier in weight. This weight or force of the air up in the sky will cause a pressure upon the matter below it. The denser (weight, more force, more pressure) and more heavier air will effectively "sink" around and cause the less dense and less weight matter per unit of volume to be forced out (here upward) or "displaced" from its position, and cause it to move upwards towards the less dense air.

If the air pressure within a balloon or bubble remains the same as it rises upward, the size or volume of that balloon or bubble will actually increase due to the decreasing pressure upon its outer surface as it rises into a less pressure region.

When water freezes when being equal to or below 32°F = 0°C, it is one of the few substances that expands slightly when it is frozen to be a solid and no longer a liquid. This happens because these molecules that are loosing energy and joining together to form a rigid crystal structure with a slightly larger volume than that of the water. Ice will therefore actually be less dense (and "lighter") per unit volume than liquid water. Ice (ie., solidified water) is much harder than liquid water, but it is slightly less dense. If a piece of ice is placed in liquid water, the ice will effectively rise upwards (vertically), much like a less dense hot-air balloon would in the air, or a bubble of air in the water. By the force of gravity that applies some energy and induces motion, colder, more dense air or water will then sink to a lower level than less dense, warmer air and water. That which is less heavy in weight on average will be naturally forced via the pressure (from the weight, force) upon it to essentially move upward, as the more dense and heavy substance takes its position. This can be conceived as exchanging positions due to density, weight and pressure. Ice will float on water, and the relatively small amount, or percentage, of ice that is above the surface of the water is due to that ice has weight and is only slightly less dense than water. A unit and-or given volume of water will expand by about 5% in volume after it freezes. Now consider removing that 5% amount of volume, and we see that a unit of volume of ice then weighs about 5% less than the same volume of water. Since ice is much more dense than air, the ice will not be caused (by a net force due to a pressure difference) to move and rise upward and into the air to a lower air pressure region like a hot-air balloon would.

Air or any other gas that is twice as hot in temperature will have twice the kinetic energy, and therefore it will have twice the pressure or force upon the inside surface of a balloon to inflate or expand it. If the container is solid, rigid, non-expandable, then volume will not increase, but the pressure of the internal gas and its inner container wall or surface will increase when its temperature is increased. Excessive high pressure within a container can potentially cause a container to burst open abruptly and cause damage and-or injury. A lesson is to check how much maximum internal pressure a container is rated for before it is filled and-or (possibly) heated.

For a balloon filled with heated gas that is energized and expands in all directions, it could be said that the volume (**and not the mass and-or its weight**) of the gas has doubled when the temperature doubles. The space between the atoms or molecule of the heated gas has increased. If the container cannot expand, and is rigid, then the pressure of the gas will increase when its temperature is increased.

For a balloon, when the heated gas looses energy and looses temperature, the pressure upon the inside surface of the balloon will decrease, and the balloon will get deflated. For a balloon that was inflated with compressed air, it will generally remain inflated to the same volume for much longer and when the (internal, external) temperature difference is minimal and-or held constant. The volume of an inflated balloon can change if the outside, external air pressure on it increases or decreases, and-or the temperature changes such the temperature and pressure of the gas in the balloon changes. If an air filled balloon is placed in water, the internal volume of that air will decrease as it goes deeper into the water where more water pressure is upon the surface of that balloon or "bubble". The air in that balloon will get more compressed the deeper it goes in the increasingly pressurized water.

side, that the volume of it will be half because the gas molecules are compressed or pressurized closer together. A pressure gauge will display twice the pressure.   In a reverse manner, when a volume of a gas is doubled and still having the same mass (ie., amount, weight), the internal pressure of it will be half.   This relationship is true when the temperature of the gas is constant (ie., the same). Hotter gas has higher kinetic energy, hence it exerts more pressure upon itself (internal pressure) and the inside surface of its holding container.

(Jacques) **Charles's Law** published in about 1802 in France states that when the pressure (or weight as shown in the above figure) of a gas is constant, that the volume is determined by the temperature of that gas. A heated gas will have more kinetic energy and will want to expand a container or volume it is in when heated. Charles's and the **Robert brothers** also developed the fist hydrogen gas, manned balloon in 1783.

Boyle's Law and Charles's law are two fundamental **Gas Laws** that consider volume, pressure, and temperature, and of which can affect the others two values unless one is held constant. This book briefly discusses these laws here, so as to have a practical, minimal understanding for chemistry, and for hot air balloon and-or airplane flight.

According to Boyle's Law, the product of pressure = (p) and volume = (v) is a constant (k) for a specific type of gas (which has a certain density) and at a constant temperature.   For a certain gas (and its natural density) at a certain or constant temperature, and at or under (ie., subjected to) two different pressures and volumes, the equation is:

(p1)(v1) = k = (p2)(v2)   : **Boyle's Law** , p = pressure , v = volume, and when the temperature is constant

Since the product of each side is a constant of (k), and if one factor of a side of the equation increases, the other will decrease by the same factor. Therefore, the factors of (p) and its corresponding value of (v) are inversely related.

Also note from this equation, mathematically and in reality, the "inverse ratios" of both pressures (p), and both corresponding volumes (v) of a gas:

$\dfrac{p1}{p2}$ = $\dfrac{v2}{v1}$: for these values, please consider a non-expandable, rigid container having a constant volume and holding a certain amount or volume of the substance such as a gas within it.

If one ratio (ex., p2/p1) went up by a factor 2, the other corresponding ratio (ex., v2/v1) will decreased by a factor of 2. If pressure doubled (increased by a factor of 2), the corresponding volume was halved (decreased by a factor of 2).

For the volume and pressure of a substance in a rigid container with a constant volume: At standard pressure (14.7psi, at sea level) and temperature (72°F) or STP conditions, air will exert a pressure of 14.7psi any place on a surface, including a container of any constant volume that is then sealed, and the air within it will still have the same pressure of 14.7psi. This rigid container and-or volume can have more air placed within it such as due to a pressure such as from an external air compressor forcing more compressed air inside, and while using a one-way ball valve so that the internal, pressurized (ie., having higher pressure) air cannot flow back out and then decrease the internal air pressure. In this type of situation with a rigid, constant container volume, both the internal air pressure and the internal volume or amount of the air gas matter will increase, but the volume of the rigid container will not.   If the internal pressure (force/area) doubled, perhaps as displayed on a pressure gauge, the volume or amount of the internal gas has doubled. This volume or amount of gas per volume is called the density of the substance or gas, and is calculated as: density = mass / unit volume = mass / volume .   For a compressible substance such as a gas, pressure and density are directly related. For example, if the pressure increases by a factor of 2, the density of that gas will increase by that same factor of 2. Another consideration is that for non-gas substances such as many liquids and metals, they can have a large pressure applied, and yet their density does not change by much or at all, and these substances still have the same density = mass/volume.   The reason for this is that gas particles (atoms, and or molecules) have much more space between them than liquids and solids do, and when a pressure is applied to a quantity or volume of this gas, those particles will be forced to move closer together, and which results in more mass per unit of volume. Liquids and solids have particles that are closer together and held by positive and negative electro-static forces due to electron sharing at close distances, particularly with a metals that have more of a dense, closely packed and uniformly arranged atomic structure which is often called a crystal.   Liquids are often said as being held together, particularly at its surface, by "surface tension

For ex: 1 mol / 22.4L as= x mol / 1L   , we find x mol = 0.044643857 , hence: 0.044642857 mol / 1L

The compression piston in a Diesel type of combustion engine for vehicles briefly compresses the (fine mist) air and fuel mixture (previously created in the carburetor) to a very high pressure, and therefore, its temperature rises to a very high value and automatically ignites the air-fuel mixture without the need for an electric spark, or (electric, hot) "glow plug".   At the core of a **Diesel engine**, created in about 1893, is a concept based on a very old and amazing method to light or start a fire which is often called a "fire piston". This fire starting concept was invented around 3500 years ago in the southeast Asian island areas such as the Philippines. With this device, a volume of air in a cylindrical container is compressed by a hand powered piston or plunger, and the air becomes heated to a high temperature by that high pressure.   This hot, self igniting and combusting air is used to create a nearby hot glowing ember particle in a small piece of cloth or fiber that can then be used to create a fire to do beneficial things. This ancient concept was used and adapted in about 1893 by **Rudolph Diesel** (1858-1913), from Germany, when he made his combustion engine. The fuel this combustion engine used was refined (ie., filtered, processed) petroleum (ie., crude, unrefined oil from beneath the ground) based, but using various plant based oils is also possible. It could be said that the combustion engine is one of the results, derivative or product due to the "**industrial revolution**" phase of the world where much science, technology and communication assisted each other and they all began to increase exponentially, and until this day in 2022.

Fueled powered combustion engines could then be used for vehicles and machinery, and inspired the transportation revolution with automobiles ("cars" , ie., carrier, carriage, and a form of chair) and paved (flat, hard), vehicle efficient roadways for quick and low cost, practical transportation to locations not easily or possibly accessible by other methods of transportation. This combustion engine replaced many of the smaller steam engines, however most trains would still use a very powerful (boiling water, high pressure, high energy and power) steam engine for many years into the future. For his automobile, Diesel also used some of the concepts of steam engines such as the piston, rod, valves, and the rotating wheels of a steam engine. Both steam and combustion engines convert pressure energy into mechanical energy such as for the rotation of a wheel or gear. There have been some crude attempts of internal combustion engines shortly after the year 1800, such as in 1807 by **Joseph Nicephore Niepce** (1765-1833) and his brother Claude from France, but it was not practical due to the type of plant based powdered fuel used which was not a liquid hydrocarbon, petroleum based fuel which contains much energy potential. Still, the engine showed a great concept of using (slow, pulsed) combustion to then give a propulsion or movement such as for a small boat. Niepce is most often historically credited to the **first photograph** and the basic photographic creation concepts in about 1825. These images were crude by today's standards, however this is how modern (analog, true, "film") photography (image or picture recording) and ("film") movies started, and which now includes very practical, inexpensive digital (close approximation of the analog) images. Before Niepce, some others in the 1700's have experimented with light sensitive chemicals, such as various silver based solutions, and made crude outline images of various objects.   Though it is possible to use a simple convex magnifying lens in a telescope or camera, a practical, less problematic glass lens was first used in a camera in 1832, but before this, a pinhole was used as a type of lens for great depth of field and high sharpness (low image distortion) or clarity, and this lens which only allows a small amount of light into the camera and onto the photographic paper which at those times required a much longer exposure of light onto the photo sensitive material and a very steady (still, not moving) subject to photograph, otherwise, a (visual) image "blurring" (distortion) of the resulting image of the subject recorded onto the photograph would occur.

Carl Benz (1844-1929), from Germany, created the first practical automobile (ie., "horseless carriage" or simply a "car") for the public in 1885. It used gasoline fuel and worked well, but it was relatively crude when compared to **Henry Ford**'s "user friendly" Model-T car designed later 1908. Ford's was surely inspired by the Benz car. Carl was the husband to **Bertha Benz** who is credited to making the first long distance (about 60 miles in total length or distance) car ride in 1888. This distance and relative ease of travel was incredible for those times when a horse and carriage form of transportation was still very popular, and it brought the car into the minds and "wish-list" (hopes and desires) of many. She is also credited to some practical improvements to the design of the practical car.

To help understand how velocity can increase as pressure decreases consider this. If there is a difference in pressure (ie., force) in two containers of water, the water will be forced to flow in a pipe from the higher pressure container to the lower pressure container. The net difference in force or pressures upon the water will allow the slow water at the

Ex: (+10) + (-10) = +10 (+) -10 = +10 -10 = 0 = a net force, or difference of 0

Ex: If P1 = 3   and   P2 = 10 :   : ex. 3psi and 10 psi respectively

(P1) + (-P2) = 3 (+) -10 = 3 -10 = -7 = a net force or difference of -7, and the object in the
above pipe will have a net, effective, resultant or useful
force of 7 applied to it, and the negative sign indicates
that the direction that the object will move will be in the
left direction which is the same direction that the net force
is being applied to it.

From **pressure = P = (force / area) = F / A**   , **force = F = PA**   Higher effective pressure upon and in the water per
unit area will also apply more force to the water at that area and pipe, and this potential or pressure energy will transfer
more kinetic energy to that water at the output pipe at the bottom of the tank and its velocity will increase and flow faster
into and in the region of lower pressure due to a pressure imbalance which could be thought of as a force or energy
imbalance, or potential or static pressure energy difference.   This could be thought of as that the energy stored in the
compressed or pressurized water is now converted to water with an increase in momentum and-or kinetic energy and is
now not as compressed when subjected to a high pressure, but now is decompressed and with a lower pressure and
faster speed.

Though pressure applies a force and (potential) energy to the to the sides of a pipe or container, there is no transfer of
(actual) energy to the side of the container if it is rigid and does not expand. If it did expand, some this could be
imagined as expanding a spring which also requires energy, but then stores the energy applied. A expandable container
could also possibly burst open and release (ie., transfer) that internal pressure energy to the outside, lower pressure
region. If a container does expand, its volume will increase and the internal pressure will decrease.

Faster moving liquid or gas has more (kinetic) energy per volume of it because its kinetic energy (KE=mv^2 / 2) is
more. An increase of applied **force (a measure of the application and-or transfer of energy)** causes an increase in
pressure which causes an increase in the momentum (p = mv = momentum = (mass)(velocity)) of the gas or liquid if it
sable to move from that higher pressure region.

The velocity of the gas or liquid is inversely related to the cross sectional area of the pipe. If the cross-sectional area of
a pipe segment changes by a factor of (n), the velocity of the gas or liquid will change by (1/n), or in other words, the
velocity of the gas or liquid will decrease by (n).   For example, if the pipe area (A1) decreases by a factor of 2,
n = (1/2) = 0.5 and A2 = (A1 / 2) = 0.5A1 = n A1, the velocity of the substance will increase by (1/n) = (1/0.5) = 2, and
the kinetic energy of the substance will increase by 4 since KE = mv^2 / 2 = m(2v)^2 / 2 = 4 mv^2/2 = 4 KE

Faster moving gas or liquid has more of its energy stored in its velocity and motional ability (momentum = p = mv),
rather than as a compressed gas or liquid storing (potential) energy, and therefore, for a moving liquid in the output
pipe, there is less ("static") pressure applied to the inner sides of that pipe. This is the reason why the pressure is
reduced if the velocity of the gas or liquid increases.

An airplane wing has less air pressure on its upper surface when there is fast moving air flowing over it. This concept is
formally called **Bernoulli's Principle or Bernoulli's Law** and was discovered many years before the invention of the
airplane. For the airplane, when fast air passes upon the upper surface of that wing, there is then less air pressure
(force/area) and-or force upon the upper surface of the wing than on the lower surface, and the net difference in
these two forces or pressure on each side of the airplane wing is the net or effective force applied in the upward
(perpendicular to the direction of planes travel) direction to wing of which is attached to the plane and also transfers its
upward force (ie., [upward, vertical] lift or lifting force) to it   If the wing is tilted slightly upward, some compressed or
higher pressure air beneath the wing also improves the lift force, but the increase cross sectional area of the wing in the
direction of the plane movement also increase the "drag" or resistance to forward motion of the plane and causes it to
reduce speed and-or fuel efficiency. A plane can use smaller adjustable wings called **"flaps" or ailerons** to change

Distance = Length = L = $\dfrac{Volume}{Area}$ = $\dfrac{L^3}{L^2}$ = v t   , mathematically:

From: $\dfrac{Volume}{t}$ = A v   , we have: $\dfrac{Vol.}{A}$ = vt = L , as shown above

Another related equation is: Flow Rate = pAv   and this is derived here:

Qv = Flow Rate = V / t = "volume-metric flow rate"   , or also,   Qm = "mass flow rate" = m / t

density = p = m / V   , therefore: m = pV   : V = Volume , here, p = density and not momentum

Qm = m / t = pV / t = pL^3 / t = $\dfrac{pL^2\ (L^1)}{(\ t\ )}$ = pAv = (density of substance)(travel Area)(velocity of substance)

At two locations in a pipe with different cross-sectional areas, pressures, and velocities, the flow rate is still the same constant value, and Q1 = Q2. The density of the substance is also the same.

         Q1 = Q2   : Q = flow rate

     (p)(A1)(v1) = (p)(A2)(v2)   : flow rate of a substance in a pipe with different cross sectional areas
                                         Dividing out the common factor of (p=density) from each side:

       (A1)(v1) = (A2)(v2)   Due to Q being constant, A and v at a given location are inversely related and
                                       are not generally reciprocals of each other.
                        .
    Since the flow rate Q remains constant in a pipe, if the cross-sectional area (A) changes by a factor of (n), the corresponding change in velocity (v) will be the reciprocal of this (n) value, and which is therefore: (1/n) :

    Av = (n)(A) (v/n) or= (A/n) (nv)   : the product of Av is constant for a constant flow rate

    Av = $\dfrac{n}{n}$ (Av) = nA $\dfrac{v}{n}$ = $\dfrac{A}{n}$ nv

    If the known flow rate in a pipe has changed, and usually to a lower value, then there is a restriction or blockage in it, and-or an opening, "hole" or "leak [leakage]" at some place along its length which changes the internal pressure(s) on the pipe and within the moving substance, and this will indicate a leakage of some sort.

If there is compressed air in a container, and having a high pressure, and it is used to fill a balloon, that compressed air which had 0 speed, as like a liquid at 0 speed since it is not moving or flowing, will get decompressed as it moves out of that container and toward a lower pressure region. Once an (expandable material, stretchable) balloon starts to inflate, the air in it will begin to get compressed and pressurized so as to apply pressure to the inner side surface of that balloon and will cause it to expand (in volume) or inflate as it stretches outward from the internal pressure. The expansion will continue until that internal pressure equals the input pressure and there is no net difference in pressure to permit a flow of the input air, or the possible output air from the balloon such as if it had a greater pressure.

If you were to blow air into a rigid or strong container that does not expand, the pressure of the air within the container will increase up to the pressure of the air input, and that increasing pressure will reduce the speed of the air being input into the container until that input air speed falls to 0. If an opening was then put into that container, the velocity of the air being forced out will be initially high due to that internal pressure, and then the higher (than atmospheric) pressure within the container will decline to the value of the pressure on the outside of the container, such as the standard air pressure at sea level (14.7 psi).

RUBBER SEAL

VACUUM

PLUNGER HANDLE AND ROD
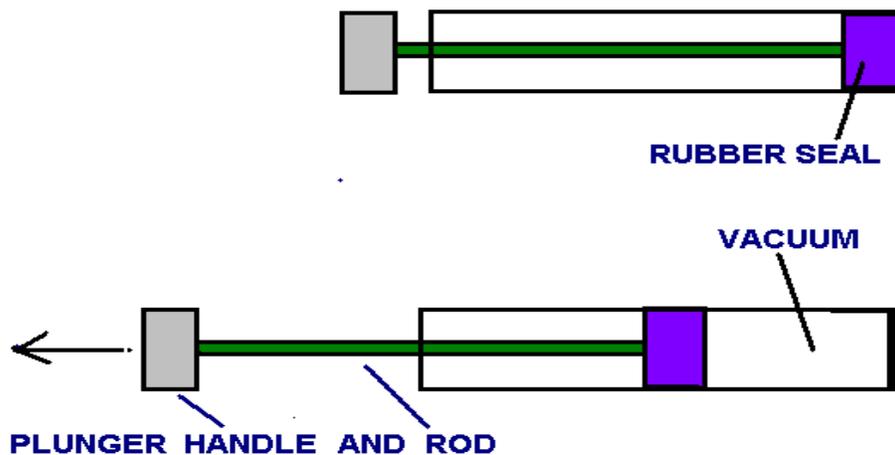
In the above figure of a plunger (ie., a compressor) and-or suction (ie., a vacuum, decompressor) device when the handle is pulled out of it, a vacuum and larger volume with a lack of air or very little air is created. The little bit of air that was initially there, at say 14.7psi, is now all within a much larger volume, less compressed, and therefore under less pressure on itself and the side walls of the device in the vacuum region. The rubber seal is expandable and expands or pushes tight enough against the wall of the container so as to maintain a vacuum or compression (such as when making a compressed air or substance) forces. After pulling the plunger out a little into the normal air pressure region it becomes apparent that it requires more force to do this (besides any friction and-or compression forces of the rubber seal). The required force by the user is due to that the normal air pressure (or force) of 14.7psi must initially be overcome with an equal amount of force, otherwise the normal air pressure will force this plunger end to move back into the plunger device because the vacuum that was created there is at a lower pressure. If the plunger had an opening, tube and-or pipe on the right side of the image shown, it could be used as a vacuum device so as to effectively suction up or "draw" a gas or liquid into a low pressure vacuum region (volume) when the plunger is pulled outward from the device in the leftward direction.

The vacuum created with the above device is a "high vacuum" with a very low amount of material remaining in it such as some air atoms that may have been put into it when initially putting the plunger into it. An opening or an out-only, one-way ball valve on the right side of the plunger container above will allow the plunger to be easily pushed into the container since there will no compressed or pressurized air causing a resistance to its motion and the full seating of the plunger. This opening can then plugged so as to use the device as a vacuum, and with a ball valve, this will happen automatically as the plunger is pulled out of the container.

The device is also very similar to that of a medical, fluid injector or remover called a syringe or "needle" that is often used by a doctor or nurse so as to give a patient some medicine. The medicine dose (the prescribed amount) needed plus a little extra is drawn into the vacuum area by the aid of the plunger, and then the plunger is tipped vertically and physically tapped so as to allow any possible air bubble to rise to the top of the liquid so as to be ejected out of the plunger and through a thin, hollow needle (essentially a rigid metal tube with a sharp point at one end) with an inner hole along its length. The slight waste of medicine is acceptable due to the dangers involved. When the plunger is forced in, the fluid or gas will get pressurized and compressed slightly (ie., reduced in volume, denser, and more so for gasses than liquids) in size, and if there is an outlet (opening) to a lower pressure region having less force acting upon that gas or fluid, then that fluid or gas will move (via the applied force, pressure [compressing the material together] and some kinetic energy given to that fluid or gas) into that lower pressure region and become depressurized and be at the same pressure as that of which it was sent out into. The above device is also the basis of a mechanical air pressure gauge to measure the internal air pressure of vehicle's tires or inner tubes.

When a person breathes in, their diaphragm muscle pulls upon their lungs and expands or increases their volume while

opening the outlet end so as the liquid will move out from it due to its weight and gravity will create a lower pressure region in and at the pipe or tube opening in the supply fluid container and causing more fluid to be naturally pushed (if it is already at some pressure) or sucked into the pipe, fluid or tube region that is at a lower pressure. A siphon is a good method to irrigate (to apply water) farms when possible and needed. This type of siphon functions much like a hand operated or motorized water pump. The word "sip" is a derivative or form of the word "siphon", and a common use of it is to take a sip of a drink from or through a straw (a small tube used to assist some people when drinking fluids).

A calibrated **barometer** ("baro" means weight or pressure, hence a pressure meter) can sense and indicate any changes in its local air pressure value and displays the current air pressure. In short, a barometer is an air pressure gauge for measuring the natural air pressure due to the above air column. A mercury (column or filled) barometer is sometimes called (Evangelista) **Torrricelli** (ie., units of tor=mm of height, units) barometer because he invented it in 1643 in Italy. He was a student of Galileo. This barometer uses a rigid glass tube sealed at the upper, top end and is filled with liquid mercury (Hg) metal. Mercury, with a higher density per volume and weight than water, is used since if water as used, the device would need to be much taller (about 33.8 feet = 10.3m, and this is derived in the topic of Air Pressure And Barometer Values Derivation, further ahead in this book) since water has less density, and therefore, less weight and less pressure (at the bottom, deepest part) per unit area of a (vertical) column of it. The tube is then tilted vertically (ie., "upside down") and will drain slightly into a bowl filled with mercury which can be used as a sensor and be affected by the air pressure upon it. The mercury will drain into the bowl until there is no difference in the force (ie. weight) or pressure (psi) of the column of mercury and the air pressure (that is on average, 14.7psi) upon the mercury in the bowl. When the pressure of the column of mercury equals the air pressure, say 14.7psi at sea level, then the pressures will be in balance and no further movement or flow of that column of mercury will take place and reduce the height of it. There will be a slight vacuum (without air or other substance) of space created near the sealed, top or upper end of the tube. This space helps the mercury have room to move upwards, rather than trying to be compressed at higher temperatures.   If the local air pressure increases, it will push down on the pool of mercury and cause it to apply more pressure to the mercury in the glass tube and cause it to raise upward in height, or decrease height if the local air pressure decreases.

With a barometer, it is possible to calibrate it and use it as an altitude sensor and altitude (height) indicator since air pressure (or "air weight", or "air density") decreases with altitude, and this was discovered in 1646 by **Blaise Pascal** (1623-1662) from France.   Other liquids, such as water, can be used to make a homemade barometer, but then that barometer needs to be larger (especially higher) in size and is therefore impractical. Normal air pressure is defined as 14.7psi = 1atm = 1013.25 millibars (mb) = 101.325 kPa , or a height of 760mm = 76cm = 29.92in. =~ 2.5ft. of mercury (Hg). This pressure is when there is 760tor = 760mmHg =~30inHg of a column of mercury (Hg). During stormy weather, the height of the mercury may reduce by 1 inch or so, and this indicates a lower air pressure, and during a hot and dry day, the height may increase by an inch or so, and this indicates a higher air pressure. Surly, hot, less dense air which can expand a balloon, is at a higher pressure due to its excited or energized state or condition with its atoms having a higher thermal (kinetic, motional) energy and can apply more force to the inside surface of the balloon. pressure = force / area = weight / area.   Since pressure is expressed as a ratio, if the numerator and denominator are multiplied or divided by the same value or factor, it creates an equivalent fraction, and yet the pressure rating is still the same. Due to this reason, the column of mercury can have any cross sectional area, such as for example, 3mm, and it is rather the density of the liquid used and the height of it which determines the pressure:

$\quad$ **P = pressure = (force / area)** = (mass)(acceleration) / (area) = (weight / area) : units of Pascals = Pa
$$1Pa = 1N / m^2 = kg / m\ s^2$$

$\quad$ **P = pgh** = pressure of a vertical column of an element = (element density)(gravitational acceleration)(height) =

$\quad$ P = (mass / volume) (g) (h) = (mass / L^3) (g) (L^1) = (mass / L^2) (g) = (mass / area) (g) = **(weight / area)**

$\quad$ work = (force)(distance) = (ma)(h) = (mg)(h) = **mgh** = Kinetic Energy joules , to raise an object in gravity = KE
$$\textbf{KE = work = mv\^2 / 2 = Fd = mad = mgh = (weight)(h)}$$
$\qquad\qquad\qquad$ Here, KE = GPE , joules

and practical (useful) upward force to carry or lift some objects, animals, and people. These balloons would be made from a lite-weight cotton cloth material with paper glued over it so as to seal it from air leaks which would release the internal air pressure and deflate the balloon and reduce its altitude.

The famous United States, statesman, inventor and one of the electric pioneer's, **Ben Franklin** witnessed the first manned (by **Jean-Francois-Pilatre De Rozier**, and **Francois Laurent d'Arlandes**) balloon flight while in France on November 21, 1783, and then some of the other initial balloon flights, and with some filled with hydrogen gas which is even less dense and lighter than heated air and will effectively allow a greater upward force (lift) than just hot air alone, and can therefore lift more weight per (same) volume of gas, and without the need to have an on-board air heater. Some of these flights went several thousands of feet high and traveled more than 10 miles away. People who knew of these balloon flights (and even later, such as with the first airplane and rocket flights) were amazed and considered them as a wonderful technological discovery and advancement for mankind. Ben Franklin soon told **George Washington**, the President of the country of United States Of America (USA), and in 10 years later, George Washington became privileged to witness the first manned balloon flight launched in United States, and this was on January 10, 1793 at Philadelphia city in the state of Pennsylvania (PA). The balloon landed eastward in the state of New Jersey. The pilot was Jean-Pierre Blanchard, and who previously was the first to cross the English Channel on January 7, 1785, and while delivering the first air-mail from England to France.

A lighter than air balloon rises upward due to a similar process that happens with an air bubble in water, or warm water that is less dense that rises upward to the water surface. Another example is when ice rises upward in water. The simple answer is that the air bubble is less dense and-or lighter than water around it. The internal pressure of the water, due to its column or weight above it and-or at its sides will move (due to its pressure or force) into (ie. displace, by force, to take the place of) the position of the less dense, less weight bubble, or denser, heavier air as for a balloon, and replace (opposite of displace) its location and it will be moved or forced to a less dense, less pressure region above it. [Fig 228]
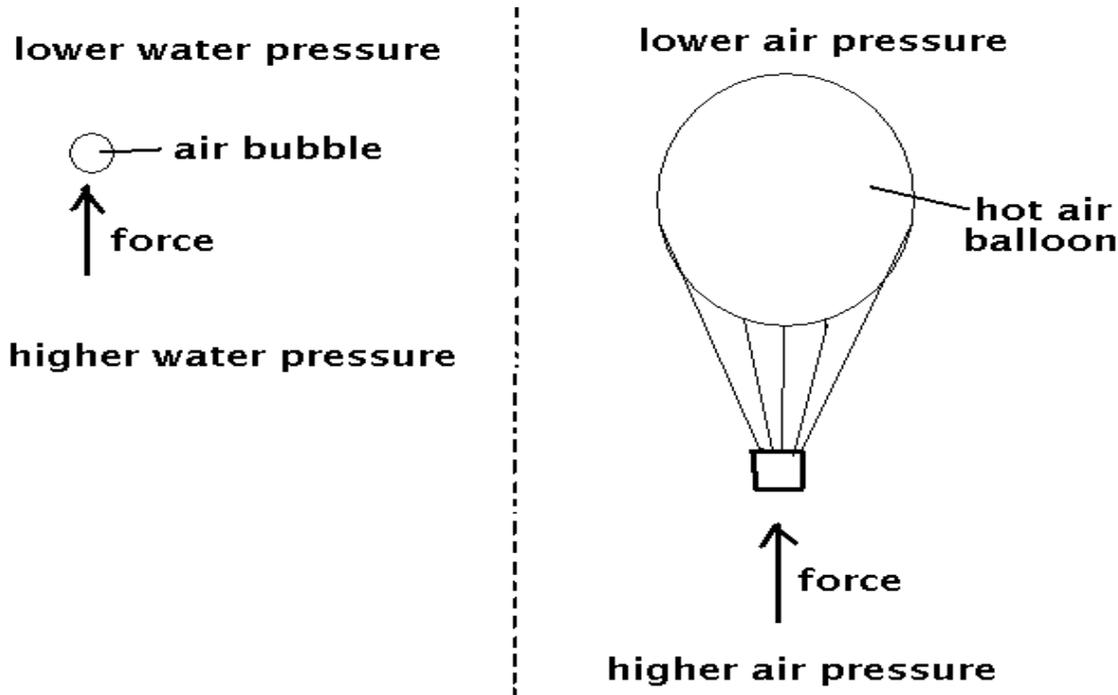
Though a balloon's construction material is heavier than air, its volume displaces (resides in its place) the air around it like how an air bubble in water displaces the water around it. Regular air in the balloon will not let it go upwards, and it needs to be filled with something lighter (less dense and weight) than the surrounding air pressure upon it. The hot air, hydrogen

weight of a cubic foot of water displaced by an object having a volume of a cubic foot (7.48 gallons) is 62.43 pounds (lbs) of weight. The weight of a liter = 1L = 1000g (gram weight, if water) = 1kg = 1000cm^3 is 2.205lbs. If you had a lite-weight and rigid box structure with 1 cubic foot of empty volume in it, it would take about 62.43 pounds of downward force so as to make it submerge below the water surface level. Due to this upward (buoyant) force, this also means that an object submerged in water will be less heavy (ie., weigh [downward force] less) by an amount equal to the weight of the volume of water it displaces and-or the (upward) buoyant force. In short, the water pressure is essentially forcing the object upward to a region of lower pressure at the surface, hence against the downward pull of gravity, and the object will then effectively weighs less in the water.

The concept of displaced water is also a very effective way to calculate the volume of an object by measuring the equivalent volume of water it displaces, which can be determined by the difference of the height levels of the water that it is placed into - such as a calibrated measuring container. The (mass, rather than weight) density of the object can be found by dividing the mass of the object by its volume. Note: weight = effective force = (mass) x (acceleration) = mg, and therefore, mass = weight / g. Knowing the density (density = mass / volume) of the object can also be used to determine the specific element that the object is composed of, such as determining if a king's crown is composed (ie., containing, consisting) of pure gold which has a certain known density of matter.

Due to being capable of displacing a large amount of heavy water, underwater balloons with a suspended basket or rope beneath it are very practical for helping lift heavy weighted objects to near the surface of the water for retrieval. For this to happen more easily, a deflated balloon must first be taken beneath the water to the object's location and then filled and expanded with air so as to displace water. The larger the air filled balloon at a given depth, displacing a larger volume of water and its equivalent weight, the more total "lift" or buoyant force it can produce. This can also be achieved by using several smaller sized balloons connected to the object, and the total lift or buoyant force will be sum of the lift forces from each individual balloon.

A submarine can remain at a certain depth of water, and without naturally rising upward into a less depth of water, or naturally moving downward to deeper depth of water. It can remain at a depth by placing enough water into its (water) ballast (ie., weight(s) used for stability and control) tanks to where the average density of the submarine structure has the density of the surrounding water. In other words, the submarine will have the same weight of the water it displaced. If it weighed more, it would sink, and if it weighed less, it will rise upward. To make the submarine rise up to the surface, the water in the ballast tanks is simply replaced with low density and weight air. Likewise, to submerge to a certain depth, a certain amount of water is let into the ballast tanks. A hot-air balloon may use some ballast weight such as sandbags used to help set a limited altitude for a hot-air balloon, or to give it some extra altitude by discarding some of it.

The amount of the upward buoyant force or " effective lift" is due to the amount of displaced water and-or the difference in water pressure on the upper and lower surfaces of the object. Due to this, even at great depths and pressures, the buoyant force on an object remains the same since the difference in pressure upon its surfaces is the same.

By the second half of the 1800's, experiments of machine powered balloons ("airships") were beginning to be made by several people, and first using steam in 1852, by **Jules Henri Giffard** in Paris city, France. An electric motor was first used in 1883 by Albert and Gaston Tissandier. In 1884, in Paris city of France, **Charles Renard** flew the first practical and controllable, electrically powered airship. **Alberto Santos Dumont** (1873-1932), from Brazil, made the first practical gas powered balloon and tested it in Paris city of France in 1898, and this balloon was propelled forward by using a spinning propeller which pulled this airship forward. He was also an early and significant pioneer in airplane flight soon after the success of the Wright brothers first combustion ("gasoline", "petrol") engine powered, controllable (ie., steerable) and sustained airplane flights. Dumont is generally credited with making the first plane with wheels (rather than curved wooded skids or ski's) to assist with take-off and landing of the plane. The Wright brothers initial flights used a weighted catapult so as to give the airplane some initial power, and wheels would of been useless in the sands at Kittyhawk. Wheels offer low friction energy losses, and practicality.

In 1848, **John Stringfellow** (1799-1883), from England, created a steam powered, unmanned airplane, but it was also crude, under-powered and very limited for sustained manned flights. It made short "hops" or distances rather than have a sustained flight, but it did indicate and helped inspire the possibilities of future advancements and powered manned

propeller is essentially grabbing and collecting some air in front of it and pushing it back towards the wings. This will also cause an equal and opposite force in the opposite direction which will cause the plane to move forward with the same force of the wind it is pushing back towards the wings. [FIG 229]
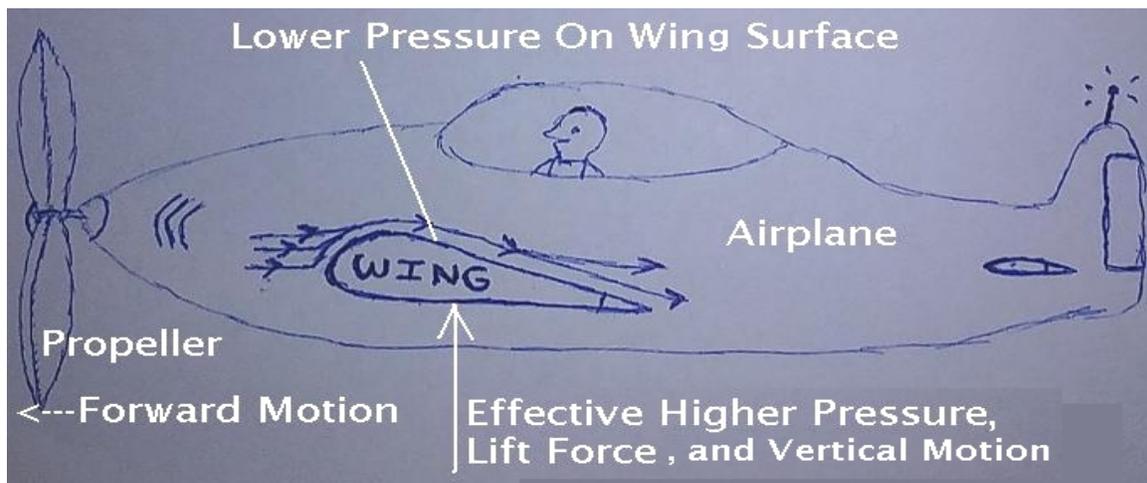
A noted good attempt, similar to **Samuel Langley's** attempt of creating and flying an airplane at about the same time frame as the Wright brothers, was from **Richard Pearce** of New Zealand. With some more assistance and resources, he could of made a plane that was practical enough to be publicly and repeatedly demonstrated. Some of his knowledge and ideas were important and eventually became helpful to other airplane makers, but his claims were generally not publicly documented by him at the time. Even the Wright brothers did many of their test flights in secluded locations so as to help keep them private from potential competitors. One concept Pearce used tor his crude and under-powered airplane was that of using the previously known concept of ailerons used for balloon airships. Ailerons ("flaps") are small movable wing extensions or segments of the wing itself, so as to help control the direction and-or stability of a flying vehicle. His flying machine and concepts are considered as a very good attempt of powered flight, but his airplane didn't have a sustained flight capability since it could only be flown in "hops" (short distances). He later acknowledge the Wright brothers first practical and independent flying achievements which include a special curved wing shape to effectively create a lifting force which is due to the difference in air pressure (a force) upon the upper and lower wing surfaces, and with the higher force (ie. usually, normal air pressure) being applied to the bottom of the wing and forcing the wing and plane vertically.

In 1916, the first airplane manufacturing company was created by the Wright brothers and was called the Wright Company. It briefly joined together with their rival, the Glen L. Martin Company so as to improve airplanes. Later, the Wright Company became known as Wright Aeronautical and focused on the manufacture of more powerful airplane engines, and specifically, improved engines based on the original 1901, high power, Manly-Balzer radial engine created for Langley's failed experimental aircraft just before the Wright brothers success. This company later joined with (Glen) Curtiss airplane company and became the Curtiss-Wright corporation. In 1916, the Boeing company started making seaplanes that could take-off and land on water, and this helped to overcome the problem of the limited number of airplane runways (long "take off" or launch surfaces where enough speed and lift is first developed) available at the time, and especially near more remote destinations in need of supplies and where no runway could possibly be built. As of the year 2022, the Curtiss-Wright company is still in operation and making various technology products. Their internet website is: curtisswright**.**com   After the business relationship with the Wright Company, the Glen L. Martin Company continued onward and would manufacture both airplanes and rockets, and later, this company joined with the Lockheed company and became known as Lockheed-Martin.

The first human to walk on the Moon was **Neil Armstrong**. He was born in 1930 in Ohio state (where the Wright brothers are from) and was already flying an airplane in 1946 before he had a car (automobile) license. Orville Wright lived long enough, up to Jan 30, 1948, to know of jet powered airplanes, and the increasing dreams and possibilities of advanced airplanes and rocketry. Neil took some small samples of the Wright brothers first airplane to the Moon and back in honor of the Wright brothers and their achievement for all of mankind. Neil mentioned that he never met Orville in person and

Aviation Discoveries Before The Wright Brothers First Manned Airplane Flight

The glider (unpowered, no engine, manned (piloted) airplanes) makers, and later, the Wright brothers naturally based some of their work on the previous knowledge of others. Scientists and-or inventors have no need or desire to invent the wheel, so to speak, but rather try to understand and-or improve it. The Wright brothers also used their own imagination and observational discoveries noticed in nature, such as with the flight ability of bird wings and their movement, various "propeller" seeds of trees, flying insects, various types of fish (ie. their tail-fin being the "rudder"), boat sails, windmills, hot-air balloons, ("box") kites (ie,, with wing surfaces), and (non-machine powered) gliders (essentially large kites, capable of a lifting a pilot and sustaining a reasonably long horizontal movement and direction, particularly in the direction of any oncoming wind). Before the first self-powered airplane flight, the Wright brothers made and experimented with manned gliders beginning in 1900 at Kitty Hawk, North Carolina state, and it would only be a matter of time and experiments such as new wing designs and their increased lift efficiency found with the aid of a small wind tunnel they had built.   A wind-tunnel is structure that consists of a powered fan blowing (ie., moving) a fast and controllable current of air (ie., wind) into and through a long rectangular box and expelling it over the wing models and-or airplane shapes and surfaces. The wind-tunnel was first created in 1871 in England by Francis Wenham and John Browning. The Wright brothers used patience, persistence, calculations and technology (such as the engine itself) till that first airplane could be made and then flown. Unlike a balloon at the mercy of the windspeed and direction of the wind, the airplane allows people to fly in any direction needed, and to any practical landing location (ie., a designated flat, "runway" surface). The Wright brothers first airplane had a similar design to the 1896 biplane (bi-wing or 2 main wing surfaces. so as to slightly (about 20%) increase the total lift force) glider made by **Octave Chanute and Augustus Herring** from the United States of America, however the Wrights used a wing shape with its upper surfaces deliberately made slightly curved (sometimes called "camber") since it was first noticed by **George Cayley** who, many years previously, made the first scientific study of how things could possibly fly, and of a specially curved wing surface shape called an "**airfoil**" that improved lift. Cayley is also further discussed below in this book. The Wright brothers eventually perfected their glider to the point that it was practically a controllable airplane and without an engine to sustain longer flight. Chanute and his knowledge provided some help to the Wright Brothers before their first flight. Many modern **windmill** wings or blades for the generation of electricity also use an airfoil shape so as to help increase the efficiency of the energy collection and conversion of the available wind power to available electric power. In brief and without consideration of the airfoil shape, for a windmill ("wind generator") blade to collect the wind and its kinetic energy, it cannot be flat or at a 0° angle because it would collect no wind energy, and it also cannot be perpendicular or 90° to the wind for its force would be simply pushing the blades in the direction of the wind, and rather than rotating them. A compromise blade angle must be found by experiment, and obviously a 45° angle is a reasonable place to start and it will usually be lower than that. The best angle will produce the most output power for a given windspeed. If the blade is at a too low of an angle, then the input energy collected will be low, and some of its energy will be lost when it is deflected or diverted at an angle. If the blade is at too much of an angle, then the force applied will not rotate the blade as much, and the drag (air resistance, friction) caused by the side of the blade colliding with the air as it rotates will also reduce the speed of the blade and the power output.

After the first airplane flight, the knowledge of flying, airplanes and their manufacture expanded quickly and greatly all over the world, and with many people still being naturally skeptical and unconvinced of the manned-flight news until they have personally seen photos of an airplane in flight, and-or actually seen an airplane flying nearby.   By the end of just the first year of airplane flight, the Wright brothers were flying for several miles of distance in their airplane at a relatively low altitude. By as early as 1910, some planes such as the Wright Brothers popular "Model-B" plane were already capable of flying up to and over a mile (5280 feet) high (altitude) in the sky. The study of flight, airplane design and efficiency has continued ever since their first flight.

Rewinding back in history, the first practical and flown manned gliders were developed by the **(Otto and Gustav) Lilienthal brothers** from Germany, in about 1891 which was only a few years prior to the Wright brothers first airplane flight. Nonetheless, gliders were very important for airplane development, and are a fun way to fly. In modern times, some highly experienced skydivers at a high mountain location, or jumping out of an high airplane, put on a suit called a "wing-suit" that has a wingshape and some gliding capability which gives them an incredible fast, bird-like flying sensation and view of which is often recorded for others to experience. At some point with sufficient altitude left in their journey, these fast moving skydiving pilots will deploying a parachute so as to slow them down and land safely.

and perhaps maximum engine power and-or speed should rather have a limit.

A paramotor (a parachute type wing shape, and motor) is a backpack-like, portable, lite-weight combustion or electric engine powered flying paraglider which is essentially a parachute-glider that uses its cloth-like fabric wing, somewhat similar to a parachute, to enable the lift process.   Paramotors are generally not very fast, but can travel reasonably far and at reasonably high altitudes. Paramotors require only a very short runway, perhaps less than 30ft, and can safely land in a clearing (absent of trees, brush, fences, wires, stones, water, etc) at the on the top of a mountain after a few minutes of flight from the ground level. A paramotor is generally much less expensive than a regular airplane and-or ultralight aircraft. Having an airplane pilot license is sometimes not required, but pre-flight training by a certified paramotor flight instructor is recommended and-or required by local (ie., location) laws. Due to that there may be other planes in the sky, there are laws, rules and-or recommendations to follow, such as what altitudes to fly at, and-or not to fly at.

Today, experiencing a form of flight and being a pilot is relatively cheap with modern remote controlled small electric engine(s) helicopters (non passenger "drones") having on-board, live radio-wave ("wireless") video for steering, location, and-or "filming" (image recording) capability via a small and lightweight camera. This is a virtual ("nearly", "as if", virtually, simulated) flying experience. Being small helicopters, they require no runway and can be launched almost anyplace and at anytime after charging up its power battery. These pilots and planes often need to observe any local laws created in regard to their use, and to be fly fair and safe. I would personally recommend blade protectors or guards ("standoffs") to help prevent device damage and-or injuries to others. These devices are also used for many practical and valuable applications such as for search and rescue, movie making, aerial photography, data acquisition and map making.

Several human powered lite-weight airplanes have been designed and successfully flown at low altitude (less than 100 feet high due to having only a low "air speed" and-or forward movement, hence a low lift force). These would of made the Wright brothers proud since they use human leg power, much like riding a bicycle, and so as to turn a long propeller. The drawback is that a very long wing surface (about 112 feet) is then needed to create enough lift force. The longest verified flight as of the year 2020 was about 71.5 miles of distance in about 4 hours of time. The average speed of this flight was therefore: From distance = (speed)(time).   speed = (distance / time) ~= 71.5 miles / 4 hours ~= 17. $\underline{9}$ ~= 18 miles per hour. The pilot and human engine power for the airplane flight and record was **Kanellos Kanellopoulos** from Greece in 1988. He flew the plane from and between two islands in the Mediterranean ocean. He was previously a cyclist in the 1984 Summer Olympics sports events, and so he had a good physical condition, strength and duration to achieve this.

Several solar-electric powered and piloted airplanes have also been designed and successfully flown and they also require wings longer than most small commuter airplanes that have a combustion engine. These planes are therefore rare, costly and impractical, but it is still good to know that it has been tried and accomplished.

Before extreme high altitude airplanes, jets or "rocket-planes" were made, the only way to test peoples (mankind's, piolets) abilities in very high altitude flight near the edge of space was with high altitude balloon flights. The data from all these high altitude flights was, and still is, very useful for all the future manned high altitude rocket-planes and spaceflights. Amazingly, as early as 1800, the year Volta invented the "Voltaic pile" (ie., battery), balloon flights were already over 4 miles high and near the zone of where supplemental oxygen is typically needed for most high mountain climbers.

In 1932, in an internally air pressurized craft ("gondola", "capsule"), Swiss scientists **Auguste Piccard and Max Cosyns** were the first to reach 10 miles high (= 52800 ft). Piccard is also credited later to the creation of the bathyscape (strong container or vehicle for deep water, essentially a "submarine") which is used for exploring deep water, and with the inside occupants still at a normal air pressure for normal breathing. Piccard has therefore made many additions to the knowledge for survival in both low and high pressure environments. His son **Jacques Piccard** and **Don Walsh**, from United States of America Navy, are among the few explorers to ever reach the deepest part of the ocean which is in the **Mariana Trench** region known as 'Challenger Deep' (named after a ship that discovered this very deep area) in 1960. This deep location is in the west Pacific Ocean near the island of Guam. While a typical space capsule or container only has to endure a relatively low (air) pressure, say 14.7psi, from the inside and upon its inner surface, the Trieste bathyscaphe ("deep-ship)", that is much like a specialized submarine vehicle, had to endure the extreme danger of thousands of pounds per square inch from the outside water pressure at the depth of about 35800 feet (6.78 miles deep).

[This space for book edits] Please make a helpful donation to the author of this book.

after a rocket with the spacecraft is launched vertically, it will be steered to climb higher towards space at about a 45° angle with respect to the ground or horizon for much of the flight, and then later, to a lesser angle so as all of its fuel energy is converted to a kinetic energy and velocity of the spacecraft in the horizontal direction (ie., parallel to Earth's surface), rather than the vertical or higher direction, and once the desired orbit altitude is reached.

The downward or vertical drop (downward distance) over (ie., below) the horizon (horizontal, level or straight line) of the Earth at its surface (ie., 4000 miles, the radius from the center of the Earth) is about 8 inches per mile. To orbit the Earth near its surface, a horizontally traveling object will need to likewise drop by the same value of 8 inches per mile. For much higher altitudes of orbit, both the force of gravity and the curvature (ie., vertical drop per horizontal distance) of the Earth gets less from the center of Earth, and therefore the gravitational acceleration caused by that amount of continually applied force of gravity is less. Here, in more distant or higher altitude orbits, the orbiting object will then fall (vertical drop distance) a less amount of distance per amount of time, perhaps just 1 inch per mile of orbit travel, and therefore, that object can actually go slower in the theoretical linear, horizontally direction and maintain its altitude of orbit. [FIG 230]
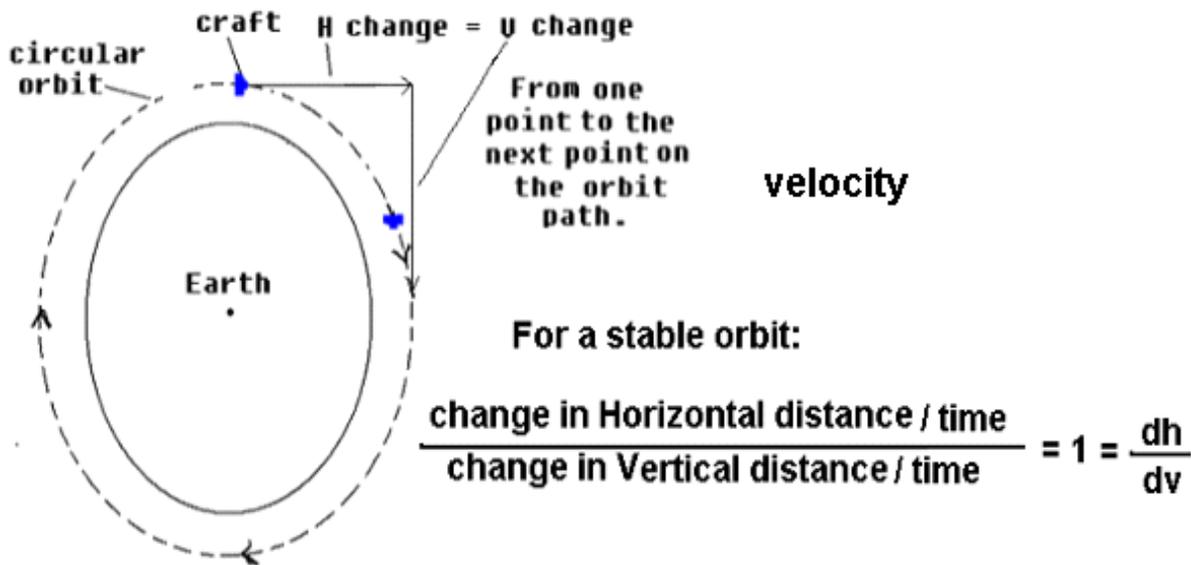
The amount of vertical drop per unit of time depends on the gravitational acceleration = (change in velocity / change it time) at that altitude.

At low Earth orbits, the gravitational acceleration due to the Earth is only slightly less than at ground level.

Likewise another equation similar to that shown in the image above is:
(change in vertical distance / time) / (change or drop in the curvature of Earth below the horizontal / time) = 1

Here is a derivation of the 8 inches per mile drop in vertical height below a straight or level horizontal line on the Earth's curved surface, and that the specific vertical drop from that line is due to its amount of curvature (below Earth's level, straight horizon) at that radius (here, 4000mi), and here specifically, at the surface of the (theoretical, averaged, smooth, level terrain) Earth. [FIG 231]

effectively remain the same value.

The larger the distance or radius from the center, the greater the circumference, and for a given angle, the corresponding or subtending segment or arc of the circumference actually gets "flatter" (less of a curve) or more straight-like or more linear or line like. It could be said that the curvature at a farther radius value is less per degree, and therefore, there will be less effective drop below the horizontal line per mile at that radius or orbit distance, and the craft will not have to go as fast to effectively regain the altitude it lost due to gravity of which is also actually less for farther radius orbits.

Due to that the force of gravity upon two objects or masses also depends on the distance between them, a higher altitude spacecraft will have less force of gravity constantly applied to it, and therefore, it will not accelerate as much, and the value of the gravitational acceleration at that orbit altitude will be less:

Note: Circular orbit distance or altitude from the center of the Earth = Radius of orbit = Ro =

Distance from the center to the Earth's surface   +   The altitude of orbit above the surface of the Earth

Earth's radius   +   Ae
Ro =   4000mi   +   Ae   : Ro = radius of orbit, from the center of orbit

Altitude (mi)	g (m/s^2)		Altitude (mi)	g (m/s^2)	: and Ro = Radius of Orbit = Orbit Radius:
					: Altitude is that measured from Earth's surface
0	9.81		1000	6.25	: Ro= 5000 m = 4000 mi + 1000 mi, from Earth's center
100	9.33		2000	4.33	
200	8.87		4000	2.426	: = 8000 mi = 4000 mi + 4000 mi, from Earth's center.
300	8.47		8000	1.075	: = 12000 mi = 4000 mi + 8000 mi, from Earth's center
400	8.09		20000	0.268	: g = 26.8 cm/s^2 = 0.0268 m/s^2
500	7.73		25000	0.183	: r = 29000mi = 4000 mi + 25000 mi , g = 18.3 cm/s^2

Due to the inverse square law of forces, if the distance increases by a factor of (n), the result is a force value of: force / n^2.   distance2 / distance1= n. At Ro = 4000 miles from the center of the Earth, g = 9.81m/s^2, and at twice this distance which is Ro = 8000 miles from the center of the Earth, n = Ro/Re = 8000mi/4000mi = 2, the value of (g) is: (g) = 9.81m/s^2 / n^2   =   9.81m/s^2 / 2^2   =   9.81m/s^2 /4 =~ 2.45 as shown in the table above.

(go) at a certain radius from the center of Earth = (g) at Earth's surface / ((orbit distance from center, mi) / 4000 mi)^2
go = (9.81m/s^2) / (Ro/Re)^2 : A formula for the gravitational acceleration at a given radius of orbit of Earth
go = (9.81m/s^2) / n^2   : gravitational acceleration at that **o**rbit , n = ratio of actual orbit distance, to Earths radius

For an object to remain in a stable orbit, the force of gravity (Fg) upon an object (with mass = M2) must equal the centripetal force (Fc) upon that object. The **centripetal force** upon an object can be thought of as equal to the equal and opposite force (ie., centrifugal force) that is pushing the object away from the other mass, here Earth which is pulling upon it with gravity. When these forces are equal or "in balance" with no net difference, the object will not move closer or further but remain in a stable or constant orbit distance. As noted previously, the force of gravity is less for higher orbits. If you were to attach a string to an object and move it as if in orbit about you, the tension in the string would be equivalent to a centripetal force.

Fg   =   Fc   : When the are equal: Fg - Fc = 0

$\dfrac{G\ M1\ M2}{r^2}$ = $\dfrac{M2\ v^2}{r}$: G = universal gravitational constant, **M1 = mass of Earth** in kg
                    M2 = mass of orbiting object in kg, v = velocity of orbiting object in m/s^2,
                    r = radius of orbiting object from Earth's center, in meters = 3959 mi
                        = radius of Earth + altitude of orbit
                    : If the forces are the same, the accelerations must be equal to (g) at that alt.
                    : F = ma = Fc = M v^2 / r , hence (a) = v^2 / r and (m/s)^2 / m = m / s^2

down, perhaps by using a rocket engine to provide a force in the opposite direction to its forward travel, it will then not have enough centripetal force (Fc) to balance (ie., 0 net force, 0 difference) the gravitational force Fg upon it. Fg will be greater than Fc. The altitude of the spacecraft will decrease, and it will usually have some remaining horizontal velocity if it was not completely slowed down in horizontal motion to 0 mi/s. The spacecraft will then also have a downward vertical component to its motion and it will accelerate toward the Earth due to the net **e**ffective downward force (F**e**) of gravity now acting upon it **:** (F**e** = Fg - Fc).

In order for the spacecraft to make a relatively soft landing at say 0mph, all of the high kinetic energy of it must be expelled or removed, and much of this energy will be lost as friction (resistance to movement, and converted to heat) and-or pressure with Earths (air) atmosphere, all due to the high speed of the reentry flight. Some kinetic energy (particularly the velocity factor) will also be lost with a parachute's upward drag force. Some spacecraft even have a rocket motor or engine to reduce its downward motion and kinetic energy during landing, and some spacecraft may deploy a type of "air bag" (ie., a type of strong material, inflated balloon) at the bottom area of the spacecraft to help make a softer landing. A spacecraft could reside and travel in a lower orbit if it can be made to have the natural orbit condition of Fc = Fg, and at a lower orbit, the velocity needed for a given mass will be higher since the force of gravity (Fg) is higher at lower altitudes.

The author **Jules Verne** (1828-1905) was probably fascinated by the more modern telescope views of the Moon, advances in technology, and considered Newton's theory of orbiting in about 1865 while writing his book called 'From Earth To The Moon', and then afterwards, writing another book called 'Around The Moon' in 1870. In 1902, before the first practical airplane was invented, and inspired by Verne, the first science fiction and outer-space type movie called "The Voyage To The Moon" was created by **Georges Melies** (1861-1938). This surely helped inspire the imaginations of people and the future of science and flight. Shown in the movie, the passengers entered a large projectile and were launched out from within a large cannon (a propulsion and aiming tube for the projectile). This idea was most likely based on Verne's books which seems to consider the previously known (Isaac) "Newton's Cannon (or projectile)" theory about how to orbit the Earth. While returning from the Moon in the year 1969, Neil Armstrong (the first person on the Moon) mentioned Jules Verne. Verne's book had essentially and reasonably predicted some similar details of the actual ventures to outer space and the Moon. Verne also wrote many other highly popular science-fiction books, some of which were later made into popular movies such as: 'Around The World In Eighty Days', 'Journey To The Center Of The Earth' and 'Twenty-Thousand Leagues Under The Sea'. A contemporary to Vern was **Herbert George (H.G.) Wells** (1866-1946) who wrote books such as: 'The Time Machine' in 1895, 'The War Of The Worlds' in 1898, and 'The First Men In The Moon' in 1901. These writers turned actual, known scientific explorations, discoveries and facts, and mixed with their own "vision" (ideas about the future), imagination, dreams and fantasy into a "mixed literature" called science fiction (fictitious, fictorious, meaning probably "fake", "made-up", "exaggerated" and often with some facts being real or true so as to make a known and believable connection with reality), and of which can even inspire real science, invention and research in an increasing, circular-like growing manner.   Some of the earliest and significant science-fiction, outer-space adventure type of movie series were: Buck Rogers (literature by 1929, movies by 1930, TV by 1950), and Flash Gordon (literature by 1934, movies by 1936, TV by 1954), and which was obviously inspired by the entertainment success of the Flash Gordon and previous media about science fiction. Some other popular, and relatively modern science fiction writers are: **Arthur C. Clarke**, **Isaac Asimov,** and **Robert A. Heinlein**. Arthur C Clarke who wrote book such as the 2001 A Space Odyssey, was also an early visionary of man-made satellites for (re-transmission) relaying communications on Earth. Robert A. Heinlein, was a writer and visionary who included many science facts and social ideas in his science fiction writings. Isaac Asimov, a writer and visionary that promoted the concept that robots should only be made to help humanity. In late 1958, the military launched such a communication satellite which mas mainly for proof of concept and experiments. About three years later, **Telstar-1** was the first public communication satellite, and it was launched on July 10, 1962, and which soon inspired a popular instrumental song called Telstar in late 1962.   Telstar was placed in a relatively close to Earth elliptical orbit, and therefore had to be tracked (ie., aimed at) more by receiving antennas here on Earth, unlike some of the more modern communication satellites placed into a very high altitude, geostationary orbit (ie., has the same rotational or angular velocity as Earth's axial rotation) about Earth. The Telstar mission publicly demonstrated that its technology was practical for public communications by relaying some TV (electronic TeleVision) and telephone signals.

A modern large and heavy rocket can enter a high velocity (about 17100 mph =~ 4**.**75miles/s after starting out at 0 mph and then gaining more and more velocity), low altitude (about 220 miles high, such as for the ISS - International Space Station), orbit in about 9 minutes of time.   It is sometimes mentioned how many seconds of rocket fuel and-or the

A company recently created (about 2021) by the **Elon Musk** (born 1971, in Africa) has currently manufactured a (stationary and-or portable) satellite transceiver system called **Starlink** which is for delivering and-or improving communication services such as for the popular Internet (electronic, digital - 1's and 0's) communication system for undeveloped, difficult and-or distant regions on Earth. Though each Starlink satellite launched for this system is relatively expensive, the users transceiver system is relatively inexpensive and has a relatively low monthly fee. He has helped employ many people, and with his earnings, he has branched off into the exciting field of space and rocketry of which also employs many people, and so as to help benefit all the people of Earth in some way, either now or in the future. As of the year 2022, Musk is still launching many small communication satellites into a relatively low altitude orbit about Earth on his SpaceX company rockets, and is associated with several other popular companies such as Nasa, Paypal (for digital payments using the Internet communication system), Ebay (a popular selling store location on the Internet), the Boring Company (for tunnels), and Tesla (for electric automobiles, power storage systems such as electric batteries).

Many of today's rockets to space use various types and combinations of solid or liquid fuels such as gasoline, nitrogen oxide, methane, liquid hydrogen and liquid oxygen. On March 16, 1926, the first successful liquid fueled rocket launch was made by **Dr. Robert Goddard,** (1882-1945) from the state of Massachusetts in America, and who was a scientist and teacher, and who theorized about rocket construction and space flight. His liquid rocket concepts began in about 1909 and his rocket fuel was gasoline and liquid oxygen that was then mixed in the rocket engine during launch. Later, he would launch his rockets at a more practical and larger test range (ie., a large unused land area) site at Roswell, New Mexico. The best rockets he made did not go very high (about 2 miles high), but it was still enough to prove they had potential abilities such as high-altitude research, and perhaps, for space research. Using liquid fuel allows the fuel flow to be controlled with pumps and valves, whereas it is much more difficult to do this practically with a solid rocket fuel. Rocket engines need to produce a controlled ("controlled explosion") combustion of just some of the total amount of rocket fuel, and so as to produce a practical force in just the (forward, travel) direction of the rocket body or mass. Goddard is credited to the first modern methods of steering (guiding, guidance, aiming control, stability such as with gyroscopes) rockets, and creating the concepts of multi-stage rockets that are essentially a rocket attached to the top of a another rocket. A heavy stage can be discarded once its internal fuel is used up, and this will significantly improve the efficiency of the rocket. As a youth, and even before plane flying was proven to be practical in 1904, Goddard was highly interested in all forms of flight such as with birds, kites, and hot-air balloons. In 1908 Goddard thought that cameras could be sent to and back from the orbit of other planets. Goddard, in 1912, was also one of the first pioneers in vacuum tube construction and its uses at about the same time as its first certified original inventor - **Lee DeForest.** Goddard's rocket studies and discoveries would eventually lead to the development of rocket-engine powered airplanes ("jets") and then the rockets to outerspace. For a historical note, "blackpowder" is a solid, granule fuel developed for small rockets and was invented by the Chinese in the early 10th century AD, and today, it is used in some legal and-or permitted (due to potential dangers of fire and-or injury) "fireworks" which are colorful, decorative light and sound displays up in the sky during festive occasions. The first hot gas, and here, particularly a primitive steam engine that could produce (here rotating) motion via the force and-or pressure of the hot gas is credited to Hero of Alexandria in the first century AD. His (steam) engine is commonly known as the aeolipile which can be translated to English as "wind ball", "wind sphere", "wind spinner", and possibly a "wind engine". This engine eventually would eventually lead to things such as steam trains for transportation on rails (ie., metal rail roads), and turbines (ie., mechanical energy transfer blades, or energy transducers) for electricity generation. Rocketry is also a creative hobby, and some people joyfully build the available kits for purchase and launch them in relatively safe, open (no nearby man-made structures and-or dry flammable grasses) and permitted locations. Some of the rocket contemporaries to Goddard are **Hermann Oberth** (1894-1989) of Hungaria-Germany, **Konstantin Tsiolkovsky** (1857-1935) of Russia, and **Robert Pelterie** (1881-1957) of France. Oberth was also involved with the first rocket plane tests in 1929, and was a mentor (ie., a teacher) to **Werhner Von Braun** (1912-1977), (aka, ["Verner"] "Von Brown" as an English language derivative or interpretation of) of Germany. Braun and some other members of his rocket team became prisoners of war to the Americans (USA) and Russians (Soviet Union) in WW2 (1939-1945), and some of his rocketry knowledge would eventually became useful for the USA space program directed by NASA (National Aeronautics and Space Administration, of the USA). Since his youth, Von Braun was a joyful and artistic visionary of the future of spaceflight, space research and exploration.

The first practical **jet engine** for airplanes is credited to **Frank Whittle** (1907-1996), from England, in 1930, and the planes with jet engines would eventually be simply called as "jets". The basic concept of a jet engine is to use a highly

rocket fuel or energy needed to keep it moving at the same orbit speed.

Speeds of several miles per second are common for spacecraft such as for the **Apollo-11** craft moving at about 7 miles per second on its way to the first manned Moon landing on July 20, 1969, and this is considerably more than the 0.64 miles per second that the Moon orbits about the Earth. **Neil Armstrong** and **Edwin ("Buzz") Aldrin** were in the Moon landing craft called "Eagle", and **Michael Collins** was the pilot in the Moon travel, orbiting and return to Earth spacecraft called "Columbia". Each was born in 1930, the same year planet Pluto was discovered. The launch of Apollo-11 to the Moon was on July 16, 1969, at 9:32AM, Eastern U.S. time, from Florida state, and returned to Earth eight days later on July 24, 1969 in the Pacific ocean. The Apollo-11 moon spacecraft ("Eagle") landed in the Sea Of Tranquility ("a mare", a relatively flat, dark basalt (ie., a hardened lava stone) region) on the Moon at 4:17 Eastern U.S. time.

Shortly before the Apollo-11 mission and just 64 years after the Wright brothers first airplane flight, the **Apollo-8** mission craft was the first manned craft to arrive at and orbit the Moon on December 24, 1968, three days after launching from Florida state of the United States. Some of the initial verses from the historic book of Genesis from the Old Testament Bible book were read and broadcast to the people of the world on December 24, 1968 by the astronauts in their spacecraft. This first manned mission to the Moon was profoundly important for all the later missions. Their craft didn't land on the Moon, but it was needed for proof of concept, telemetry (remote or distant sensing and-or measuring), communication, photographs, and many other initial tests, modifications and improvements before the future landing missions to the Moon. This mission could be thought of as mankind's rehearsal for the Moon landing. The world celebrated vanguard (first, leading) crew was **Frank Borman, Jim Lovell, and Bill Anders**. Their mission included lunar photography images so as to find some practical landing locations for the future landing missions. They orbited the Moon for a total of 10 times during nearly a day of Earth time. All of the Apollo missions that went to the vicinity of the Moon would land (ie. softly, "splashed-down") in the Pacific Ocean, and usually near various retrieval ships and helicopters.

In March, 1969, Apollo-9 tested the lunar landing craft and the spacesuits for the manned Moon walks while in orbit about the Earth. In May of 1969, Apollo-10 was a similar mission, but it also tested the lunar (Moon) lander at a height of only about 10 miles from the lunar surface. Through several years of time and Moon landings until the last Apollo space program Moon landing on December 19, 1972, a total of six crafts and twelve people landed and walked on the Moon's surface from Apollo-11 mission and up to Apollo-17 mission. Many scientific instruments such as a (Earth based) laser-light reflector (ie., like a mirror) surface were placed on the Moon's surface for studies to be performed when needed such as to calculate the distance to the Moon. Rocks and dirt were collected during all of the 6 manned landings, and this material was brought back to Earth to be studied until this day and with more modern and advanced instruments and analysis concepts. These missions were made possible by the accumulation of knowledge, technology and talents of many people since ancient times to these modern times, and as a recognition and goodwill gesture for the people of the Earth, a piece of Moon rock that was brought back from the Moon was honorably given to each country and its people.

Outer-space has no pressure (0 psi) since it is a vacuum and generally void (empty) of any gasses (such as air) or fluids to induce or apply any physical force and-or pressure upon an astronaut or spacecraft. The air pressure used within the astronauts spacesuits as they walked on the Moon during an EVA's (Exra-, or External Vehicle Activities) was a relatively low value at about 4.9 psi so as to reduce the total amount of force necessary to move their arms, legs and fingers inside a "pressure stiffened" (like the surface of a balloon constantly pushing [ie., a force] outward and trying to retain its shape) spacesuit. The required amount of oxygen needed for them to breath properly was supplied from (cold, condensed) liquid oxygen tanks in the spacesuit's backpack structure, and their exhaled carbon-dioxide gas was removed from their helmet (solid head covering) area. A spacesuit is also a temperature insulated and controlled environment to sustain life, and also has radio communication installed in it. With a lower pressure in the spacesuit, astronauts must also avoid any decompression problems when they enter a region of different pressure such when they go back into their pressurized spacecraft (LEM, Lunar Excursion Module, the Moon landing craft). An air-lock is a small room an astronaut can go into and close so as to be slowly adjusted to a desired air pressure (such a 0 psi, 4.9psi or 14.7psi standard air pressure [STP]). If pressure changes happen too fast, dangerous forces and damage can result. Within the living quarters of a spacecraft, the pressure is maintained at about 14.7psi = 1 (std) Earth atmospheric (atm) pressure, hence the same value as the air pressure near the Earth's surface at common sea-level that most people are commonly used to and-or require for a long duration of breathing.

April 26, 1962, and without transmitting any data due to system failures. **Ranger 7,** from the United States, was much more successful, taking the first good resolution photographs of the Moon as it approached and impacted its surface on July 31, 1964. **Luna 10** became the first man made, artificial satellite to orbit the moon on April 3, 1966.   It mainly measured solar radiation levels, but it did not have a camera system to take photographs. **Lunar Orbiter 1**, from the United States, was the first satellite to photograph the Moon while in orbit about it, starting on August 18, 1966. A total of 5 Lunar Orbiters were launched at later dates up to late 1967. These high resolution photographs would be used to locate possible landing sites for the Apollo program to land people on the Moon in the late 1960's and early 1970's.   As of the year 2022, scientific spacecraft have reached all of the known planets in the solar system, including Pluto and some asteroids, and much precious data and photographs were collected and sent back to Earth via radio communication signals. At these distances from Earth, the received signals are very weak and just slightly greater than the strength of all the ("background", natural, man-made) radio noise from space and the Earth. Much of this noise is effectively eliminated by using radio antennas that have a parabolic shape of which will then focus and concentrate the weak incoming signals from a relatively small area and-or direction in the sky and space. These signals will then be electronically amplified, and have as much noise as possible removed so as to record and study the desired signal and data contained within it more effectively and accurately.

Space debris (ie., junk, garbage, parts) from rockets and satellites is dangerous and has become an increasing safety concern due to the damage that it can cause to other space objects such as satellites, spacecraft, and astronauts because it has a high amount of kinetic energy (KE), mostly due to its high velocity (ie., speed), and that energy can be converted to large values of force when a collision happens, and can do much damage.

Some noted first successful flights and missions:

Mariner 4 - Launched in late 1964 in USA, and reaching closest to Mars on July 15, 1965, the Mariner 4 space-probe (research spacecraft) was the first human-made craft to arrive nearby to and (crude, low tech and resolution) photograph Mars. These images showed Mars as having an oceanless, cratered surface. Mariner 4 came within about 6000 miles of Mars, but did not orbit it, but simply flew past Mars and went into a Solar or Sun oriented/focused elliptical orbit about the Sun. For electrical power, such as for the camera and radio communication with Earth, it utilized solar-panels and a battery. Data was saved onto a tape recorder, and it took about 9 days to transmit the initial volume (about 20, low resolution photos) of data back to Earth since the (digital, binary) signals used a low bit rate, and which does help reduce possible communication and-or data errors.

Venera 7, from Russia, became the first spacecraft to safely land on Venus, and on December 15, 1970.

Mariner 9 - On November 14, 1971, Mariner 9 became the first spacecraft to orbit Mars.

Mars 2 and **Mars 3** mission by Russia were the first to attempt a landing on Mars. Mars 2 crashed on Mars, and it is thought that Mars 3 landed safely on Dec 2, 1971, but the imagery was not functioning properly. Nonetheless, their main mission platforms were still orbiting which sent many valuable photos back.

Mariner 10 was the first spacecraft to flyby and photograph Venus, and on February 5, 1974.   It later took photographs of Mercury.

Venera 9 , from Russia, was the first spacecraft to orbit Venus, and on October 22, 1975.

Pioneer 10, launched March 3, 1972 in the USA**,** became the first craft to flyby (at about 81000 miles) Jupiter on December 3, 1973, and of which it obtained good resolution photographs. The flyby of Jupiter and its gravity attraction also increased the speed of Pioneer 10. **Pioneer 11**, launched on April 5, 1973, also made it to Jupiter on December 2, 1974 and used a gravity assist of Jupiter so as to direct its course to the more distant Saturn. Pioneer 11 reached Saturn on August 8, 1979 and took good photographs. Pioneer 10 and Pioneer 11 both continued on a path out of the solar system. By 1979, and due to their electrical power being depleted, communication to these spacecraft had ceased.

Viking 1, and then **Viking 2** soon afterwards, were the first spacecraft to successfully land on Mars and take many photos

time)) which leads to a higher velocity (= at) and more distance (= vt) per unit of time.   In theory, even if the lift force was just 1N, perhaps the rocket being very massive and-or weighty, the rocket would still move vertically upward, but it would be very slow due to the relatively low amount of the net or effective lift force applied, and therefore providing only a low amount of rocket acceleration, and therefore a low speed (velocity) and the rocket would probably "run out" (ie., used up, drained, not have any left to use) of fuel well before it reached orbit. From f=ma, a=f/m, and if the force or energy applied low, the acceleration will be low. It simply takes more energy (ie., fuel) and-or work to lift (ie., f=ma, and work = fd = mad) a more heavier (due to more mass) object a unit or certain distance, and this leaves less fuel to do other things such as lifting the rocket higher (against the downward force and pull of gravity upon it) and reaching orbit.

rocket engine vertical thrust force > 1 is needed for the rocket to move or be lifted upward or vertically
rocket weight   : Thrust to weight ratio   ,   weight = a downward, vertical force

A rocket will launch (begin to move vertically) when its fuel is combusted in the rocket nozzle or chamber, and the net sum of the force from each atom and molecule with their high kinetic energy that strikes or collides with the bottom of the rocket will be used to created the net vertical lift or thrust force. After some amount of time, the same combustion or thrust force of the rocket engine will then be used to produce both a vertical or lift force, and a horizontal force to move the rocket into an (elliptical or circular-like) orbit about the Earth.
.
In place of the term (rocket engine thrust), the term (rocket engine vertical force) can be used. In place of the term (rocket weight), the term: (gravity induced force on the rocket mass) can be used.   For the rocket to move upward or vertically, the upward or lift force must be greater than the downward force of gravity which is the weight of the rocket, hence their difference must be greater than 0. These forces (mathematically equal to f=ma) will cause a corresponding acceleration of the mass such as a rocket, and the vertical acceleration must be greater than the downward acceleration due to gravity. so as the rocket can have a net acceleration vertically and move upward. In other words, the upward force applied to the rocket must be greater than the downward force or weight of the rocket, and for that rocket to move upward or forward into the air. If an object, perhaps a plane, was at a constant altitude, the upward force would equal the downward force of gravity which is the weight of the object. The (instantaneous and-or constant) lift force generated by the wings must equal the downward acceleration or fall of the plane due to the force of gravity, and for it to remain at the same height or altitude. Regardless of the mass and-or weight of the object, the gravitational acceleration in the downward direction is a constant of 9.8m/s near the Earth's surface.

As the rocket uses up its limited amount of fuel, the mass and weight of the rocket will get lower, and the thrust to weight ratio will then be increasing, and this improves the efficiency of the fuel energy. The rocket will then accelerate and go faster since the same combustion force is now applied to a smaller mass or weight and it will accelerate. From F=ma we have: a = F / m   = (change in velocity) / (change in time).   Some rockets may have an additional stage (vertical, additional rocket section(s)), and the thrust to weight ratio may change, and mostly due to the rocket utilizing different, perhaps fewer and-or smaller, less powerful rocket engines.   The rocket stage(s) with the empty fuel tank are then quickly discarded as useless weight to keep lifting.   As the rocket climbs higher, there is less fraction with the air because the air pressure and-or density is reducing. This friction and-or force against the forward thrust of the rocket is sometimes called the rockets dynamic pressure, and at a certain point or altitude during the rocket flight, this friction or dynamic pressure force will be at maximum dynamic pressure or drag due to the rockets speed and the air pressure at that certain altitude. Surely the air pressure is relatively low at several miles high, but the rockets velocity is a high value when the rocket is impacting the air molecules which will then gain kinetic energy from the impact from the rocket, and this causes a reduction of the kinetic energy and-or speed of the rocket. The pointed tip of a rocket reduces air resistance (ie., friction) or drag.

Though the final efficiency of the rocket fuel system may be as low as 1%, the social value is very useful, such as if the rocket successfully placed a communication and-or weather satellite into orbit.

How much thrust can a rocket engine make? To find this value, the rocket engine is tested to check for flaws, stability, efficiency, and the force (ie., thrust) it can produce. The rocket engine is started and the amount of fuel used and the corresponding force it produces is measured and recorded as like data points for a graph. This data will be used to determine things such as the acceleration to expect [ie., a = F / m = (engine thrust) / (mass of the rocket)] and the

large rocketship having enough fuel can then constantly accelerate at say a constant value of 9.8m / s^2. Objects in the craft that would otherwise be floating as weightless (but not massless), would then be moved toward the floor(s) and they will then experience a force (ie., artificial gravity) upon it much like how the entire rocket-ship is experiencing a force upon it from the rocket engine.

The formula for the escape (from gravitational, free-fall orbit) velocity (Ve) is:

$$Ve = \sqrt{\frac{2\,G\,M}{r}}$$: **Formula for escape velocity (Ve)**
  G = universal gravitational constant
  M = a spherical-like or uniform mass being orbited by the rocket or satellite   ,
   r = radius or distance from the center of the mass being orbited

Letting g = local gravitational acceleration = $\frac{G\,M}{r^2}$, Ve can be expressed as: $Ve = \sqrt{2\,g\,r}$ =~ $1.414 \sqrt{g\,r}$
   at an altitude or (r)

When (r) increases, the effect of gravity upon the orbiting body is less, and it is easier for it to then escape from orbit, and its escape velocity will be lower. Just the same, for an object to go into an orbit about a mass, it must be going at a velocity less than the escape velocity. Another way to consider escaping the effects of gravity is to accelerate away from the mass being orbited by an amount equal to the local gravity at that altitude. At the surface of Earth (where r equals the radius of the Earth) and in a low (altitude) Earth orbit (LEO), if the rocket is accelerating at slightly greater than 9.8m/s^2, then it will eventually escape from Earths gravity. For higher or farther orbits, this value will be lower since the local gravitational acceleration is a lower in value. Even though an object is said to have escaped the orbit of a body or mass, that object will always be slightly influenced by the gravity of that mass and every other mass depending upon its distance from the object according to the law of gravitation of two objects and-or masses. Even though the force of gravity may be very weak upon another object and-or mass, a constantly applied force such as gravity or a small amount of thrust from an engine will cause an acceleration and therefore a constant increase in velocity unless it is in a circular orbit about a mass such as a planet or moon. For orbiting conditions, the (inward) centripetal force due to gravity is equal in value to the (outward) centrifugal force of the object wanting to move away from its orbit as it would if the gravity ceased. While in orbit, the net or effective force upon the orbiting mass is 0:   Fc = Fg   , and Fc - Fg = 0

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provide a force equal to its weight, that object can hover (be suspended, not moving) in the air, or land slowly when the force created by the rocket engine is slightly less than the weight of the object.

Ex. How much energy is needed to make a 1kg mass travel at 100 km/hr , and-or 100km/hr faster?

First, velocity or speed = $\dfrac{distance}{time}$ = $\dfrac{100km}{1hr}$ = $\dfrac{100km}{3600s}$ = $\dfrac{0.02778\ km}{1s}$ = $\dfrac{27.78m}{1s}$

If KE out = KE in , = KE = $mv^2 / 2$ = (1kg)(100 km/hr)^2 / 2 = (1kg)(27.78 m/s)^2 / 2 =~ (1kg)(772 m^2/s^2) / 2 =

= (mass)(m/s^2)(m) / 2 = ((mass)(acceleration)) (distance) / 2 = (force) (distance) / 2 = N-m / 2 = work / 2 = 386J

COMMON EQUATIONS FOR MOTION

The study of motion is also called kinematics, and the equations for motion are also called kinematic or
kinetic equations since the word: "kinetic" is an old Greek word that means movement. These equations include one or more variables: distance (a change in position or location in reference to or from another location) or displacement = (d), velocity (v) or speed (s), time (t), and acceleration (a). Also included in this discussion is when the acceleration is gravitational acceleration (g) and when an an object in "free-fall" (in the force of gravity only) from a height.   Gravitational acceleration gets its name from that the force (the application of energy) of Earth's gravity is constantly applied to objects, and will therefore constantly transfer more (kinetic) energy to that object and therefore cause the object to increase its speed and which is called an acceleration.

The most common and basic equation for motion is the distance equation:

distance = (speed)(time) = (velocity)(time)   : this is a linear equation of the form y=mx. Speed or velocity is the
                                                                                       rate of change of distance with respect to time:
           d = v t   and mathematically:
           v = d / t   : speed or velocity is the rate of the change in distance with respect to time
            t = d / v   **:** time of travel

The units for distance (d) are usually: miles (mi), feet (ft), meters (m) or kilometers (km)
The units for time (t) are usually: hours (hr) or seconds (s), but may occasionally be minutes (min.)
The units for speed (v) are therefore a distance unit divided by a time unit. Speed is a ratio value that can be considered as the average distance moved per unit of time.

If an object is at rest, not moving, its velocity is 0. For an object to start moving, some force must be constantly applied until a certain desired speed is reached and then the force no longer needs to be applied. Between 0 and that certain speed, the object is increasing in speed. It is said that its speed or velocity of the object is changing per unit of time, and this is called an acceleration of the object and-or its speed:

          ending velocity = initial velocity + change in velocity   : this will be shown to be: v = initial velocity + (at)
                v2   =   vo   +   v1

A change in velocity per unit time is called acceleration (a). Acceleration is the rate of a change of speed or velocity with respect to time.

          Let: (change in velocity) = (v2 - v1)   , if v1 = 0 , (change in velocity) = v2
          Let: (change in time) = (ending time - starting time) = (t2- t1) , if t1 = 0 , (change in time) = (t2 - 0) = t2

           Letting: v=v2, and   t=t2:

Ex. How long will it take a free-fall object starting out with 0 speed, and to travel downward a distance of 9.8m = 32.2ft ?   (Note that 1m = 3.281ft   and   1ft. = 0.3048m)

$t = \sqrt{(0.203874)(9.8)}$ seconds
t = 1.414 s   : it took longer than 1s because its initial velocity or speed was 0.   and:
When t = 1s: d = 4.905m =~ 16.1 ft when the object starts at 0 speed.
This value is half of 32.2 ft

Ex. If an object of any size mass is traveling horizontally and at any velocity, the force of gravity will constantly give that mass some kinetic energy and cause it to move vertically downward at an increasing speed when the force is constantly applied. The constant gravitational force will cause it to accelerate in velocity by 9.8 m/s^2 = 32.2 ft/s^2. The distance it actually moves per unit of time, say 1 second, depends on its velocity at that time and which is constantly changing and increasing.   Starting at 0s where its velocity is 0m/s, how long will it take for the object to move 1ft = 30.48cm = 0.3048m?

d = vt = (at)(t) = at^2 = gt^2   , mathematically:

$t = \sqrt{\dfrac{d}{g}} = \sqrt{\dfrac{0.3048m}{\dfrac{9.8m}{1s^2}}}$ =~ 0.176 s

Ex. If an object, vehicle or spacecraft started out at 0 m/s and ended at 100 m/s after 10s of time, what was its (assumed constant) acceleration during that time?

Clearly the object got faster in speed or velocity during that time of travel. Since it got faster and gained kinetic energy, some energy must of been applied and transferred to it, and this is usually done by an external force, or an internal engine producing a force. If the force was constantly applied during this amount of time, the acceleration also had a constant value during this amount of time .   When a force is applied to an object, it will accelerate and gain speed or velocity and kinetic energy.

acceleration = a = (change in velocity) / (change in time)
            = (v1 - v0) / (t1 - t0) = (100 m/s - 0m/s) / (10s - 0s) = 100m/s / 10s = 10m/s^2

ending velocity = (initial velocity) + (change in velocity)
            v2   =   v0 + v1
            v2   =   0m/s + at   : from: a = (change in velocity) / (change in time), if v1=0 and t1=0,
                                          a = (v1 - v0) / (t1 - t0) = (v1-0) / (t1-0) = v1/t1 = v / t , and v=at
            100m/s = at

            acceleration = ending velocity / time =   v / t = 100m/s / 10s = 10m/s^2

The velocity of the craft after 5s is:

 v2 = v0 + v1 = 0m/s + at = 0m/s + (10m/s^2)(5s) = 50m/s

The velocity of the craft after 8s is:

 v2 = v0 + v1 = 0m/s + at = 0m/s + (10m/s^2)(8s) = 80m/s

If the craft had a mass of 1000kg, what would be the force needed to accelerate it at 10m/s^2?

 Form: (force) = (mass)(acceleration) = f = ma   , we have:

        f = (1000kg)(10m/s^2) = 10000 kg-m / s^2 = 10000 Newtons (=N) of force

If energy is used (ie., transferred, work) to raise an object or mas (m) to a height (h) on Earth, its potential energy is:

PE = work = fd = (ma)(h) =
= mgh Joules :mass in kilogram units, g=9.8m/s^2, height with meter units, and its kinetic energy is: KE=0 Joules when not moving. If it is then allowed to free-fall under the force of gravity only, all of its potential energy will of been converted or transformed to its (available) kinetic energy just before it reaches the ground at height=0:

Ex. How much does GPE does 1kg and-or each kilogram of mass of an object have at 100 km high in altitude have?

GPE = mgh = (1kg)(9.8m/s^2)(100km) = (1kg)(9.8m/s^2)((100km)(1000m/km)) = (1kg)(9.8m/s^2)(100000m) =
= 980000 J =~ 1000000J =~ 1MJ . Since this equation is linear to altitude or mass, an object twice as
high or twice as massive will have twice the GPE and-or require that much energy to lift it that high.

KE max = PE max :Just before reaching the ground where PE=0, and all of its GPE energy was converted to KE.

$\dfrac{mv^2}{2}$ = mgh   , from this we can solve for v, and here it is the object's velocity just before reaching the ground:
Note that mathematically, after multiplying both sides by 2: mv^2 = 2mgh , and
after dividing both sides by m: v^2 = 2gh = (acceleration)(distance) = (M/s^2)(M) =
(M^2 / s^2) = (M/s)^2 = v^2   ,   (here, let M = Meter units of distance)

$\dfrac{mv^2}{2}$ = mgh   , from this we can solve for v, and here it is the object's velocity just before reaching the ground:
Note that: mgh = (weight)(h)

v = $\sqrt{2gh}$ = at   : velocity of a free falling object just before impact or reaching the ground.
The square root of (2g) is a constant.

velocity (meters/second) = $\sqrt{(2)(9.8)m/s^2}$ (height m)   = $\sqrt{(19.6)\ (height)\ m^2 / s^2}$

velocity (meters/second) = 4.427$\underline{2}$ m/s $\sqrt{height\ m}$: use a height that is an absolute value of meters

velocity (feet/second) = 8.02$\underline{5}$ ft./s $\sqrt{height}$: use a height that is an absolute value of feet

h = $\dfrac{v^2}{2g}$: (2)(g) = 2 (9.81 m/s^2) = 19.62 m/s^2 = 64.4 ft/s^2

g = v^2 / 2h

In general, for a moving object to go slower, it must loose some KE, and to go faster, it must gain some KE. To change speed, an acceleration (or deceleration) of the object must first take place. To cause an acceleration of an object, a force must be first be applied to the object so as to increase or decrease its KE. Force is the application of energy to an object and can be measured as: Force = (mass)(acceleration).   and   acceleration= Force / mass

Using the above formula, mass can be defined as: mass = Force / acceleration , and if there is less acceleration of the object when a given force is applied, the more mass of that object. More massive objects have more **inertia** or resistance to a change in movement, and are therefore the more difficult to accelerate it to a given speed or velocity for a given amount of force.

Most "weighing scales" such as most modern digital scales for home and everyday use, do sense the weight or (gravitational induced) force of the object placed upon it, however, the result displayed is its corresponding mass value. After all, grams and kilograms are (in the modern, technical sense) amounts of mass and not weight. A 1kg of mass actually weighs 9.8 Newtons (of Force) on Earth. In a reverse manner, a 9.8N weight (ie., a force) corresponds to 1kg of mass on the Earth. The machine or "scale" is essentially calibrated (ie., adjusted, set) to automatically divide the weight

power = energy / time = 1000J / 10s = 100 J / s on average = 100 watts on average

We see that the actual height, mass or weight does not take part in the power equation, but those are rather part of the total energy needed and used. Power is an (average) measure of the total energy usage and not of mass or weight.

Ex. If it took 1000J of energy to lift an object to a height of 1 meter in only 5 seconds of time, the average energy used per second is:

power = energy / time = 1000J / 5s = 200 J / s = 200 watts.

Something, such as a machine rated with a "higher wattage" or power than another machine, can get things done (ie., do work or a task) quicker (a less or shorter amount of time) because more energy is available and-or can be delivered per unit of time or per second. More energy can be used to create more force so as to apply and transfer that energy. More "wattage" (output, available) means more energy available (output), used and-or required (input) per second.

Ex. If a device uses 10 watts of power and is used for 1 minute, how much energy did it use?

From: **power = energy / time** , watts   we have:
         energy = (power)(time) , joules   since 1min = 60s of time:

         energy = (10 watts)(60s) = ($\underline{10\ joules}$)(60s) = (10)(60)joules = 600 joules of energy
                                              s

The force applied to or available from an object is: F= ma = mg.   The localized or concentrated force at the site or area of the impact is called the **pressure (P)** applied from that force: **Pressure = Force applied / area**. , P = F / A = force per unit area. If a force is applied to a very small area, the pressure and transfer of energy can be very high and more useful for certain things, such as for a hammer to pound a pointy nail into wood, or pointed wedge to split wood. If a hammer is used to apply force to a nail, that force is effectively concentrated, focused, available to, and applied to a small area at the pointed tip of the nail which has a very small surface area. Note that the total energy available at the hammer just before impact is not actually amplified or increased at the tip of the nail. Energy out = Energy in. What is rather amplified is the force per unit area = pressure. Pressure is essentially an amount of concentration of force. This is much like how a large amount of light is gathered, concentrated or focused by a telescope lens or mirror. The total amount of light received is not increased or amplified, but the concentrated light energy into one spot or area makes the light energy per unit area much higher as if it were actually amplified, and objects will then appear as being brighter. It could also be said that the energy density = (energy / area) is now higher. A funnel shape can be used to concentrate audio or sound waves so as they are effectively louder from the source. With the hammer and nail, the total amount of available energy is being applied to and transferred to a small region or area, hence the energy is effectively being concentrated into one spot or area so as to be more useful to do certain things such as making the tip of the nail go more easily into to the wood. Extra:

$$P = \frac{F}{A} = \frac{(mass)(a)}{m^2} = \frac{N}{A} = \frac{N}{m^2} = \frac{N\ (m^{\wedge}1)}{m^2\ (m^{\wedge}1)} = \frac{N\text{-}m}{m^{\wedge}3} = \frac{N\text{-}m}{Volume} = \frac{Work}{Volume} = \frac{Energy}{Volume}$$: mathematically
                                                                                                                                                          and:

(Pressure)(Volume) = Energy   : = (F / A) (V) = $\frac{F\ V}{A}$ = $\frac{mg\ (d^{\wedge}3)}{d^{\wedge}2}$ = mgh   : such as for a column or height of liquid
                                                                                                        that applies pressure on a surface,
                                                                                                        PE, Static or Potential Energy

Energy = (Pressure)(Volume) = PV = pghV = mgh = KE = $\frac{mv^{\wedge}2}{2}$: p = density of the substance = mass / V ,
                                                                                              Ex.: p = mass per cubic meter = kg / m^3, and:
                                                                                              KE = Dynamic or Kinetic Energy , due to motion

         In a pipe with a flowing liquid passing through it, the Energy and flow rate (Volume / time) of that liquid is

result reduces rapidly when the denominator increases. This is like the inverse of n^2 and where the result increases rapidly as expected for the power of a value greater than 1.

Note that the equation of: $\sqrt{1/n}$ has the format of the square root of a reciprocal or inverse of (n). If (n) increases greater than 1, its reciprocal value is less than 1 and decreases, and the square root of that value will also be less than 1. Shown in this book is a similar mathematical discussion for Newton's Cradle and kinetic energy.

[FIG 231A]

BERNOULLI'S PRINCIPLE
With flowing water in a pipe :

A1	A2 < A1	A3 = A1
v1	v2 > v1	v3 = v1
P1	P2 < P1	P3 = P1

FLOW RATE = Q1   **Q2 = Q1**   **Q3 = Q2 = Q1**
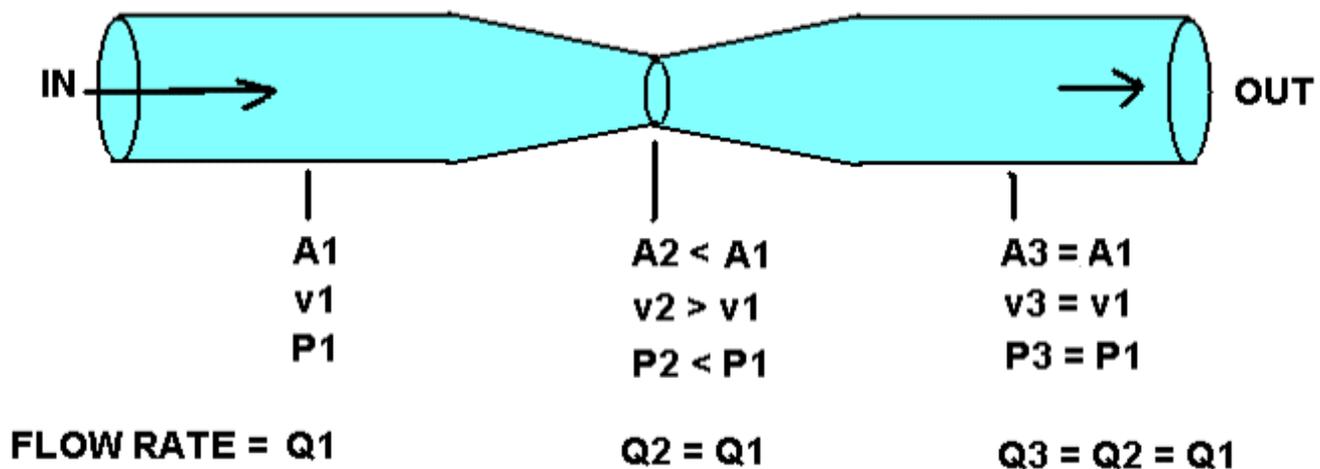
Before we move on to Bernoulli's Equation, there is a way to calculate the velocity of a fluid in a pipe by knowing both the flow rate (Q) and the area (A) at a location along the pipe:

The flow rate (Q) expresses the volume (V) of the fluid that will pass a point in one unit of time: $Q = V/t$, The Volume of this fluid can be calculated from the equivalent geometric volume within that section of the pipe. Here the pipe is a tube or cylinder shape, and has a circular cross-sectional area and is equal to $A = (pi)r^2$. The cylinder shape has a height=length=distance that the fluid will travel in that unit of time:

Volume = volume of a cylinder = (cross-sectional area of the pipe or cylinder)(distance) $= V = Ad$

Expressing both volume expressions (V, and Ad) with respect to time so as to have a time rate:

$Q = \dfrac{V}{t} = \dfrac{Ad}{t}$, therefore: $Q = \dfrac{Ad}{t} = Av$, and:

$v = \dfrac{Q}{A}$ = velocity of a fluid in a pipe = (flow rate) / (cross-sectional area of the pipe)

Flow rate can be physically determined by letting the fluid flow into a container, and then dividing by the time it took to reach a volume: $Q = $ Volume / time $= V/t$

= mass/volume, however water is very difficult to compress, much like a solid.

Water that was raised to a height in a tower or dam by using energy (ie., work), can store (gravitational) that input energy as potential energy (PE), and when that water can flow downward from that height, its energy is converted to kinetic energy and can provide the force to turn a turbine wheel of a mechanical device or an electrical power generator. The kinetic energy and-or velocity of this water will be at its maximum value just before it reaches the bottom of the pipe. In theory, this KE per volume of water at the bottom, will equal the PE per volume of water at the top height, and the total amount of available energy is the same: KE = PE.

Ex. If a water tank up high on a water tower is storing 100J of (Gravitational) PE = Fd = (weight)(height) = (mass)(g)(h). The energy per volume is then: PE / V = (m)(g)(h) / V = (m/V)gh = (density)gh = pgh,
Since p = m/V, m= pV, and the amount of energy stored in relationship to the volume of the liquid is:
PE = mgh = pVgh . For the fluid of water, it has a density of 1g/cc = 1000g/ 1000cc = 1kg/L. g=9.81m/s^2.

If it takes 50s of time for all that water to flow through a pipe to an electric generator, then that stored 100J of PE would be converted to 100J of KE energy in an ideal (low power losses, full energy conversion) system. The rate of using this energy would be:

Rate, or time rate, and power of using this amount of energy in 50s is: $\frac{100J}{50s} = \frac{2J}{1s}$ = 2 watts of power
P = 2w

For an ideal energy system and-or conversion, 2 watts of electrical power would be generated and-or available. Also in an ideal system, the flow rate (Q = Volume / time = V / s) is constant and the available and generate power will be constant, but since the water is draining from the container, it contains less stored potential energy, and less depth (distance from the surface to bottom) of the water as time increases, changes, progresses or passes, and therefore, the water pressure at the level of the outlet pipe is constantly being being reduced if that volume drained out is not resupplied into the container. Since the (water or fluid) pressure in the pipe is less, the force and the application of that potential or pressure energy will also get less, and the available and generate power will be less. As the fluid in the container drains, it will loose both its Gravitational Potential Energy (GPE) and the available (static) pressure energy and-or force, but as the water reaches the bottom of the tower, it will regain, convert or transform that GPE as kinetic energy (KE). If the fluid moved a distance of 1 meter per second in the pipe or middle (a low resistance and-or friction region) of a water steam, it is then said that its velocity = v = distance / time = d / t = 1 meter per second. In theory, and for an ideal system, a turbine or power transfer wheel with a circumference of 1 meter distance, and placed into this flowing water, would turn or rotate once each second if the friction or force against its movement was low, hence it would have a maximum rotational velocity or rate of 1 revolution (360°) or cycle per second =1 rps = 60 rpm.

To calculate the rps value: From v = d / t , v / d = 1 / t = frequency = cycles = revolutions,
ex. cps = hertz = hz

$$\frac{v}{d} = \frac{v}{c} = \frac{1}{t} = \text{hertz} = \text{rps} = \text{frequency}$$
: c = circumference = 2(pi)(r)
: d = distance
When time has units of seconds (s).

Ex. If the water velocity is v = 1m/s = 3600m/h = 3.6km/hr, and circumference of the wheel is c = 0.25 meters:
rps = velocity / circumference = v / c = 1 (m/s) / 0.25 m = 4 (1/s) = 4 hertz = rps

For torque (effective rotating or twisting force, equivalent to an amplified linear force) applied to this wheel or turbine:

From: Pressure = Force / Area = P = F / A , F = PA.

Pressure = PE : Potential Energy (ability to release energy, stored energy)

From: **Energy = (Pressure)(Volume) = PV** , **Pressure = Energy / Volume = "energy density"**

Energy = (Force / Area)(L^3) = (F / L^2)(L^3) = FL = **Fd = Fvt** = work = energy : L = length = distance
Energy = (Pw)(time) = Fd = Fvt , joules , note mathematically:
Energy / t = **Power = Fv or = (Torque)(v)** : Power in watts , F in newtons , **linear v in m/s** , and:
Fn = Pw / v = E / vt = E / d and **Pw = Ev / d** and **d = vt = Ev / P** , hence: **t = E / P** : P is a rate

Pressure = <u>force</u> = <u>Energy</u> = <u>J</u> with units of: joules / m^3 , also: **F = J / d = joules / meter**
           Area   Volume   V

(F / A) = (J / V) = ma / d^2 = J / d^3 , mathematically:

mad = (force)(distance) = J = Work = Energy

Kinetic Energy density of a moving liquid = <u>__KE__</u> = <u>mv^2 / 2</u> = <u>mv^2</u> = <u>pVv^2</u> = <u>pv^2</u> = Dynamic
                                       Volume   V   2V   2V   2   Pressure
                                                                        p = density
For the last term above, the units are : <u>mass</u> (m/s)^2 = <u>kg m^2</u> = <u>kg__</u>
                                           V   m^3 s^2   m s^2

We see that Dynamic Pressure is equivalent to the Energy (KE) density of a volume of a moving liquid, and that its equation contains the density (p = mass / volume) of the liquid's. The denser it is, the more energy it will have at a given or particular, constant velocity.

Note: P = pressure = force / area :"force density" , p = density = mass / volume :"mass density"

Dynamic pressure can be thought of as the pressure (= force/area) of the moving liquid pushing (applying force, applying energy) on the liquid in front of it. From the above equation, we have:

KE = (Volume)(Dynamic Pressure) : "(dynamic) pressure energy" of a moving volume of liquid
KE = (V)(pv^2 / 2) = (mv^2 / 2) : here p = density of the substance (liquid or gas) , and that
                                           p can be derived here as: p = m / V

E = Work = Fd joules = mad = m (change in v) / (change in t) d = m (ch. v / ch. t) v t = m v^2 / 2
This is an average energy value since v0=0 and it will take time to reach the velocity of movement.
It could also be thought of as due to an equal and opposite force, say friction, receiving energy, and half the energy input does not get transferred to the mass. If the distance was a height, then: E = W = mah joules or= mgh joules in Earth's gravity of 9.81m/s^2

Bernoulli's Equation:

P = pressure , p = fluid density , v = velocity , g = gravitational acceleration , h = height

At each location in a pipe:

Pt = (Initial Pressure Energy) + (Kinetic or Dynamic Pressure and Energy) + (Potential or Static Pressure and Energy)

Total Pressure At Location 1 = Total Pressure At Location 2 , and:

P1 + <u>(p)(v1^2)</u> + (p)(g)(h1) = P2 + <u>(p)(v2^2)</u> + (p)(g)(h2) **: Bernoulli's Equation**
                 2   2

The basics about collisions of objects:

A brief discussion on this topic was previously given in this book with Newtons's Balls, and here is some more to consider:

For an object: **weight = F = mg** , and a constant mass and constant gravitational force induced acceleration (a=g) of the mass will produce a constant weight of an object.   weight = mg = (a vertical, downward force) = F = ma

If object1 will collide with object2:   : or mass1 will collide into mass2 :

   Object1 is moving and has **momentum** (p=mv) and kinetic energy (ke = mv^2/2) and will collide (impact, contact and apply force to) with object2 which is at rest. Object1 will then loose some of its energy by transferring it to object2. More specifically, Object1 will apply a force (ie., the application of energy) to object2 and some of its energy will be transferred to object2. Much of this energy will be kinetic energy. As object1 looses kinetic energy and decelerates, object2 will accelerate. During this process, due to the concept of equal and opposite forces, when object1 collides with object2 it will be as if object2 is applying the same amount of force to object1, and in the opposite direction, and therefore object1 will decelerate.   The impact force (application of energy) is related to the amount of acceleration (f=ma) or deceleration of an object, and also determines the amount of energy transferred. Remember that: power = watts = Joules/second. If the energy transferred is high and-or the time is a low value such as for an impact, the power can be a high value for a brief amount of time, especially if the acceleration or deceleration is quick such as during an impact or collision.   A brief and large pulse or release and transfer of energy can cause significant damage to both objects if the collision is not "elastic" and where much of the input energy is converted to heat.

   The force of an impact depends on the total kinetic energy available in the objects and the directions they were moving, such as towards each other, one being still, or both going in the same direction with one being faster and one being slower than the other behind it. The kinetic energy in an object is due to its mass and velocity. Total momentum = p = mv, and total kinetic energy = mv^2 / 2 and this equation which is similar to that of momentum is needed since if the velocity of the object doubled, its kinetic energy is actually four times more, but its momentum is only doubled. Momentum can be considered as a measure of an objects motion and to stay in that motion, rather than its stored energy.

   The momentum of a mass is: **momentum = mv**   : a change in velocity will create a change in momentum, this
                                                          will not be instantaneous, but will take some amount of time
                                                          for it to accelerate from one velocity to another.

   F = ma = m (change in velocity) = (change in momentum)   : showing force as (equal to) the amt. of change in
                    (change in time)   (change in time)   momentum (ie., motion ability) with respect to a
                                                                                          change in time. Mathematically:

   (change in momentum) = (force)(change in time) = (N)(s) = (ma)(s) =   (kg m/s^2)(s) = kg-m / s = (kg) (m/s) = (m)(v)
                                                                                          , and where (v) is the change in velocity.

   A force (the application of energy) will cause a mass to change its velocity and-or accelerate (or decelerate) during the time that the force or energy is applied to that mass. The more mass of a moving object, the more potential or kinetic energy it has, and therefore the more energy needed to change it to another velocity. As an example, mass1 is a small mass, and mass2 is a larger mass, and both have the same velocity. Since momentum = (mass)(velocity) = p = mv, momentum1 is less than momentum2. Less energy is needed to slow down a mass with less momentum. If mass1 was traveling faster it could then have as much momentum as mass2.

   More energy is needed to slow down a mass with more momentum or kinetic energy. If the force applied to a mass is not much Newtons, or it is applied for only a short amount of time, the total energy applied and transferred (or effectively removed from it so as to slow it down) to it so that it will have energy to do things was not much, and it will then take more time to apply and transfer more energy to slow it down. Still, regardless of the rate of energy use (ie, work done) or transfer of energy, the application (force) and transfer (power=joules/second, watts) of a same total amount of energy will be required to slow a mass with kinetic energy down to a certain speed or stop = 0 speed.   In short, the

at the appropriate locate on both of their masses so as it will be caused to then travel directly to the intended pocket.

More equations for distance, velocity and kinetic energy:

For an object that is already going a certain velocity (v1) and then uniformly (through the entire distance) accelerates to another velocity (v2), the distance traveled is equal to the average velocity (Vav) times the time (t) or duration of travel:

distance = (speed)(time) = (velocity)(time)   : here (time) is the total time (t) of travel
distance = (average velocity during the acceleration(s)) (time)   ,   (average velocity) = (total distance) / (total time)

distance = $\dfrac{(v1 + v2)}{2}$ t   ,   since v2 = v1 + change in velocity = v1 + at   ,   t = $\dfrac{(v2 - v1)}{a}$

distance = $\dfrac{(v1 + (v1 + at))}{2}$ t = $\dfrac{2v1t + at^2}{2}$

distance = (v1)(t) + $\dfrac{at^2}{2}$: total distance = initial distance + change in distance   : *
                             : mathematically: d / t = (average velocity) = v1 + at / 2

$\dfrac{distance}{t}$ = average velocity = $\dfrac{v1 + v2}{2}$ = (starting velocity + ending velocity) / 2   ,   mathematically:

distance = $\dfrac{(v1 + v2)}{2}$ (t) = $\dfrac{(v1 + v2)}{2}$ $\dfrac{(v2 - v1)}{a}$ = $\dfrac{v2^2 - v1^2}{2a}$,   mathematically:

From the above equation (*) for distance, if the initial velocity (v1) is equal to 0, such as for a still object dropped from a height, then the equation reduces to:

distance = at^2 / 2

Work = energy = (force) (distance) = (ma) (at^2 / 2) = m(a^2t^2) / 2 , joules

Since **v = at** , and squaring both sides: v^2 = (at)^2 = (a^2t^2) , and substituting this into the above equation:

Work = energy = KE = $\dfrac{mv^2}{2}$, joules   : KE = Kinetic Energy , motional energy

Note for example, that if a moving object gains a KE value of 10J, then it took, or will take or require, 10J of input energy so as to give it that amount of KE. If you know an objects KE, you can find its velocity from:

velocity = $\sqrt{\dfrac{(2)(KE)}{mass}}$, (meters / s)

For a given or constant amount of energy (KE), if mass increases, the velocity decreases, and this is an inverse relationship.

Energy = Work = Joules = (force)(distance) = (ma)(d) = Fd = N-m = Joules   ,   mathematically:

Force = (mass)(acceleration) = ma = Newtons = $\dfrac{Joules}{meter}$ = $\dfrac{J}{m}$ = N   : J = N-m : Netwon-meters

F = $\dfrac{J}{m}$ = $\dfrac{J}{d}$ = $\dfrac{Energy}{Distance}$ = $\dfrac{PV}{d}$ = $\dfrac{Pd^3}{d^1}$ = Pd^2 = P A   : mathematically, P = F / A , or by:
                                        P = pressure , F = force , A = area

AN EXAMPLE OF FUEL USAGE AND A MOVING VEHICLE

The average car vehicle weighs about 4000 lbs =˜ 1814 kg and has a rated distance to fuel usage of 20 mpg = 20 miles per gallon = 20mi / 1 gal = 32.18<u>7</u> km / 3.78<u>5</u> L , and after dividing both sides of this fraction by 3.785, we can reduce this fraction to: 8.504 km / 1L.   1 gallon = 128 fluid ounce = 128 fl.oz. = 3.785L   and   1L = 33.814 US fl oz.

Ex. If a car is rated as having 20 miles per gallon, to travel 60 miles of distance will require: total distance / (distance / gallon) = 60 mi / (20 mi / 1 gal) = 3 gallons. If gasoline is $5 per gallon, the total fuel cost is: (cost per gallon)(number of gallons) = ($5.00 / 1 gallon)(3 gallons) = $15.00

If the car described above is moving at 60 miles per hour = 60 mph =˜ 96.6 kph = roughly 100 kilometers per hour, the amount of kinetic energy (KE) it will have is:

  KE = mv^2 / 2 = (1814kg)(100^2) / 2 = 9,070,000 Joules

As the car travels on a consistently made road surface, it will loose a certain value of that energy every amount of time and-or distance due to the friction or collisions of the tires ("wheels") and the road surface. To maintain speed and-or the same amount of KE, the car will require a certain amount of fuel so as to accelerate (in velocity) back up to the desired speed or velocity of 100 km/h. The faster the car, the greater the distance it will travel per unit of time, and the greater the friction losses per unit of time, and therefore, it will take more fuel and-or energy per unit of time to maintain a higher velocity.

Actual or current fuel used for a certain situation (ie., a certain vehicle , speed or velocity) can be described as:

  (fuel usage) = (x gal / hr)   : a fuel to time rate of using the fuel , this depends mostly on the speed , or
  (fuel usage) = (x mi / gal)   , and this is equal to the cars (typical or average) fuel rating
  (fuel usage) = (x gal / mi)   : this is the reciprocal of the above, but here, a fuel to distance rate

 At 60 mph, a car rated at 20 mpg will require a fuel usage = (x gal / hr) of:

    From: total = (60 mi / h) = (fuel rating) (fuel usage rate) = (20 mi / 1 gal) (fuel usage rate)   , mathematically:
    fuel usage rate = **(**60 mi / 1hr**)** **/** **(**20 mi / 1 gal**)** = 3 gal / 1 hr   , and-or: 3 gal / 60 mi

    This value of 3 gal / 1 hr can also be found using an equivalent fraction or "proportion equation":

     20 mi = 60 mi   , solving for X we have: X = 3   , and if the amount of time was 1 hr, we have: 3 gal / 1 hr
      1 gal   X gal

 3 gal = 3 gal   = 0.000833 gal = 0.107 oz =˜ 3g   =   1oz__ =˜ 28 g__ =   1g _   : 1oz = 28.35g
 1 hr   3600s   1s   1s   1s   9.346 s   9.346 s   0.333s

 100 km / hr = 100 km / 3600s = 1 km / 36s = 0.0277<u>8</u> km /s =˜ 27.8 m / s =˜ 91.21 ft / s
 60 mi / hr = 60 mi / 3600s = 1mi / 60s = 1 mi / 1 min = 1 mi / 60s = 0.01667 mi / s = 88 ft / s

 For the car to travel at a velocity of 60 mi / hr =˜ 100 km / hr =˜ 88 ft/s , the amount of fuel usage per second to maintain that speed or velocity will be about   3 g / s =˜ 0.107 oz / s = 6.47 oz / 60 s = 6.47 oz / min. The useful (actual combusted fuel portion that is actually converted to forward thrust or movement) energy in 3 grams of fuel per second is what is needed to replenish the KE lost per second due to the weight and friction(s) (road, bearings, etc) at a velocity of 60 mi / hr =˜ 100 km / hr . Technically much of the fuel to a typical car engine is eventually wasted as heat due to mechanical friction such as with the "drive train" or "power train" (the mechanical parts to transfer the energy from the engine to the wheels), and wheel bearing friction , and a low combustion efficiency, etc. An average car engine system is only about 35% at producing KE or forward motion from the fuel, and therefore, there is much wasted fuel and-or energy. The miles per gallon rating is the actual result or distance to expect after the fuel and-or efficiency losses.

and sold as a pressurized liquid gas in a hard metal tank, and having a l large amount of mass and stored energy. When the valve to the tank is open, it will reduce the pressure inside the tank, and the liquid propane will boil and create propane gas, such as used for heating and cooking.

Plastics are long-chain hydrocarbons, and which are often made from crude oil (petroleum) liquid into a solid material. Plastics are made applying further chemical processes to some of the refined crude oil products mentioned above, and can account for about 4% of the crude oil volume. Considering average and-or typical values: For harder plastics that sink in water, their density is about 1.28g/cc on average, and for soft plastics, their density is about 0.91, and for the foam types of plastics, such as styrofoam that is often used for insulation or flotation products, their density is about 0.1g/cc and which is much less than that of water which has a density of 1.0 g/cc

As of about the year 2020 or so previously, the oil industry is said to be at "peak oil", meaning that the known supply will most likely diminish as the years go by, and as the large, natural reserves of crude oil in the ground get removed and depleted. Other problems can reduce the supply of oil and its products, and we need to find alternatives to crude oil.   It is difficult to obtain an equivalent energy density per gram, cc or weight of diesel or gasoline, but it is a goal to be considered when manufacturing modern super capacitors, batteries, etc. Where will these electrical energy storage devices obtain their energy from, and it also needs to be efficient, sufficient and with renewed or naturally replaceable quantities so as to be available at a low cost to many people in need?   Solar and wind power and-or generate electricity seems to the "green energy" or "free" and renewable (replaceable, unending, reliable) method for now as of the year 2023.

A FLYWHEEL AND ITS MECHANICAL AND KINETIC ENERGY STORAGE

A concept and mechanical device that is somewhat related to the concepts of the energy or power of gears or wheels is called a **flywheel.** A flywheel is essentially a rotating wheel, disk or object that can accumulate and store kinetic energy. A "spinning top" novelty device is a type of flywheel. The energy in a flywheel can then be used later to transfer and-or generate some mechanical and-or electrical power when needed. A flywheel can also be used to help maintain a desired rotational speed and-or timing for a machine, and this special version of a flywheel is sometimes called a "governor", and may be used for something such as to help control (ie., govern) the amount of fuel to an engine so as to keep a steady rotational speed of it and-or to maintain a certain speed of a vehicle that is moving forward. Controlling the speed of a machine or technical can be done by controlling a fuel intake valve or possibly an electric switch or resistor. A governor is therefore both a (speed, rotation, power) sensor and controller.

The amount of kinetic (movement) energy a flywheel can store depends on the mass of the wheel and how fast it is spinning (ie., its rotational or angular speed or velocity). Remember that kinetic energy is related to mass and the square of the velocity.

The larger the mass of an object, the more difficult it is to get it to have a certain (linear or rotational) velocity and-or stored kinetic energy value. A larger mass is then said to have a larger inertia (ie., resistance) to movement. The larger the mass of a flywheel, more torque (twisting force) will be required to get it to move (here, rotate). It will also need some extra initial or temporary force (input torque) so as to overcome more friction (on the bearing, pivot, axle or axis) due to the weight of that mass. As long as energy is being input into this system, input force or torque can be applied to the mass of the flywheel. Constantly applying an input force will constantly apply input energy and will cause an acceleration (change in velocity, here an increase), and the flywheel will rotate faster as it gains and stores more kinetic energy.

Torque = Twisting or Rotational Force = (Force)(leverarm) = (mass)(acceleration)(leverarm)

If the mass is increased, the more energy will be needed for it to have a certain amount of rotational kinetic energy. The amount of input energy required to maintain the rotational kinetic energy and-or velocity of the flywheel (or any other object in motion) will be very small if the friction and-or other energy losses (such as using its energy to rotate another gear or wheel) are small.   If 100J of energy are lost each second in this system (ie., = 100 J/s = 100watts of power loss, work and-or transfer), in order for the flywheel to maintain its speed and-or kinetic energy value, an input of 100J of energy per second must be constantly applied to

1.5708 m^2 = (pi) r^2   isolating r^2:

r^2 = ___1.5708___ = 0.5   after taking the square root of both sides so as to solve for (r):
          3.14159265

r = √‾0.5‾ = 0.707107 m   : at 70.7% of the initial radius from the center point, the area and-or mass of a
                                 circle and-or disk will be half.   Incidentally, 0.707 is also half of the square root
                                 of 2, and which equals about 1.414....

If a uniform or complete (without gaps, spokes, and a larger rim mass) disk had a mass of 4kg, a radius of 20cm, and rotated at 5 revs (ie., revolutions) per second, how much kinetic energy is stored in that rotating disk?

   At 0.707 of the radius will be half of the effective mass storing the energy.

   (0.707)(20cm) = 14.142cm

   For the linear velocity of a mass or point (ie., "effective point of equivalent mass") at 14.4cm from the center, we will first calculate its linear distance per revolution, and that is equal to the "effective" circumference of the circle or disk:

   c = 2(pi)r = 2(pi) 14.142cm = 88.858cm

   In 1 rev, a point at that radius will effectively travel 88.858.cm. If we multiply the number of revolutions by a value, then the distance will increase by that same value:

   Let: 1 rev equates to 1 circumference (equivalent linear, one-dimensional) distance
              1 rev = 1 circumference distance   multiplying both sides by x, for a particular number of revolutions:
            (x) 1 revs = (x) circumference distance
             x revs = x (2)(pi)(r) distance   ,   using the given values, and dividing both sides by 1s, so as to have a rate:

            5 revs = 5 (2)(pi)(14.142cm) = 444.28 cm   = 4.4428 m / s   :an angular velocity converted to its corresponding
              1s   1s   1s   linear velocity

   KE = _mv^2_ = (4kg)(4.4428m/s)^2 ~= (4kg)(19.74m^2/s^2) = 39.48   newton-meters = 39.48 Joules of energy
            2   2   newton-meters = (F)(meters) = (ma)(meters)

It is clear in the above formula that if the (speed, rotational or angular) velocity of a flywheel doubles, then the kinetic energy stored within it will increase by 4, but it will then likewise then take four times the amount of its previous input energy for its velocity to then double:

   KE at v=1 is: mv^2 / 2 = m1^2 / 2 = m(1) / 2 = 0.5m   : here, m = mass

   KE at v=2 is: mv^2 / 2 = m2^2 / 2 = m4 / 2 = m2 = 2m

   KE new = _2m_ = _2_ = 4   , From this, a generalized formula for KE due to a change in rotational velocity is:
   KE old =   0.5m   0.5

   (KE new) = (ratio of velocities)^2 (KE old)   = (Vnew / Vold)^2 (KE old)

Another, and more general formula for the kinetic energy stored in a flywheel is: **KE flywheel = m w^2 r^2 / 2**
, where m = mass in kg, w = angular velocity in radians per second, and r = radius. Surely, to increase the radius, mass must be increased, but in theory, if the mass is a constant, perhaps by thinning the flywheel and then increasing (r), the KE of that flywheel wheel increase by 4 if r is doubled, and even if the angular velocity remains constant.   Note, if r=1, and then doubles: r^2 = (2r)^2 = (2(1))^2 = 4 , and KE will increase by the square of the increase in r.

MORE ABOUT THE INVERSE SQUARE LAW

This is perhaps a simpler verification that as the distance doubles from the source of energy, that the intensity at that distance is reduced by 4. A factor of 4 times less equals a factor of 1/4 = 0.25. [FIG 232]

point source or source very distant
and considered as a point
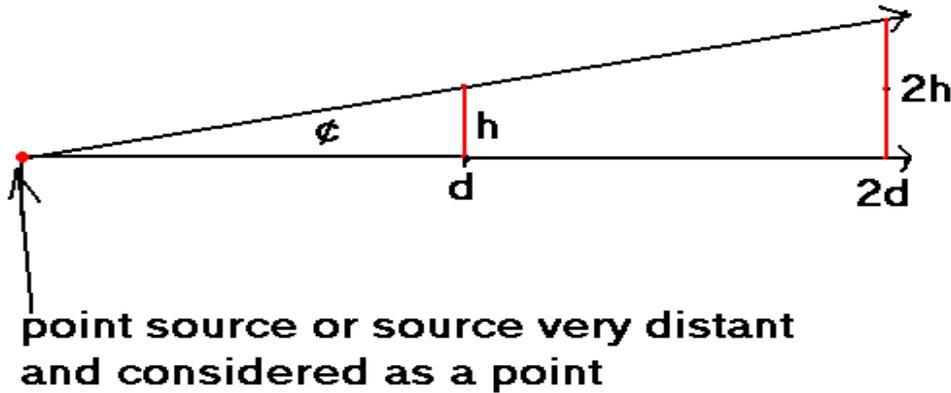
At twice the distance from the source or vertex point of this right-triangle, the height will double. This is easily verified when you remember the simple and amazing fact that a similar triangle can be created by extending the sides of that triangle, all by the same factor, and here, it is 2. A right triangle, and half of a true distribution angle from a point source is used for this analysis, but the results are still the same.   Incidentally, for this figure, if (d) is considered the center of a circle, this figure with similar triangles can be used to verify the fact that an inscribed angle is half of a central angle in a circle.

Now consider that these height values are the diameter of a circle area or side of a square area. At twice the distance, this length doubles and the area would increase 4. If h=1, this value squared would be an area of: 1^2=1. At 2h=2(1)=2, and this value squared would be an area of: 2^2=4. This is (4/1)=4 times the area than at 1d = d from the source. The energy from the source that reaches the area at distance (2d) sums up to the exact same amount of energy at the area located at (d), but it is then spread out becoming a more sparse, thinner, dimmer or weaker concentration, and is then actually 4 times less per unit area as that of the area at (d). 4 times less means a factor of 4 less which equals a division by 4: (1/4)=0.25 , which equals a multiplication by 0.25 = "a quarter of". In a reverse manner, it could just as easily be said that at half the distance to the energy source, that the energy will increase by 4 per unit area.

A simple formula for all this would show the inverse relationship of intensity (I) and distance (d), and specifically the relative distance in terms of the ratio of the distance in question to (d). Also, its more than an inverse relationship, (I) is inversely related to not only the distance, but to the square of the distance. This is the meaning of what is called "an inverse square relationship". In a pseudo-type of equation using a relative value, this information can be initially expressed as :

I = 1 / d   1 = 100% of the true, theorized or measured intensity at that location. When distance doubles to 2d, the intensity = I per unit area decreased by 4. This can be mathematically represented as:

(In the above drawing, consider a square area with (2h) as the length of its side. This area is then 4 times more than at (1h), and all the energy or intensity at (1h) has expanded outward into to 4 times the area, with each unit of area receiving less, and is actually (1/4) only.)

$$\frac{I}{4} = \frac{1}{2d}$$

When the distance tripled to 3d, the original or starting intensity decreases by a factor of 3^2 = 9 per unit area:

$$\frac{I}{9} = \frac{1}{3d}$$

Ex. If the energy intensity at a distance of 2 mile from a source is measured to be: $I_0 = $ 25 units / m^2, what is the energy intensity at a distance of 1 mile, or at half (1/2 = 0.5) the distance to or from the source:

c = (distance in question) / (reference distance) = 1 mile / 2 miles = 0.5
c^2 = 0.5^2 = 0.25

$I_1 = I_0$ / c^2 = 25 units per square meter / 0.25 = (4)(25 units per square meter) = 100 units per square meter

It could be said that when the energy intensity increases by 4, that distance is at halfway (1/2 = 0.5) to the source of that energy.

Ex. At what distance from the current location and distance is the intensity level at just 50% = 0.5 = 1/2 = half of the current intensity (Io) level:

I_1 = (0.5)(I_0) = $\dfrac{I_0}{c^2}$: in some cases I_0 can also be set or considered 100% = 1. Solving for c, the distance ratio:

c = $\sqrt{(I_0 / (0.5\ I_0)}$ = $\sqrt{1/0.5}$ = $\sqrt{2}$ = 1.414

Since: c = (distance in question) / (reference distance)   then: (distance in question) = (c)(reference distance)

If the initial distance is 3 miles, the new distance where the intensity is at 50% is: (1.414)(3 miles) = 4.242 miles

Ex. At what distance from the current location and distance is the intensity level at just 200% = 2.0 = double or twice of the current intensity level:

I_1 = (2.0)(I_0) = $\dfrac{I_0}{c^2}$: in some cases I_0 can also be set to 100% = 1. Solving for c, the distance ratio:

c = $\sqrt{(I_0 / (2.0\ I_0)}$ = $\sqrt{1/2.0}$ = $\sqrt{0.5}$ = 0.707

If the initial distance 3 miles, the new distance where the intensity is at 200% is (3 miles)(0.707) = 2.121 miles

If the intensity increased by 400% = 4 , you went closer to, and halfway to the source of that energy or electro-magnetic field, and the distance to the source is then the same distance that you initially went to be closer. As you keep halving the distance to the source, the intensity will keep increasing by 4.

I_1 = I_0 / c^2   If I1 is 4 times that of the reference intensity Io :
$4I_0$ = I_0 / c^2   After solving for c:
c = 0.5
c = (distance in question) / (reference distance) = 0.5 : = 50% , this indicates the distance was halved.

If the reference distance is unknown, a relative value of 1 = 100% can be used, and the results will also be relative since (c^1) is a relative (linear factor) value.

Ex. If I_1 was equal to I_0 increased by 50% = 0.5 of its initial reading that was farther from the source, write an expression for I_1 :

I_1 = I_0 + (increase in I_0) = I_0 + (0.5 I_0) =   Io(1 + 0.5) = Io(1.5) = 1.5Io , = equivalent to 150% Io

At what distance will this intensity be located?

Laser Light Laser means Light Amplification By Stimulated Emission Of Radiation, and other electromagnetic waves, such as radar, is not usually radiated outward in a sphere or wide angle pattern, but is emitted as a narrow angle, concentrated and powerful long distance energy (here, light) beam, and the inverse-square concept will not apply to "narrow (dispersion, transmission, radiated) beam" energy transmissions.

Light (or "light flux") brightness or intensity is measured in units of **candlepower** (cp), and more modernly as candelas (cd). 1 candlepower is equal to 0.981, roughly 1 **candela**. Originally, a candlepower was defined as the light intensity (and not a candle's thermal energy) of a specifically (formula and design) constructed candle with a burning flame radiating light or energy onto a 1 square foot surface at a distance of 1 foot from the candle. If this 1 foot value is considered as a radius value of a sphere, the sphere would have a surface area (=4(pi)r^2) of about 12.57 square feet, and if each square foot was illuminated by an energy of 1 candle power, the source would be a total of 12.57 candle power of energy. The candle power unit has been redefined to a better (than using various error prone candles and lamp constructions) calibrated standard as the light energy from molten platinum shining through a 1 square centimeter opening. This amount of light intensity is equal to 58.9cp =~ 60cp. This was also refined as the light energy and intensity at a frequency of 540Thz and having a power of (1/683)=0.0014641watts =~ 1.5mw per steradian angle.   This amount of energy can also be measured by the temperature increase of a black body or surface that absorbs (and does not reflect) all electromagnetic radiation, including heat and light, and its temperature will increase because the light energy absorbed was transformed into thermal energy.

As indicated above, the frequency of light and its specific corresponding energy used for the definition of a candela corresponds to a green color which is supposedly the color that the human eye is most sensitive to.   Don't confuse the light energy of 1 candela with the emitted heat energy due to 1 burning candle, that is, a candela is a measure of the light energy only and not a candles heat energy which is a much higher value than its light energy, and is actually about 30 to 40 total watts of heat energy.

A lumen (lm, a Latin word for light. A related word is "luminosity" = "brightness" or "light measure") is equal to the amount of visible light energy emitted from a source for 1 second in a light beam having or limited to, or measured at, the dimensions of 1 **steradian** (sr or sd, "a solid or 3 dimensional [1 radian] angle", like a cone shape or direction and with sides having an angle of 1 radian from its vertex, extending outward from the center point of a sphere, and which will also intersect a circular area equal to (r^2) on the surface of the sphere) from a source of 1 candela.   1lm = 1cd within and onto a (ideally a black body, no reflection of energy received) (spherical) surface intersected by that 1sd angle.   If a point source of light radiates in all directions, a sphere pattern, equal to: (surface area of a sphere) / (r^2) =
= 4(pi)r^2 / r^2 = 4(pi) = 12.56637 steradians, the total amount of lumens emitted is: lumens = (candelas)(12.5_7_).

1 Lumen = 12.57 candlepower or candelas of light energy per second. This is essentially the total light energy from
          1 candle. 12.57 candle power will radiate 1 lumen through a 1 radian wide (solid) angle = 1 steradian.

Lux = Lumens of light energy per or upon 1 square meter of surface = lm / m^2

1 foot-candle (fc) = total energy or light upon a spherical surface having a radius of 1 foot, and with a source of energy of
     1candle at the center. There are calibrated light meters ("photometers") available that can measure light intensity
     using small sensors, and a reasonably well lighted room will have about 55 foot-candles measured near and upon
     its surface.

Due to energy expanding outward from a point source in all directions, the greater the distance it is measured or seen at, the less intense the energy or light becomes per square unit of area of surface or measurement, according to the inverse square law (energy per distance) concept. Still, the total or collected energy emitted or measured per solid angle, such as a steradian angle from a source of energy is always the same no matter how far from the source.

From an ideal point source of energy which is being radiated in a sphere pattern, and since there are (4)(pi) ~= 12.57 steradians available in that sphere pattern, each steradian will contain:

   total energy available / 12.57 = 100% / 12.57 steradians = 1.0 / 12.57 sd =~ 0.07955 sd =

THE SPEED OF SOUND

For the value of the speed of sound (a compressed air wave traveling outward from its source) in the air, first consider that a person is repeatedly, visibly striking an object and periodically (same time) producing a sound every 1 second of time , and that it takes 1 second of time for that sound to reach the listener. The speed of that sound can be calculated from knowing the distance equation: distance = (speed)(time), therefore mathematically: speed = (distance / time). If the distance was 1100 feet, and the time was 1 second, the speed of sound is about: **speed of sound ~= a distance of one-thousand and one-hundred feet per second = (1100ft / 1s).   1100 ft/s = 0.335.28 km/s = 335.28 m/s**

How long will it take sound to travel 1 mile = 5280 feet?

From d= v t , t = d / v   ,   t = (5280 ft / 1) **/** (1100 ft / 1s) = 4.85 s   = roughly 5s

How long will it take sound to travel 1 kilometer = 1000 meters?

From d= v t , t = d / v   ,   t = (1 km / 1) **/** (0.33528 km / 1s) = 2.98<u>3</u> s =~ 3s   ,

Many equations involving electromagnetic or sound energy radiation include an inverse-square relationship of the intensity and distance to or from a source. For example, for the forces of gravity, magnetism, electric force intensity, light intensity, and sound intensity. Sound is caused or created by a physical ("mechanical") force or pressure upon something. For example, an elector-mechanical vibrating speaker will giving air some kinetic energy, and then this compressed air or pressurized air that contains energy is transferred outwards and through a medium such as less pressurized air as a mechanical force, vibration or wave.   A force applied to air will cause a compression and then a decompression ("rarefaction", reduction) of the small particles (atoms) of air matter as the energy is radiated or transmitted outward form that source, and this will then propagate (distribute, transmit, send) the sound energy (air compression, pressure, force) and affect other nearby particles of air or other matter like a "chain reaction" or "domino effect" of collisions, and at the same frequency as the source vibration and-or force.

Sound and vibrations travel faster in denser materials since the energy transfer is more efficient. There is more material or matter per unit volume to transfer sound or vibration energy, hence reducing the time of energy transfer from particles to particle.

A modern, closer and accepted estimate for the velocity or speed of sound in a 20°C = 68°F air temperature, about room temperature, is typically considered as 1125 feet per second = nearly 343 meters per second = 767 miles per hour, and this value increases fairly rapidly as the density of the material (transfer medium) that the waves pass will through increases.

The speed in sea-salt water is slightly less than 1 mile per second at about 4921 feet per second = 3355 miles per hour = 1500 meters per second.   4921feet ~= 5000 feet. 1 mile = 5280 feet. In dense metals such as aluminum and steel, the speed of sound has an average of about 16400 feet per second = about 3.1 miles per second = about 5000 meters per second.

The speed of sound also increases due to temperature since the molecules and-or atoms of the transmission medium (air, water, metal) are moving more quickly, and therefore can receive and transfer (or transmit) the sound energy in a shorter amount of time. Typically, the speed of sound changes by 0.6 m/s = 1.9685 ft/s = about 2 ft/s for each 1 degree increase (change) in Celsius temperature, or about 1.1024 ft/s for each degree change in Fahrenheit. A change of 1 degree Celsius temperature corresponds to a change of 1.8 degrees Fahrenheit.

The ratio of an objects velocity (ie., speed) to that of the speed of sound in air is called the Mach Number of that object. It was created by the Austrian physicist Ernst Mach in about 1887.   It is basically how many time faster an object is going as compared to the speed of sound. Mach studied many things related to shock and (air) compression waves when an object travels faster than sound. The speed of sound can vary in air, depending on altitude and density of the air, but for general, practical or relative comparative reasoning, it can be considered as the speed of sound at sea level, and at 20°C

Pressure (psi)	Typical Altitude (ft)	Boiling Point Of Water	: Approximations for all values listed here.
4.36	30000	152°F = 67°C	: Mt. Everest 29032 = 5.4985 mi ~=5.5mi =~ 30000 ft , and
5.5	25000	163°F = 73°C	the total change in boiling temperature from sea level is
6.6	20000	174°F = 79°C	about 60°F. The corresponding change in air pressure is
8.3	15000	184°F = 84°C	10.34 psi. The mathematical relationship between the
10.1	10000	194°F = 90°C	boiling point and altitude is **nearly** linear at about:
11.7	6000	201°F = 94°C	Approximate rate of change = - (1° / 500 ft) = - (2° / 1000 ft)
12.7	4000	205°F = 96°C	
13.6	2000	208°F = 98°C	
14.7	0	212°F = 100°C	: sea level air pressure and boiling temperature point of water

A quick and simple estimate, especially for the lower altitudes, would be for every 2000ft higher in altitude, the air pressure will drop by 1psi., and the boiling temperature of water will decrease by about 4°F =~ 2°C . Another quick and simple estimate is that the boiling temperature of water will decrease by 1°F for every 500 feet higher in altitude, or 2°F =~ 1°C for every 1000 feet higher in altitude.   For conversions, 1foot = 0.3048 meters. Multiply feet by 0.3048 to convert to meters.   Multiply meters by 3.2808398 to convert to feet.   1000ft = 304.8m,   1000m = 1km = 328 1ft.   From this discussion, a reasonable estimation formula can be:

Boiling Point Of Water Estimation Formula (at altitude) = 212°F - 4°F (altitude feet /2000 feet)
Boiling Point Of Water Estimation Formula (at air pressure) = 212°F - 4°F (14.7psi - psi)

It will take less energy and time to boil water at a higher altitude where there is less pressure or compression applied to the water, and this will also decrease its density slightly. At low pressures upon water, it can actually expand and start (cold) boiling and off-gassing steam (water gas, water vapor, water molecules) since the molecules are not held together as much. Less energy is less joules of energy. Less energy per unit of time is less power (ie., rate of energy use or transfer) needed. P = energy / time = joules / second , and with units of watts. Now consider at sea level, for water to boil, the molecules of it must be given more kinetic energy by heating that water, and then when the water pressure becomes greater than the air pressure upon it, it will steam and boil into the lower pressure region of air that is at standard air pressure of 14.7psi.

The formula for air pressure at an altitude can get complicated with many variables to consider, and of which you may not have any data for their current values, but here is a simple estimation or approximation formula for the typical air pressure at a certain altitude less than 30000 feet, and the result differs by only about 5% or less than a real typical previously measured and-or current actual current value which can also vary due to many variables such as the air temperature:

air pressure = $\dfrac{14.7 \text{ psi}}{3.2\text{^(height ft / 30000 ft)}}$: **A Simple Air Pressure Approximation Formula**
For Heights <= 30000 Feet
Height above the surface of the Earth = altitude.
A lookup table is also an acceptable alternative.

Or by converting 3.2 to an equivalent power of (e):   e^1.163151... = 3.2   and   e^(1.163151)^x = 3.2^x

air pressure = $\dfrac{14.7 \text{ psi}}{(e\text{^}1.163151)\text{^(height ft / 30000 ft)}}$ = $\dfrac{14.7 \text{ psi}}{e\text{^}(1.163151 \text{ (height ft / 30000 ft)})}$ = $\dfrac{14.7 \text{ psi}}{10\text{^}((0.50515) \text{ H/30000 ft})}$

The value of 14.7 psi is also called 1 ATM ("1 atmosphere" of pressure) which is the normal, average and standard pressure of the atmosphere (air) at sea level.   14.7 psi = 1 ATM (typical "atmosphere (air) pressure" at sea level) = (nearly) 1 bar = 101 kPa , and Pa is a unit of pressure called a pascal, see the note below. The value of 3.2 is a compromise value so as to be more accurate near the ("center height" of this scale and formula) 15000ft height, and when not using other variables such as temperature. It was derived from that at 30000 feet, the known pressure is about 4.36. 14.7 / 4.26 = an increase by a factor of about 3.37. Therefore, 4.26 is about 3.37 times lower than 14.7   4.26 / 14.7 =~ 0.304**3** =~ 30%. 4.26 psi is the air pressure at the top of Mt. Everest, and is about 30% of the air pressure at sea level.

1 **ATM** =1 ATMosphere = ~ 29.92 inches in height of mercury =~ 30 inches in height of mercury
1 ATM = 14.7psi = 101325 pascals ~= 100000 pascals   : 1 ATM = the pressure of 1 atmosphere.

Ex. At tank of some gas is noted as having an internal pressure of 5 ATM = (5 ATM)(14.7 psi / 1 ATM) = 73.5 psi

1 **bar** of pressure = 100 Kpascals = 100KPa = 100000Pa = about 14.504psi , and this is just slightly less than 1atm

14.7psi = 14.7 pounds per square inch = 14.7 pounds per 6.4516 square centimeters, with about 2.2785 lbs / cm^2 which
            is equivalent to about: 1.034kg / cm^2   =~ 1kg / cm^2   =~ 9.8N / cm^2 =~ 10N / cm^2 =~ 100000 pascals

STP = Standard Temperature and Pressure, and it depends on the substance, location, conditions, experiment, etc.
For the atmospheric (air) pressure at sea level: STP is: 0°C = 32°F   and   14.7psi = 1 atm = 1 bar   . To quickly convert
from Kpa to psi, divide the Kpa by 6893 , (about 7000), for ex.: (100000Pa/1psi) / (7000Pa/psi) = 14.3psi

As mentioned previously in this book, a device called a barometer can measure the local air pressure at various levels of
height and weather conditions which can affect air pressure. By knowing the average air pressure at a certain height or
altitude, a barometer can actually be used to also effectively measure (by calculation) altitudes after rearranging the air
pressure formula to solve for height.

A formula for water pressure at a certain depth is:

   water pressure = 14.7 psi + (depth ft)(14.7 psi)   : 33 ft ~= 10 meters, each extra 10 meters or 33ft of depth will add
                                                      33ft   another 14.7 psi to the pressure. With this equation, the values of
                                                                           pressure and depth have a linear mathematical relationship. For
                                                                           every 33 ft deeper, the water pressure increases by 14.7 psi =1atm.
                                                                           The pressure at 33 feet deep is: 1 ATM +14.7 psi = 29.4 psi
                                                                           = 2 ATM. Each foot adds: (14.7psi/33ft) = 0.4455psi per 1 foot.
                                                                            Note that 14.7lbs = 6.668kg.   and   1 in^2 = 6.4516 cm^2

At 100ft , psi =~ 60 ,   At 200ft, psi =~ 104 , At 1000ft, psi =~ 460   ,   At 1km = 1000m = 3281ft, psi = 1476 ,
At 1mi = 5280ft, psi =~ 2367.   To use the above formula for meters of depth, first divide the number of meters by 0.30479
to convert it to its equivalent feet units value.

The above formula is linear (ie., water pressure in a proportional mathematical relationship to depth) because water does
not compress very much due to its fairly high density, molecule structure and could be considered as already being a
highly compressed gas of steam [water vapor], but the formula for air pressure (ie., air pressure in relationship to height)
is non-linear because air compresses much more than water because air is usually a gas that has a low density, except
for liquefied air.   The air pressure and-or density of the air at sea level is relatively high due to depth of it in the
atmosphere region and-or column of the air and its mass above it, and the air pressure decreases relatively rapidly as the
altitude increases.

The minimum depth that is considered as extremely dangerous for a diver to dive or go to without a special pressure
protective diving depth suit or rigid surrounding safety vehicle, such as a "submarine", is 130 feet. Here, the pressure is
about 58 psi. This is about the pressure of a typical 26 inch bicycle (air pressure inflation) tube in the tire.   At this depth,
and for greater depths, a minimum amount of "decompression" time is needed so as to avoid decompression sickness
(sometimes called "bends") as the diver travels upward to a lesser depth .   Here, expanding in volume compressed
nitrogen gas bubbles in the body, due to less pressure on them as the diver goes upward in the water to a less depth, are
given some time to be safely eliminated from the body.   The diver must ascend slowly at calculated amounts of time.
People in high altitude, internally pressurized craft, where the air pressure is low at high altitude, may experience similar
effects of a divers decompression sickness if the craft becomes unpressurized. Consider the fact that water boils faster or
more easily in a lower pressure environment such as on a high mountain. There are fish and other life forms living their

A SCHEMATIC DIAGRAM OF A BASIC ELECTRIC CIRCUIT

A schematic diagram or representative drawing of a simple electric (electrons with kinetic energy and electric charge) circuit (path, course) is shown below, and Ohm's Law is also used for some analysis and calculations of it. For electrons to flow, it needs a complete path to do so, and this is called a closed circuit, as opposed to an open circuit where a wire got cut open or something. A short circuit is where the electrical path has been shortened for some reason, perhaps by a piece of metal or wire. This short or shorted circuit is very dangerous for it can damage your device, drain the power supply, and-or cause smoke, and fire. Usually a circuit will cause some resistance in a circuit to be less or bypassed, and therefore , a high current will then be flowing in it.

Below, (I) is the identifier name of the variable for electric current and which is measured in units of amperes (A). Ir represents the amount of current through the resistor. V is used for voltage which electrically gives (electric) force and therefore transfers (kinetic, motional) energy form the battery (energy) supply to the electrons to flow. R is used for resistance which limits the flow of current and prevents the battery from draining of its energy too fast.   Vr represents the voltage across the resistor. Conventional current flow is where the electrons are considered as coming out of the positive terminal of the battery, and this is also mathematically practical since the increase in energy of the electrons is then is given a positive sign rather than a negative sign. [FIG 235]

$I = 0.5A = 500ma$

Showing "conventional" or commonly measured current flow.

I
current

$V = 5v$
battery

$R = 10$ ohms
resistor = r

Power lossed in the resistor
$= P_r = V_r\, I_r = (5v)(0.5A) = 2.5$ watts

~electrical conductor or "path" such as a metal wire
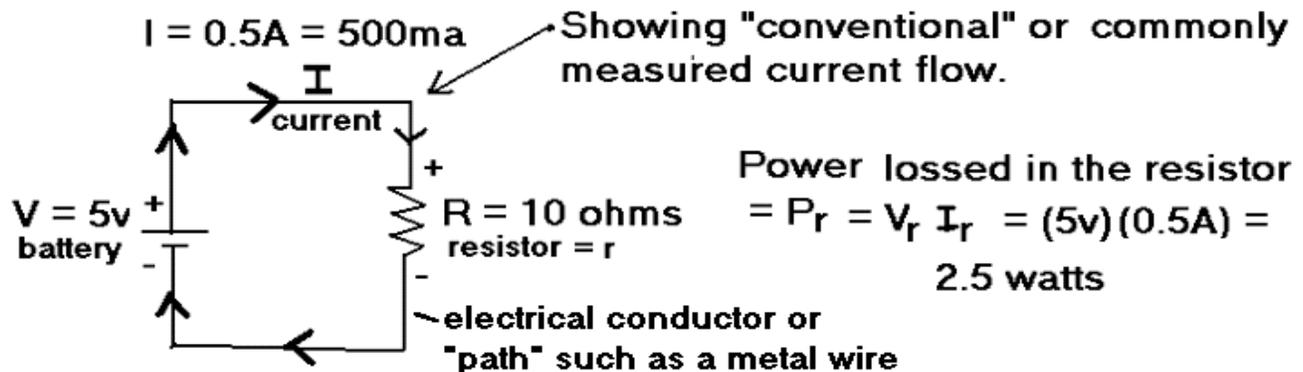
I = voltage applied to total resistance / total resistance
$I = 5v / 10$ ohms $= 0.5A$

$V_r = (Ir)(Rr) = (0.5amps)(10ohms) = 5$ volts

The power lost in the resistor or resistance can also be considered as equal to the amount of power delivered to that resistor, and-or needed by that resistor to function properly, such as a heater, etc. Generally, in many circuits, heat is an indication of not only lost power, but of completely wasted power and energy. Because energy has a price and resources associated to it, a circuit and-or device should be designed so as to have a low waste of power, hence have a high energy efficiency or usage:

  Power Input = Power Output + Power Losses

  Efficiency = _Power Output_ = _(Power Input - Power Losses)_   : divide Efficiency by 100 to find the
          Power Input   Power Input   equivalent percentage value

Since the efficiency value is a pure unit-less number, you can also use relative or percentage values so as to be non-specific in the actual detailed values, and have a useful generalization or basic value that easily express what is happening in a system(s) or device, and could even be used for comparing systems and costs. Here, power input would be expressed as: 100% = 1.0 as a relative or comparative value.

Standard USB 4-wire color codes:

    RED = +5Vdc
    BLACK = - , negative and-or battery-circuit ground

    White = Data, positive signals   (not needed in the above circuit, cut and insulate it)
    Green = Data, negative signals (not needed in the above circuit, cut and insulate it)

* If your supply voltage is not the same as 5v, then you need to calculate the value of the resistance needed.

A technical note about diodes and LED's which are a special type of diode: Before a diode "turns on" when the rated barrier voltage is finally reached, and its ("active" or controlled) resistance is very low and begins conducting like a short circuit (ie., a wire or closed switch), a diode will have a very high resistance of which is determined by the applied voltage (dropped or lost) across it. As the applied voltage across a diode increases from 0v, the diodes effective resistance will increase from infinity ohms, such as that of an open circuit and-or reverse (voltage) biased diode, and to a very low value of resistance, perhaps 0.1 ohms. In this manner, a diode can behave much like a voltage controlled adjustable or variable resistor and-or voltage divider. In place of a resistor, 3 plain (non-LED) diodes in series can drop about: 3 (0.65v) = 2v

later, others would add better tuning controls (such as vacuum tube electric oscillators and tuners) of the transmitter and receivers so as to significantly reduce the bandwidth (ie., range) of frequencies transmitted for each transmitted pulse. With a capacitor and inductor, the radio waves could be tuned (adjusted) to be a certain frequency. The radio receiver would then also have to be "tuned" (for maximum signal strength reception) to this transmitted (limited) bandwidth or frequency which also electronically rejects or filters [remove out] all other frequencies. This "reduced (frequency) bandwidth" transmission and reception method allows many more radio-wave transmitting stations and their transmitted and received information from interfering with each other.

Even when tuned to a certain frequency so as to be efficient, much of a transmitted radio waves energy is often radiated or transmitted in a 360° directional pattern so as to be available to anyone nearby with a radio receiver, but then with much energy actually often wasted, but it will be available whenever needed as "wireless communication" without a direct wire connection. The amount of the transmitted radio wave energy received by a distant receiving antenna is very small and this amount is impractical to use directly to power most things, and it must then be greatly amplified so as to be useful such as in an amplified radio receiver. The exception to this is with crystal radios (receivers) which can function without any extra voltage source and-or signal amplification as long as very electrically efficient and sensitive (tuned) antenna, circuit, and earphones such as the (high-impedance, low voltage and power needed, very sensitive) carbon or (piezo) crystal types are used. As of the year 2021, carbon earphones are difficult to find and-or are expensive, but a piezo-crystal earphone can still be found and purchased for a few USD dollars each.

The assigned main frequency for a radio station is called the carrier frequency or wave since the input or information signal, such as an audio signal (ie., energy) is converted or transformed to a corresponding electric signal (ie., electric energy, voltage and current) equivalent in a microphone, and which will essentially be applied (ie., to modulate, change, adjust) to and "carried" on, integrated or combined with as part (such as the intensity or "amplitude" of) of that one main transmitted radio wave. At the receiver, the microphone signal will be removed out from the carrier wave signal in a process called demodulation, and it will eventually be heard through an output audio speaker.   [FIG 236]

Block Diagram Of Radio Communication
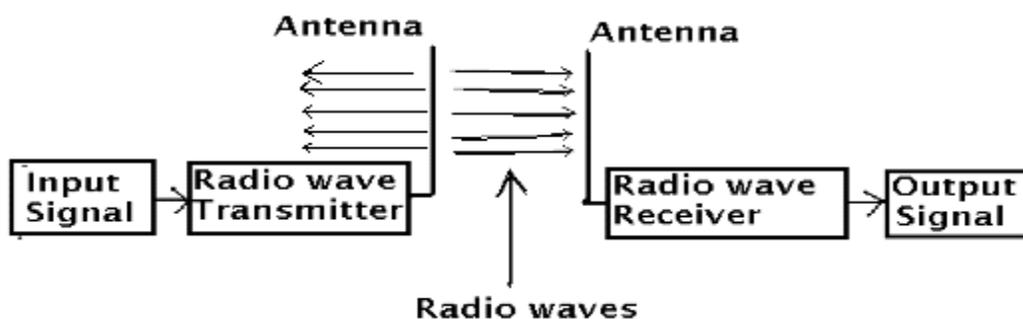
Though an antenna appears as an open circuit, the electric field created in the transmitter still causes electrons in the antenna wire to move by coulomb or electric forces, and a magnetic field at a right angle is then created with respect to the electric field created.   A radio wave (an electro-magnetic [fields] wave and transmission of energy) is composed of an electric field and a magnetic field at a right angle to the electric field. The (RF or rf, radio frequency) signal received by the reception antenna is often very small and needs to be greatly amplified so as to be useful and-or heard. Often the input and output signals are ("low frequency", LF) audio signals such as speech and-or music.   Today with modern "computer phones" with internet capability, transmission towers can send both audio and computer data using (digital, as in high power quick pulses, as opposed to an analog form) radio waves. In a microwave oven, the (microwave) radio (rf) signal is relatively close to the food substance, and therefore of a relatively high strength so as to apply energy to the food molecules and cause them to gain kinetic energy and its associated heat due to collisions and infra-red (ie., heat) radiation.

It is interesting that radio wave energy is composed of an electro-magnetic (electric and magnetic) fields perpendicular (90°) to each other, and does not have electrons with kinetic energy to transfer energy from one location to the other as

COMMON CLASSIFICATIONS AND ALLOCATIONS OF VARIOUS FREQUENCY RANGES

The following is often called the (standardized) electromagnet frequency bands ("spectrum") or range classifications:

C = speed of light in a vacuum =~ 299,792,458 m/s =~ 300,000,000 m/s = 3(10^8)m/s =~ 186,000 mi/s = velocity = v

f=frequency or hz, = 1/ (time of each cycle)   ,   w=wavelength in meters , hz = hertz = cycles per second = cps
frequency (f) and wavelength (w) are mathematically, inversely related in value to each other, and are the factors of C or the speed of the wave.

 C = fw ,   f = C / w , w = C / f ,   : Ex. If n is a multiplier or factor: At: nf, w = (1/n)w = w/n.   At: nw, f = (1/n)f = f/n.
                                                    C is a constant, and (f) and (w) are inversely related to each other mathematically.

Wavelength is a real physical distance between each cycle or corresponding peaks of a wave of energy at a certain frequency and medium that the wave or energy is traveling through. Wavelength is not the same meaning as the speed of sound or light transmission through a medium (material, such as air or a vacuum). A 1000hz sound pulse will travel faster through a medium of metal than in air, yet the frequency and its corresponding time (t=1/f) is always constant. Wavelength of a wave is determined by the frequency and speed of travel, hence wavelength is also determined by the medium it travels through.

Ex. For sound waves that travel at about 1125ft/s = 343 m/s in air, the wave length or wavelength of a 1000 hz sound
     wave is from:

    distance = (speed)(time) = wavelength

wavelength = (speed)(time) = v t
wavelength = (speed of sound)(1 / frequency)
wavelength = (speed of sound) / (frequency)   = v / f = v t , For rf, light or radio: w = vt = v / f = c / f

  w = (speed of sound) / (frequency) = (speed of sound)(1/ frequency) = (speed of sound)(time) = (v)(t) = d
     w = (1125ft/s)(1/1000c/s) = (1125 ft/s)(0.001s/c) = 1.125 ft / c   : ft / c = feet per cycle, or simply feet

       Note how the unites were expressed: (c/s) for frequency,   and   (s/c) for time , hence each has the mathematical
       inverse or reciprocal of the others units, and since frequency and time are inverse values of each other.

Sometimes a wavelength can be physically seen and-or demonstrated such as for sound waves in a pipe filled with a gas and a line of holes in it, and which will indicate where and the amount of sound or air pressure is in the pipe, and that distance or length between wave peaks or the troughs seen can be measured as the wavelength, and the corresponding frequency of it can be calculated using:

From: wavelength = distance = (speed)(time) = (speed)(1/frequency) = v / f , we have:

       frequency = (speed / wavelength) = v / w   : for radio waves , speed = v = C = 186000mi/s = 300000km/s

It is good to memorize that: wavelength = distance = (speed)(time) = w = v t = v / f , and that t = 1 / f and   f = 1 / t

Extra: Ex. If the physical wavelength of a certain (rf) radio wave is say 0.1m , the number of these waves per unit
        distance or meter is simply found by division of this distance into the unit distance of 1 meter:

       number of waves per meter = "wave number" = 1 meter / wavelength = 1 / 0.1 m = 10

145.80Mhz (transmitted by the ISS,
   receives at 144.49Mhz and 145.20Mhz)
                     2.06m

Shortwave, space communications, such as for the
ISS (International Space Station). Includes voice and data
such as **SSTV (Slow Scan TV images)** of which images can
be processed by a computer program (software) into an audio
file. This file can then be sent using radio communication.
The audio signal in the received radio signal (which has
the image data coded into audio signals/tones being carried
in/on as part of the high frequency radio signal) is sent to the
computer mic input or other audio input of a computer of which
an SSTV program can then process back into an image.
Amateur ("HAM"), shortwave radio stations often experiment
with using SSTV, and it is slow, but effective. There are many
low-cost programs and standardized methods for data
communication via radio that you should obtain and have
available as a radio operator/communicator, and before any
grid-down or emergency situation.

156.8 Mhz

A common emergency radio frequency. VHF CH 16. Please
monitor, listen and-or communicate with this frequency, and
then alert authorities if needed.

174Mhz 220Mhz   1.5m at 200Mhz   VHF TV CH 7 to 13

300Mhz to 1000Mhz=1 Ghz   1m at 300Mhz   UHF TV, Ultra High Frequency. 406 to 406.1 Satellite emergency.
                     0.5m = 50cm at 600Mhz   Digital TV radio waves are 470Mhz to 850Mhz
                                         FRS=Family Radio Service = 462Mhz to 467Mhz FM, 2W max
                                         FRS uses FM modulation which is less noisy than AM mod.
                                         GMRS = General Mobile Radio Service, is similar to FRS, but
                                         is more for business and requires a license to operate, and also
                                         can use "repeaters" (ie., transmission relays) to extend their
                                         range.   FRS has 22 channels available as of the year 2021.

1Ghz to 100 Ghz   30 cm at 1 Ghz , Radar   ,   3 cm at 10 Ghz, Microwaves (microwave ovens)
 10^9 hz   and with 0.3 cm = 3mm at 100 Ghz = 100(10^9)hz. , 1(10^11)hz
                          1 Ghz =   1 billion hertz = 1,000,000,000 hertz = 1 (10^9) hz = 1 giga-hertz

2.4Ghz to 2.485Ghz   12.5cm at 2.4Ghz   "Bluetooth", short-range radio communication system for data.
                          12.1cm at 2.485Ghz   Devices with Bluetooth can share files and audio as digital data.
                                                   There is also offline WiFi and-or BlueTooth phone apps.

100Ghz to 1000Ghz = 1Thz   1mm=1000nm at 300Ghz   Thermal Infra Red (IR, heat electromagnetic radiation)
                          1mm = 1(10^-6)m   1 Thz = 1 trillion hertz = 1(10^12) hz. Thermal energy is due
                                                   to the kinetic energy of particles, and this thermal energy
                                                   also radiates some (infra-red) light or electro-magnetic energy.

10Thz   to 100 Thz   30um to 3um   IR (infra red to barely visible red color light)
= 10^13 hz to 10^14 hz   3um = 3 micrometers = 3(10^-6)m = 3000nm

1000Thz   580nm=Red, 545nm=Green, to 440nm=Blue   Visible Light (**R**ed, **G**reen, **B**lue, "rainbow" spectrum =
= 10^15hz   "**R**OY **G** **B**IV" = Red, Orange Green, Blue Indigo, Violet
= 300nm = 300(10^-9)m = 3(10^-7)m   The color of an object is due to that all the other colors of the full
                                                 spectrum or white colored light are absorbed by the atoms on its
                                                 surface, and that light energy is converted mostly to thermal (heat,
                                                 kinetic) energy such as for photosynthesis in plants. The reflected

that material. A refraction effect is somewhat similar to the effect of having a mirror direct the light to another direction. When the light bean exits the transparent medium it will refract again at an angle that is determined by the new medium and its density. If a bean of light enters a medium straight-on, at no angle, then it will not refract at an angle, but will pass directly through it like a straight line passing through it, and without any refraction. The direction change can be estimated and-or thought of as being like that of having two wheels at the ends of an axel or line moving forward, and if one wheel slows down, there will be a direction change due to a rotation or torque effect about the slowed or stopped wheel acting as a type of pivot or rotation point.

Reflection and refraction are two different concepts. With reflection, the light does not enter the substance, but rather "bounces" off it, and it could be at an angle. The angle of reflection will be equal to the angle of incidence (incomming). light.

The speed of light and-or an elector-magnetic wave will actually be reduced (ie., be slower) while going through a medium such as glass or water, and this is due to the electrons in the glass atoms gaining some kinetic energy and creating their own electro-magnetic field which will then interfere with electro-magnetic field and-or energy of the light wave. This physical interaction is what also causes refraction.   Some of the energy of the light will be converted to heat (ie., kinetic energy) in the medium, and therefore, its intensity will continue to lessen as it travels through that medium. Beer's Equation or the Beer-Lambert Law states the relationship of the absorption (A) of light or energy through a transparent medium, and it includes the length (L) of travel, concentration (C) of the medium and molar absorptivity (e): $A = e\,L\,C$. A is a unitless number, usually called the Absorpton-coefficient of a material. A has a linear or direct relationship to e, L and C.   The molar absorptivity coefficient value (e) depends on the specific substance or medium, and the wavelength of light used, and is measured and found by experiment with a unit length of a test sample, say 1cm: $e = A / LC$. The (molar) concentration, and which can also be thought of how much material is dissolved in a unit of a clear liquid, and this similar to a density value, but in terms of mol of mass, instead of mass, and with units of: mol / Liter = mol / $(10\ cm)^3$. Determining the amount of light passing through an object is a quick method to see if a standard(s) is being met in various chemistry, food and other industrial processes.

MAJOR OBJECTS IN OUR SOLAR SYSTEM

(Most of the values below should be considered as approximate and rounded.)

Object	Diameter In Miles	Average Or "Mean" Distance From The Sun In Miles (perihelion + aphelion) / 2	Time Of 1 Orbit About The Sun	Eccentricity Of Orbit (elongation)	Time To Rotate Once, "Day Length" On Axis In relation to an Earth day
Sun	865,000	0	0		30 days at its equator, due to sunspot observations
Planets:					
Mercury	3,032	35 million = 0.37AU	88 days = 0.241yr	0.206	59 days
Venus	7,520	67 million = 0.72AU	225 days = 0.616yr	0.007	243 days
Earth	7,918	93 million = 1AU	365.256 days = 1 year	0.01671	˜ 1 day = ˜24h = 23h,56m,4s
Mars	4,210	142 million = 1.53AU	687 days = 1.88 years	0.093	1.026 days = 24h, 37m , 26s

* Largest "Minor" or "Dwarf" Planets, most within the main asteroid "belt", zone or region located between Mars and Jupiter: Much of this matter is thought to be from two or more planets colliding, and the resulting pieces colliding over time.

Object	Diameter In Miles	Average Or "Mean" Distance From The Sun In Miles	Time Of 1 Orbit About The Sun	Eccentricity Of Orbit	Time To Rotate Once
* Vesta	326	220 million = 2.37AU	3.6 years	0.0887	0.22 days = 5.28 hours
* Ceres	58<u>8</u>	257 million = 2.76AU	4.6 years	0.0758	0.38 days = 9.12 hours
* Pallas	339	258.54mill = 2.78AU	4.61 years	0.23	7.8 hours
* Hygiea	276	292 million = 3.14AU	5.56 years	0.112	27.5 days
Jupiter	87,000	484 million = 5.2AU	11.9 years	0.049	9.93 hours
Saturn	72,000	887 million = 9.54AU	29.5 years	0.057	10.55 hours
Uranus	31,000	1.784 billion = 19.2AU	84 years	0.046	17.23 hours
Neptune	30,000	2.8 billion = 30.1AU	165 years	0.009	16.1 hours
Pluto	1,478	3.67 billion = 39.5AU	248 years	0.248	6.39 days
* Eris	˜ 1,446	˜ 9 billion =˜ 96AU	˜560 years	0.436	˜ 26 hours

The distance values given above are distances from the Sun, and since Earth is 93 million miles from the Sun on average, the closest or shortest distance from Earth to another object is less this distance.   For example, the closest distance to or between Earth and Mars is calculated as a difference from the reference position of the Sun:

The difference between the distance to Mars from the Sun, and the distance to Earth from the Sun is:
142 million miles - 93 million miles = 49 million miles =˜ 1.53AU = 1.0AU =˜ 0.527 AU .   This is the (average or typical) distance between Earth and Mars.

Since Mars is a distance of 0.527AU from Earth, it will take a radio signal (8.333 min. / 1AU) (0.527 AU) = 4.39 minutes to reach Mars from Earth, or vice-versa, when this closest approach or minimum distance condition exists.

A satellite launched in 2006 arrived at **Pluto** over 8 years later on July 14, 2015 and took close photos of Pluto as it quickly flew past Pluto, and without orbiting Pluto. These were the first close-up photos of Pluto for even the best telescopes could not resolve much detail of any other planet farther than Saturn. The photos revealed that Pluto is a red-pink and white colored solid (not gaseous) planet with a surface having much ice, high ice mountains, and contains some atmosphere. Because it is fairly small and not a gas covered giant planet like Jupiter, Saturn, Uranus and Neptune, and is in a relatively high tilted orbit about the ecliptic plane (solar plane of Earth's orbit about the Sun), some consider that Pluto is not an actual planet, but rather a large asteroid or planetoid like Ceres. Many people still firmly believe Pluto is a planet since it is

$$e = (aphelion / average) - 1 \quad , \quad and:$$
$$(perihelion\ distance / average\ distance) = (1 - e) \quad and \quad e = 1 - (perihelion / average)$$

The eccentricity of the Moon's orbit about the Earth is about: e = 0.055, and its average distance from Earth, or the center of the Moon's elliptical orbit along its major axis distance (aphelion + perihelion) or plane is 240000 miles.

Some common astronomical terminology and facts for elliptical orbits:

AU = Astronomical (distance) Unit. This is the average distance that Earth is from the center of the Sun, and is used
      to simply indicate relative (ie., multiples of) or comparative distances that objects are from the Sun or other
      objects. Light or radio waves will take about 500 seconds = 8.333 minutes to travel 1AU = 93 million miles.

aphelion = An objects most distant location from the Sun. ("apo" means distant, and "helion" is from "helios" for Sun,
      A common use of "apo" or "ap" is "apart", meaning separated from).   apogee = aphelion

      The elongation or eccentricity (e) of Earth's orbit about the Sun is about 0.017. Earths average distance
      from the Sun is 93 (10^6) miles. The aphelion or farthest Earth is from the Sun is:
      Earth's Aphelion = (Earth's average distance from the Sun) + (Earth's average distance from Sun)(eccentricity) =
            (Earth's average distance) (1 + 0.017) =~ 94.6 million miles

      Mar's Aphelion = (Mar's average distance) (1 + 0.093) = (142)(10^6) (1.093) ~= 155 million miles

perihelion = An objects closest location to the Sun. ("peri" means near, a common use of "peri" is "perimeter").
      perigee = perihelion

      The elongation or eccentricity (e) of Earth's orbit about the Sun is about 0.017. Earths average distance
      from the Sun is 93 (10^6) miles = 93 million miles.   The perihelion or closest Earth is from the Sun is:

      Earth's Perihelion = (Earth's average distance) - (Earth's average distance from Sun)(eccentricity) =
            (Earth's average distance) (1 - 0.01671) =~ 93 (10^6) (0.9833) = 91.45 million miles

      Earth's aphelion - Earth's perihelion = a difference of only 3.2 million miles

      Mar's Perihelion = (Mar's average distance from Sun)(1 - 0.093) = 142 (10^6) (0.907) = 128.8 million miles
      Mar's Aphelion = (Mar's average distance from Sun)(1 + 0.093) = 142 (10^6) (1.093) = 155.2 million miles

Average Orbit Distance From Or To The Sun = (Aphelion + Perihelion) / 2   , [FIG 237]

about the Sun are more or less slightly tilted by up or below by a small angle from the solar plane of Earths travel about the Sun, and they will also appear to travel across the night sky in the same general (ecliptic, east to west) location, path and direction as the apparent movement of the Sun, Moon and stars across the sky from east to west. This east to west apparent motion of the stars and planets is due to that the Earth is rotating to the eastward direction, and the generally fixed stars and planets then only appear to be traveling westward at night.

inclination = The angle between the actual orbit plane of a planet and Earth's solar plane (ecliptic).

magnitude = The apparent (logarithmic scale) or relative brightness of an object in space as seen or appears (only) from Earth and the distance to it. Brighter objects have a lower, possibly negative, value.   An actual relative brightness value = 1 / (2.512^m) : m = magnitude. For reference, the apparent brightness of the star called Vega is 0. The star Polaris, the "North" or "Pole (axis)" star has a apparent brightness of about: -3.6. Planet Venus has an apparent brightness of: -4.4. The brightest star in the night sky is Sirius with a apparent brightness of about -1.4. The apparent brightness of the Moon is -12.5. The apparent brightness of the Sun is -26. Surely this numbering system is not so intuitive, and seems backwards, and this is due to that the Greeks considered number 1 as the highest or most important for this system, and the multitude of less brighter objects would be assigned to the next integer values higher than 1 and up to include 6.   On basic star charts, brighter stars are generally and simply indicated as being bigger even though size is a different concept than brightness. Generally, objects with a magnitude of 7 or more are not visible to the unaided eye, and a magnification system (binoculars, telescope, etc) is needed to gather a larger area of light, concentrate it and then focus a brighter image of it onto the eye or an image sensor such as that in a digital camera.

WARNING: Do not view the Sun with or without any type of lens, mirror and-or telescope, and instruct others not to do so because of the high possibility of permanent eye damage due to the intensity of the light and-or concentrated light. Telescopes use a thick light filter to view the Sun.

With this logarithmic apparent (visible by the unaided eye) brightness system, each next noted magnitude appears about 2.5 times dimmer. Where is 2.5 or 2.512 from? 2.512 is the 5th root of 100. 2.512^5 = about 100. The modern apparent magnitude formula was created by Norman Pogson in the mid 1850's. Note that Vega, in the original ancient system, had a magnitude of 1 meaning it is a star of most or significant importance. With the modern system, Vega is considered to have a magnitude of 0.

For an increase of 5 magnitudes, the apparent visible intensity or "flux" is estimated at about 100 times less. For a decrease of 5 magnitudes, the apparent visible intensity or "flux" is estimated at about 100 times more.

Are stars brighter for astronauts in space? The Earth's atmosphere reduces the intensity of received sunlight. Most estimates put the intensity of sunlight before reaching Earth's atmosphere is about 1370 watts/m^2, and will be 1000 watts / m^2 max. on Earth's surface. The difference in these values is 370 watts/m^2 and represents a loss of: 370w/1370w = 0.27 = 27% , hence the intensity of the sunlight and-or starlight in space is about 27% brighter. In terms of a change in apparent, visible magnitude, it is only a small change.

resolution = In terms of photographs or other images, resolution is a word describing the linear distance (such as a width) that the smallest discernible picture element (or "pixel") represents of that object being photographed.   A more or "higher resolution" image means greater or "finer" detail is visible in that image, meaning a pixel corresponds to a smaller actual distance, and the image is a better representation of the true image or actual object. In terms of eyesight, resolution is rather an angle since an object, or even an image like a photograph, can be closer or farther away and another method to describe resolution is needed, and that is with an angle. Brighter objects tend to be able to be seen from great distances such as a candle or star in the darkness, but what is the minimal discernible distance or spaces between two of them in terms of angle? It depends on the person and how good their personal vision is, and an estimate is (1/60) of a degree = about 0.017° ("angular diameter" [ie., max. width]). This angle is also equal to 1 arc-minute in the DMS angular system.   The "(lettered) eye charts" at eye doctors offices take many things such as this into account so as to help determine how well your vision is. "20/20" is often spoken as the eye or vision test result value

* The distance to the center of the Milky Way Galaxy, from the position of Earth in it, is about 26000 light years. The diameter of the Milky Way Galaxy that out Sun is in, is estimated to be 75000 to 100000 light years across. The attractive force that holds the orbiting stars in a galaxy is often a black-hole which is a dense mass so large, its gravity force even stops light from going outward from it. The center of the Milky Way Galaxy is located in the direction of the constellation (ie., a unique pattern, group or arrangement of stars) called Sagittarius. This center is obscured by what can be called clouds of dust and debris, and can be viewed by some large radio and infrared (ie., heat) light capable telescopes. This black-hole or theoretical dark star with a very high level of gravity of which not even light has enough kinetic energy to be emitted from, and is called Sagittarius A. This black hole has been imaged over several years of time using special infra-red sensitive telescopes. This black hole currently has a several high velocity stars in an elliptical orbit about it. This black-hole was discovered in 1974 by radio astronomers noticing more than the average amount of star radio noise and-or or energy coming from this location. Radio astronomy was formally started in 1931 by **Carl Jansky** of the **Bell Labs** company, and who built a radio-telescope which was basically a large antenna construction on the ground that was tuned to receive a high frequency. This antenna (ie.,here, a radio energy receiver) could be rotated so as to be pointed in a desired directions in outer-space. He later noticed that there was always more radio noise in the direction of the Sagittarius constellation in our Milky-Way star galaxy.

* The latest estimated distance to the (current) "North Star" = "Pole Star" Polaris is about 323 light years. Polaris is actually a 3 star system. Polaris actually differs by a small angle of $0.735°$ from the true location of Earth's north pole axis. The farther a person goes more south on the southern hemisphere of Earth, the less stars visible around the Polaris region since the Earth's bulge at the equator is essentially blocking the view. This book contains basic star maps. Throughout the year of Earths travel about the Sun, the stars will appear to have a counter-clockwise (ie., to the left from the top reference) circular orbit about Polaris which will still appear (nearly) fixed at its location, and this circular path is due to that Earth is rotating on its north to south, polar axis. As Earth orbits about the Sun during the yearly orbit, all the stars at day and-or night will shift in position along the solar plane (ie., the east to west, "ecliptic" line), and about Polaris by about (360°/365 days) =~ 1 degree per day.

* Due to Earth's 365 day, or yearly, orbit about the Sun, we will gradually see other stars in the Milky-Way galaxy at night.

* Most Comets have a highly eccentric (ie., "off from center". elongated, non-circle) orbit about the Sun.

* Using the distance to the Sun, and the diameter of the Sun, and using some right angle trigonometry, you can calculate the apparent width angle of the Sun as seen in the sky from Earth or any other planet. The Sun appears to be 0.5 degrees = half a degree, wide in the sky from Earth. For an example, knowing the distance from the Sun and the diameter of the Sun, you may solve this value for Mars. Another similar calculation is from the trigonometric concept that if you double (a factor or multiple of 2) the distance of the base of a right triangle, the height corresponding to that distance will likewise double (2). If the height is to remain constant, such as the diameter of the Sun, if the distance doubles (2), the angle will be divided by that same factor (2). For Mars, being a factor of 1.5AU from the Sun, the apparent angle of the Sun seen in the sky from the surface of Mars will be divided by that same factor. For Earth, the apparent solar angle or size is about 0.5°. For Mars, it is about 0.5° / 1.53 = 0.3<u>3</u>° on average since orbits are elliptical and the distance to the Sun changes varies.

From Earth's surface, the Sun appears to have a relative area of: (pi)(r^2) = 3.14 (0.5^2 units) = 0.785 degree units.
From Mars's surface, the Sun appears to have a relative area of: (pi)(r^2) = 3.14 (0.327^2 units) = 0.336 degree units.

Since Earth is at 1AU from the Sun, and Mars is at 1.53AU from the Sun, the average distance to Mars from Earth is: (1.53AU - 1.0AU) = 0.53AU. In terms of miles, this would be: (0.53)(AU distance) = (0.53)(93 million miles) = 49.3 million miles. This minimum distance is always changing to a larger value as the planets orbit about the Sun, especially when Mars is on the opposite side of the Sun and its distance from Earth is:
Earth's distance to Sun + Mars's distance to Sun = 1AU + 1.53AU = 2.53AU = 235.3 million miles.

The ratio of the relative apparent Solar areas of Mars to that of the Earth is: 0.336 units / 0.785 units = 0.43
The relative solar radiated energy (light, heat) reaching Mars will therefore be about 43% that of the Earth.

space as to where the Big-Bang originated at.

How the speed of light was first determined or estimated

The speed of light was once thought to be so high that it could not be known. In 1676, **Ole Romer**, (1644-1710), which is pronounced with a long-o sound such as used for "row", and may be spelled as Roemer. and he was an astronomer from Denmark, and who made the first close estimate as for the speed of light. With a telescope, he was studying the orbit of one of Jupiter's moons called Io, and noticed that it took about 1.7 days and would also periodically go behind Jupiter at this same time of 1.7 days, hence it was therefore a constant and could be seen to happen at a certain predictable time on Earth.   The distance (93 million miles) to the Sun as known today, and which is also the radius of Earth's orbit about it was already previously estimated to be a lesser value. The diameter of Earth's orbit about the Sun is twice this distance, hence 186 M.mi. Roemer noticed that Io was "late" and that there was a several minute difference when Io was to go behind Jupiter. This happened when the Earth was on the opposite sides of the Sun every six months of time. Roemer understood that this time difference (16.7min, change in time, modern estimate) was most likely due to that light had a longer distance to travel and therefore took more time to reach Earth. Roemer's estimate for the speed of light was about 130000 miles per second, and this value is about (130000mi / 186000mi) =~ 0.70 = 70% of the speed of light, and is fair or good estimation considering the (limited) knowledge available at those times.

In a more modern estimate and calculation of the speed of light, and using more accurate values, we have a closer estimate of the modernly accepted speed of light of: 186000 miles / second:

speed = distance / time = (change in distance) / (change in time)   we have:

(2 x 93 million) / 16.7minutes = 186 million miles / 1002 seconds = 0.1856287 million miles / second
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ = 185,628.7 miles / second = (c) = light speed
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ = 299792458 meters / second = m/s

The above value for the speed of light is for when it is traveling through a vacuum medium that has no matter in it. Just like the speed of sound can change (increase) traveling through mediums (matter) such as air, water or metals, the speed of light can also be altered (slowed, decreased) as it travels through a medium such as glass. Though the frequency(s) and-or the colors seen of the light will remain the same, the wavelength distance of the light frequency(s) will be decreased in the glass and will resume light speed when it exist the glass material and into another material such as air.

Time	Distance Light Will Travel (approximate) = (speed) (time)		
1 second	186,000 miles =	300,000 km =	**"1 light-second"** = distance light travels in 1 second
1 minute	11,160,000 miles =	17,960,279 km =~ 18 million kilometers = **"1 light-minute"** = (c)(1 min)
1 hour	669,600,000 miles =	1,077,616,742 km =~ 1 billion kilometers = **"1 light-hour"** = (c)(1 hr)
1 day	16,070,400,000 miles =	25,862,801,808 km = **"1 light-day"** =~ 26 (10^9) km = 26 billion kilometers
1 year	5.88 trillion miles = 5.88 (10^12) miles = **"1 light-year"** ~= 9.5 trillion km =~ 9.5 (10^12) km

The distance to the Sun was first estimated by a calculations using observations and geometry before Roemer's discovery. Since ancient times, it was known that the Sun was farther than the Moon due to that during any eclipse seen, the Moon is always seen as covering the Sun. Since Roemer, the Earth to Sun distance had better approximations. Once this distance was found, the diameter of the Sun could also be estimated using basic trigonometry using the Sun's observed angle of 0.5 degrees: Tan ¢ = Tan 0.5° = opposite side / adjacent side = Sun's diameter / distance to the Sun. Sun's diameter = Tan 0.5 (distance to the Sun) = 0.008726867 (93000000 mi) =~ 811600 mi , actual value is ~ 865000 mi

Roemers analysis for the speed of light was very creative and helpful, but it needed to be verified. **Hippolyte Fizeau** (1819-1896), from France, made an estimate of the speed of light in the year 1849. In brief, his light speed measuring system used a high speed rotating disk that had holes in it near its edge, and which has a higher linear velocity than it would nearer to the center as it rotates. This device allowed for the observation and calculation of the time it took for a

ORBITAL MOTION OF OBJECTS

Most orbits of the planets around are Sun are fairly circular with a low eccentricity (elongation [a greater distance than a circular radius] or shortening [a less distance than a circular radius] near the Sun). Still, all the orbits are periodic, and will predictably repeat every so many specific number of months or years. The position of a planet in its periodic orbit can still can speed up or slow down and this can make it difficult to predict an objects position. The reason for this is that the object was traveling (orbiting) around the Sun in a natural elliptical orbit with the Sun at one foci or focus point of that elliptical orbit. It is not difficult to understand that the closer the object is to the Sun, the faster it will be traveling. This is what happens when an object falls under the constant pulling influence of the force of gravity and it accelerates faster as it gets closer (perihelion) to the source mass of that gravity. Note that a circular orbit can be considered as a special instance of an elliptical orbit. It is not difficult to then understand that an object will then be traveling slower at the opposite end (aphelion) of this type of orbit, and again this is due to the constantly applied force of gravity constantly acting (pulling) upon the object and it decelerates. In 1609, **Johannes Kepler** (1571-1630), from Germany, published significant mathematical discoveries of orbital motion. Here are the three **Kepler's Laws** of orbital motion:

1. **Elliptical orbits**:
   Planets follow an elliptical orbit about the Sun at one foci point of that orbit. The speed of the object is not constant for objects in a natural gravitational induced (caused) orbit about the Sun. Speed increases near the Sun due to the increasing force of gravity of the Sun.

2. **Equal areas per same time**:
   From the orbiting object to the foci, equal orbital (planar, surface) areas will be covered in any specific amount of time.

   This coincides with a concept called **angular momentum** which essentially means rotational or orbiting momentum. Momentum = mv can be considered as a measurement of the amount of movement, and more technically, it is a measurement of the amount of inertia (resistance to a change in motion and direction) it has and the more energy required to change it's velocity and-or direction.. In terms of an orbiting body such as a planet about the Sun, the speed and kinetic energy (KE) of the object increases as it gets closer towards closer towards the Sun due to its gravity, and this "linear momentum" increases since the product of (mv) will now be larger, however, if the product now includes the radius distance from the Sun, this value is a constant (ie. "conserved" or "maintained") for the object. **angular momentum = mvr** : v and r are inversely related, and m is a constant. Note: mv = momentum . The product of (v) and (r) is a constant, and if the mass is constant, the angular momentum (mvr) value is constant for an orbiting object. If the mass of the object changes, the angular momentum will change.

3. In 1619, Kepler found that the orbital period (P) of an objects orbit is proportional to, or directly (but not linearly) related to the objects farthest distance of the elliptical orbit. Basically, this states that more distant planets take longer to go around the Sun. For the specific and precise mathematical relationship: P^2 is directly related to the cube of (a) which is (a)^3. **For objects orbiting the same focus such as the Sun: P^2 / a^3 is a constant (ratio) value** and regardless of its mass and the time needed to make one orbit. Kepler found this by applying Newtons law of gravitation, and resulted in this equation below, and here using Earth's 1 year orbit time as a reference, and taking a ratio value:

$$\frac{(P = \text{orbital period yrs})^2}{(a = \text{semi-major axis in AU units})^3} = k = 1 \ , \ \text{or} \ = \frac{(\text{orbital years})^2}{(\text{average distance to the Sun in AU})^3} = \frac{(1\text{year})^2}{(1\text{au})^3} = 1$$

(a) is essentially equal to the (average) radius (r) distance between the orbiting object and Sun.
Clearly, because of the definition, for Earth: P^2 = k (1 AU)^3. = 1 :k is a constant equal to 1 for Earth

Saturn which has an average AU of 9.537 , P^2 =~ 9.537^3 , P = $\sqrt{9.537^3}$ = 29.45 earth years (e.y.)

With Newton's motion and gravity laws in about year 1687, it can be shown that:

P^2 = a^3 $\dfrac{4(\text{pi})^2}{G(M1 + M2)}$: If M2 is a the mass of a planet or moon, its mass, maybe unknown, is practically insignificant (as if 0kg) as compared to and-or combined (added) to the mass (Ms)

From the above equation for Fc = Fg, we have $\underline{mv^2}$ = mrw^2 , and some variables can be solved for such as:
r: r = orbit distance from the center of the Earth

v = r w = $\sqrt{GM / r}$$$: effective linear velocity for a object rotating or in circular orbit about another.
: This is also derived in the angles and rotation section in this book.
Also consider: **v = d / t = C / t = 2(pi)r / t**: t = period of orbit or rotation = T = P
Mass of Earth = Me = (5.9722)(10^24)kg , radius of Earth = Re = r = 6378km
Due to the inverse relationship, farther orbits with a larger radius will have a lower
(ie., slower) effective linear velocity of travel, and a longer time period of orbit.

For the total time to complete an orbit = period of orbit = P = T , From: d = v t , t = d / v = C / square-root (GM / r)
t = 2(pi) r / square-root (GM / r) , t^2 = ((2)(pi)(r))^2 / (GM / r) , t = P = **T = 2(pi) square-root (r^3 / GM)**

An orbiting object or mass held by gravity (ie, a centripetal force) obviously does travel in a straight line above the center point of Earth, but if the centripetal force was released, the object would leave or "escape" orbit and continue on in a straight line direction and which is perpendicular (ie., 90 degrees) or tangent to the orbit radius line at that instant.

While orbiting, the object is always, slightly changing direction (ie., rather than going straight) in every instant of time, and for a change in direction, a force is needed such as the (constant) centripetal force due to gravity at that height or distance.

Force = (mass)(acceleration) = (mass)(change in velocity / change in time) = (m)(v^2 / r):

a = (change in velocity) / (change in time)

Due to centripetal force, there is a change in direction toward the center, hence the radius (r) changes very slightly to a lesser value:

From: distance = (velocity)(time)$$with substitution:
(change in r) = (v)(change in time) = (v) (change in velocity / a)solving for (a) , we have:

a = (change in velocity) (velocity) / (change in radius)   ,   and for this situation with constant values for orbiting:

Since: (change in velocity) / (change in radius) is equivalent to the same ratio value of (velocity / radius) = v / r :

Then simplifying the equation for (a), it can then be shown that :

a = v^2 / r$$and thatFc = centripetal force = ma = m v^2 / r

Centrifugal force is an outward directed force of a rotating object or material. It could be thought of as the equal and outward (from the center) opposite force of the centripetal force. It is sometimes used to separate materials that have different densities and-or to strain (ie., remove) liquids away and out from a more solid material. A motorized machine that is designed to do this is caused a centrifuge. When material is spinning like a flywheel, the outer rim portion is going at a higher velocity than the inner portion, and therefore it has more energy and pressure. This can sometimes be seen as an increase in the height of a fluid material near the outer rim, and therefore a decrease of the height of the fluid material near the center of rotation. Centrifugal force can be used to create simulated gravity for astronauts in a large rotating spacecraft.

Estimating the orbit time of a planet, and its distance from the Sun:

To make an equation for a planets orbit, the position of a planet(s) is studied and recorded for many years and the equation is refined. An estimate for the time of orbit of a planet can be made by noting how many degrees the planet has moved with respect to the background of seemingly fixed position of very distant stars during say one or several year of

An illustration to describe many of the basic facts about the orbit of Earth: [FIG 237A]

This view in the above image is as if looking down from far above the solar plane so as to view the entire ellipse of Earth's orbit.

The two foci on the major axis of an elliptical orbit are equidistant from the center of the ellipse where the major and minor axis intercept or cross. The center of the ellipse is also the center of orbit. Here (x) or the horizontal axis is the major axis, hence (a > b) and-or (b < a).   Earth is closer to sun in the winter, but the northern hemisphere is effectively tilted away from the Sun. Using approximate values:

Perihelion distance, Earth's closest distance to the Sun is: 91.4M mi
Aphelion distance, Earth's farthest distance from the Sun is: 94.5M mi
Earth's average orbit distance = a = (aphelion + perihelion) / 2   =   92.95M mi = ˜ 93M mi

 c = Distance from the center of orbit to the focus = $\sqrt{a^2 - b^2}$ = (aphelion - perihelion)   , and:

$a^2 = b^2 + c^2$: c = foci distance is constant, and the length of the major and minor axis is constant, however,
                throughout the orbit, x or (current (a) value), and y or (current (b) value) are not constant.

a = $\frac{\text{major axis}}{2}$ = $\frac{(\text{aphelion + perihelion})}{2}$ = $\sqrt{b^2 + c^2}$ = 92.95M mi = ˜ 93M mi   :average orbit distance from Sun

2a = total distance from any point on the ellipse to both foci = 2 (average distance) = 185.9M mi   = ˜ 186M mi

$$b = \sqrt{(aphelion)(perihelion)}$$

For a practical verification of this, consider the special case of an ellipse where the aphelion = perihelion = radius = r, and this ellipse is actually a circle:

$$b = \sqrt{(r)(r)} = \sqrt{r^2} = r = a$$

In a similar manor to a previous derivation of (b) shown above: $a = \dfrac{b}{\sqrt{1-e^2}}$: the den. is < 1, therefore, a > b

If we let: $N = \dfrac{aphelion}{perihelion} = \dfrac{a(1+e)}{a(1-e)} = \dfrac{(1+e)}{(1-e)}$, also note here for example: aphelion $= perihelion \dfrac{(1+e)}{(1-e)}$

$N(1-e) = (1+e)$
$N - Ne = 1 + e$
$N - 1 = e + Ne$
$N - 1 = e(N+1)$ and: $e = \dfrac{(N-1)}{(N+1)}$: this value is always less than 1 as expected for (e)

The speed (Vp) of the orbit of Earth around the Sun will be the greatest when it is the Sun at perihelion, and the slowest speed (Va) when it is farthest away from the Sun at aphelion. The average or linear speed of Earth's orbit about the Sun can be calculated as:

$$Vo = \dfrac{distance}{time} = \dfrac{(2)(pi)(r\ mi)}{365.25\,days} = \dfrac{(2)(3.14159265)(92.9M\ mi)}{365.25\ days} = \dfrac{583.708M\ mi}{365.25\ days} = \dfrac{1.5981M\ mi}{1\ day = 24hr} = \dfrac{66588\ mi}{1hr = 3600s}$$

Vo = 18.497mi/s

Due to the force of gravity increasing nearer to the Sun, the average speed of orbit will increase by a factor of (e) at perihelion:

Vp = Vo + increase in Vo

Vp = Vo + (Vo e) = Vo (1 + e) = 66588mi/h + (1 + 0.01671) = 66588mi/h (1.01671) = 67700mi/h = 18.81mi/s

The average speed of orbit will decrease by a factor of (e) at aphelion:

Va = Vo + decrease in Vo

Va = Vo - (Vo e) = Vo (1 - e) = 66588mi/h + (1 - 0.01671) = 66588mi/h (0.9833) = 65476mi/h = 18.19mi/s

The distance between every planets can vary significantly throughout both of their elliptical orbits. For example, if a planet is on the same side of the Sun as Earth is, it will be closer to Earth, and it will be seen as bigger with a telescope, perhaps twice as big such as with planet Mars which is relatively close to Earth as compared to the more distant planets.

Most tables of information about the planets state the planets average orbit distance, and on occasion, you may need to know its actual position, and or its aphelion (farthest from Sun) and perihelion (closest to Sun) distances. As shown previously:

aphelion = 2a - perihelion = a + c = a(1 + e)   : a = semi-major axis and average orbit distance
perihelion = 2a - aphelion   = a - c   = a(1 - e)   c = distance from center of orbit to focus (Sun)
                                                                                        e = eccentricity of orbit value

Sir Ramanujan, a historic Indian mathematician with a significant volume of work and with many being practical, created a very practical formula for the approximate perimeter of an ellipse, and it has very good accuracy, and without using the somewhat complex series created with many divisions needed. It is also of note that this formula does not explicitly use the variable (e) = eccentricity, however, the axis variables (a) and (b) are used which do actually define the value of (e):

$$P = (pi)(a + b)\left(1 + \frac{3h}{10 + \sqrt{4 - 3h}} \right)$$: **Perimeter Of An Ellipse** , credited to Ramanujan from India
   $h = (a - b)^2 / (a + b)^2$
   a = major axis , b = minor axis

It is conceivable that the distance between every two successive, very close points on the ellipse can be summed together so as to find the perimeter. This is can be done with a computer using the ellipse equation and the distance between two points equation. Since the ellipse is symmetrical about both axes, only one quarter of its perimeter, such as from x=0 to x=semi-major axis, needs to be found and then multiplied by 4 to find the entire perimeter value.

The author has found a very slight correction or adjustment value for (h), and which will improve the calculated perimeter result slightly, and particularly for when the eccentricity is high, and that is: h = h + (h)(0.0016523164869) = = h (1 + 0.0016523164869) = h(1.0016523164869)

For the area of an ellipse, first consider the area of a circle being: $A = (pi)r^2 = (pi)(r)(r)$, and that we can find the area of an ellipse, by letting one variable r=a and the other variable r=b::

The area of an ellipse is: A = (pi) (semi-major axis)(semi-minor axis) = (pi)ab   : **Area Of An Ellipse**
   a = semi-major axis
   b = semi-minor axis

There is a constant ratio value of the area of an ellipse or circle to that of the surrounding rectangular area with the same dimensions as the major and minor axes, and this value is: **k = 0.785398163397448** , and this value happens to be the corresponding radian angle of a 45 degree angle, and it is also equal to (pi)/4.   **Area of corresponding ellipse = (k) (Area of corresponding rectangular area).**

The ratio of perimeters is not a constant but has a range from about 1.57$\underline{1}$ = (pi/2) to 2.0 as mentioned above, and the ratio of the two extreme values of this range is also k = 0.785398163397448. The perimeter of the ellipse within the rectangle will always be larger than the perimeter of the surrounding, corresponding rectangle. The area of the ellipse within the rectangle will always be less than that of the area of the rectangle.

One might ask, what are the dimensions of the axes and-or corresponding rectangle when **e=0.5** (ie., half of the maximum value of 1)? The answer to this is interesting, and it is when the ratio of **(b/a)** is nearly equal to: (epsilon / (pi)) = (2.718281828 / 3.141592654) = 0.865255979, and this is =~ **0.866025403** = sin 60°, and is nearly equal to the (square root of k) = 0.8662269254527578. Here, the ratio of the perimeter of the ellipse to that of the circumscribing or corresponding rectangle is about 1.5728246008$\underline{1}$ ~= (pi/2).

When the ratio of **(a/b =2)** and-or **(b/a = 0.5)**, e = 0.866025403 = sin 60° = square-root of (0.75). When **a/b=2** and-or **b/a=0.5** and-or **b/a=2,** that is, one axis is twice as long as the other, the ratio of the perimeters is a constant of: **1.61481450996017**. Another observation is that epsilon (e) is about equal to (pi) (sin 60°). When (b/a = 0.25) and-or (a/b = 4), the area is equal to (pi)(b^2).

When a=b, the ellipse is a circle and has the smallest perimeter for that value of: 2(pi)(r) = 2(pi)(a) = 6.283185307a. As (b) gets smaller for a given value of (a), the perimeter of the ellipse will decrease to its minimum value of 4a where (a) is the semi-major axis, and (b) will be equal to 0. When **a=b**, the ratio of the perimeters is a constant of: Pe/Pr = **1.570796325** = (pi)/2. The above ratio leads to the conclusion that when the ratio of (a/b) and-or (b/a) is a certain constant value, the ratio of the perimeters also has a specific constant ratio value.

When **a=b**, the ellipse will have an **e=0**, and it will be a circle shape and have the maximum area for a given value of (a).

or lost any kinetic energy, its velocity will still be the same as that of the distant orbit, and its momentum will still be the same value.   The following analysis may also be helpful to this subject: [FIG 237C]

In the above figure, an object is in a circular orbit with a circumference of C1. It takes 1hour = 1h for the object to make an orbit. From d=vt ,its velocity is:

$$V1 = d / t - C1 / 1hr = 2(pi)(r1) / 1hr$$

In terms of rotations (rots), revolutions (revs) or full-orbit velocity (Vo):

$$Vo = 1 \text{ orbit / hr} = 1 \text{ rot / 1 hr} = 1 \text{ rev / 1 hr} = 1 \text{ cycles / hr} = 360° / 1 \text{ hr} = 2(pi) \text{ radians / 1 hour} = 1 \text{ wave / 1 hr} =$$
$$= 1 \text{ repetition / 1 hr} = w = \text{angular or circular velocity}$$

If (r) is changed by (n), either as a numerical co-efficient factor or divisor, C1 is then changed in the same way by this value of (n). If (r) is decreased by a factor of 2, hence (r/2), then C1 will also decrease by 2 to be a value of say C1/n = C1/2 = C2 in the above figure. Given the same velocity (V1) and time of 1 hour, the number of orbits with each being equal in length to C2 will also increase by the same value of (n):

$$\text{From C1/n} = C2 \text{ , } C1 / C2 = n \text{ , } \text{checking: } \frac{C1}{C2} = \frac{2(pi)(r)}{\frac{2(p)(r)}{(n)}} = n$$

$$C1 = n C2$$
$$\text{same distance} = \text{same distance}$$

And this implies that during the same time (t) necessary to orbit C1 , there would be many (ie., n) orbits of C2.

From d = vt and $\frac{d}{t} = v$, Since a circumference is a distance: C = vt and mathematically: $\frac{C}{t} = v$

From C1 = n C2   and dividing each side by t, we have:

$$\frac{C1}{t} = \frac{n C2}{t} = v$$: same velocity if no forces have act upon the object, and which would then change its speed or direction.   This is mathematically equal to:

$$V1 = V2$$
$$\text{same velocity} = \text{same velocity} = v$$: linear or distance related velocity

Though the linear velocity is the same, the angular velocity = w/t is not the same.   In the same amount of time, there will be (n) orbits associated with the object and C2, and its angular velocity will be (n) times greater:

$$w = \text{angular velocity} = \frac{\text{amount of rotation}}{\text{time}} = \frac{\text{orbits}}{t}$$

A COMPUTER PROGRAM FOR THE PERIMETER AND AREA OF AN ELLIPSE

```
/*-----------------------------------------------------------------------------------------------
EllipsePerimeterAndArea.c

        A program to input the major and minor axis lengths of an ellipse, and
        the perimeter of that ellipse will be calculated using Ramanujan's
        practical formula, that is perhaps easier than using the summation
        method (by hand and pen) with more mathematical divisions needed
        to make that method of calculation.  The area of the ellipse is also
        calculated.  Other values are also calculated.   You may wish to modify
        the program to also display the perimeter values when the value of (h)
        in the perimeter algorithm is not used, and then when used.

        (c) JPA, Dec 3, 2021
-----------------------------------------------------------------------------------------------*/
#include "stdio.h"     /* for printf() */
#include "math.h"     /* for sqrt() */
#include "stdlib.h"   /* for system() */

double ellipseperimeter(double majoraxis, double minoraxis);
/* A function to calculate the perimeter of an ellipse.  The algorithm
    used is a practical equation by Ramanujan from India.  */
/*-----------------------------------------------------------------------------------------------*/
void main (void)            /* A sample program, and to test the function. */
{
double majoraxis=0.0;
double minoraxis=0.0;
double  perimeter=0.0;  /* perimeter of the ellipse */
double rectangleperimeter=0.0;
double e=0.0;              /* eccentricity */
double ellipsearea=0.0;
double rectanglearea=0.0;
unsigned char c=0;

for(;;){

system("cls");

printf("\nELLIPSE PERIMETER AND AREA CALCULATER PROGRAM.");
printf("\n(c) JPA 2021, From the Mathization ebook of math And science.");
printf("\n\n");

printf("Enter a positive length of the major-axis: ");
fflush(stdin); scanf("%lf",&majoraxis);

printf("Enter a positive length of the minor-axis: ");
fflush(stdin); scanf("%lf",&minoraxis);

if(majoraxis >= minoraxis){
    e = sqrt ( 1 - ( (minoraxis * minoraxis) / (majoraxis * majoraxis) ));
    };
```

h = ((majoraxis - minoraxis) * (majoraxis - minoraxis)) / ((majoraxis + minoraxis) * (majoraxis + minoraxis));

h=h+(h * 0.00165231179593);   /* JPA's , (h) adjustment factor, gives an exact result on a=10, b=0.000000000000001,
                                     P = 40.0000000000000. This was found by trial and error adjustment when
                                     observing the results for the perimeter.   It is reasonable on other values and is
                                     optional and-or experimental. Perhaps use it only when (a/b) >= 2 for better results.
x=4.0 - (3.0 * h);
y = sqrt(x);
x = 1 + ((3*h) / (10 + y));

perimeter = (3.141592653589793) * (majoraxis + minoraxis) * x;

return perimeter;
};
/*---*/
Here is an example of the screen output or display when using the above program:

ELLIPSE PERIMETER AND AREA CALCULATER PROGRAM.
(c) JPA 2021, From the Mathization ebook of math and science.

Enter a positive length of the major-axis: 2
Enter a positive length of the minor-axis: 1

The ratio of (minoraxis / majoraxis) is: 0.5
The ratio of (majoraxis / minoraxis) is: 2

Eccentricity = 0.866025403784439
Focus = 1.73205080756888

Ellipse Perimeter = 9.688887059761
Corresponding Rectangular Perimeter = 6

Ellipse Area = 1.5707963267949
Corresponding Rectangle Area = 2

Ratio Of Ellipse Area To The Rectangle Area = 0.785398163397448
Ratio of Ellipse Perimeter To The Rectangle Perimeter = 1.61481450996017
Press A Key To Continue, or ESC to exit:

loosen some materials faster.   To help rationalize this strange valued scale, this scale can be somewhat thought of as the relative value or concentration of [OH-] in the substance. A pH of 0 means no concentration of [OH-] as expected for an strong acid substance which has plenty of protons. A pH of 14 means an incredibly high, maximum concentration or count of [OH-] as expect for a strong base substance that has plenty of electrons.

Technically, a pH value is calculated as the negative of the logarithm (ie., an exponent value) of the hydrogen ion concentration amount. This concentration amount is measured in gram-moles per liter. The actual volume or amount of an acid substance in a solution will also determine the number of hydrogen ions (ie., protons) available and how active (dissolve-ability) that acidic solution is, and so the pH value indicated on the scale can be thought of as an inverse relative value or indication of the dissolving-ability of that specific acidic substance when it attracts electrons to it and pulls apart (ie., "dissolves") molecules.   The pH scale can also be considered as an indication, ability or potential of a (aqueous, watery mixed) substance to attract protons and-or release electrons. Bases have a high pH.

pH = - log (hydrogen ion or proton concentration in the solution) = - log [H+] = -1 log [H+] = log [H+]^-1 = log [1 / H+]

A base has a pH greater than 7,   ex: calcium carbonate, baking soda (sodium bicarbonate, pH=8), bleach, etc. Plain or pure water is said to have a pH of 7, since it does have some equal (neutral) amount of both hydrogen ions (ie., protons) and hydroxyl ions, and hence having a low dissolving ability with the relatively few positive hydrogen ions available, and the pH is considered as 7 (ie., in the middle) on the pH scale.

Pure water is said to have a "self ionization" process, and even has a very low electron (-) conductivity. In a volume of 1 liter of water, there is a concentration of about 10^{-7} mols of hydrogen ions (H+):

pH water = - log 10^{-7} = (-1)(-7) log 10 = +7 (1) = +7   : the log exponent rule was utilized here for simplification

An acid has a pH of 6 or lower.   ex: vinegar and lemon juice. Vinegar, such as clear white "5%" acetic acid and the remainder being water. Common household food vinegar is sometimes called a weak acid and has a pH of about 2.4

```
   min. pH                              max. pH
     0  ,   pH < 7     ,     pH = 7,        pH > 7   ,   14
              acid            neutral          base
```

Pure hydrochloric acid (HCL, hydrogen and chlorine atoms combined into molecules, and creating a very strong or active acid) is rated and standardized as having the lowest pH of 0. Most soaps for cleaning have a high pH, and are therefore a base substance.

If the hydrogen ion or proton activity in a solution is less than that of pure water, the solution is a base solution and its pH is actually greater than 7 according to the pH formula.

Acid substances can release or "donate" hydrogen ions (H+, protons) into a water solution. Base substances can release or "donate" hydroxide ions (OH-) into a solution. Acid solutions are highly chemically "active", and can atomically, electrically attract and break apart atomic bonds and therefore dissolve other substances such as molecules of compounds and crystallized elements (such has metals) into smaller particles and-or atoms or ions.   Consider that a hydrogen ion (H+) is electrically positive and can attract loosely bound electrons (electrically negative, -e) of other substances, and this leads to molecules, and crystals of even a single elements having their atomic bonds broken apart, and that material or substance will then dissolve by and into in the acid solution.

Note that hydrogen atoms are also the smallest atoms and they can more easily "go through" other substances and materials than the atoms of any other type of element. This makes confining hydrogen gas a problem that needs to be resolved, such as for potential spaceflight use.

For some plants to grow well, they may need or favor a particular soil pH, climate, water availability, etc. Litmus paper "(pH) test strips" can measure, indicate and help determine the pH (via color of the red and blue dye-chemical coated

level of acid so as to aid digestion by reducing all types of possible germs, and for pre-processing or preparing the nutrients in the food for absorption into the body through the walls of the intestines.

Bases, such as sodium hydroxide (ie. "lye" = "caustic soda"), can be used to break down fat molecules in a process called saponification (non-formally: that which produces a sap-like textured substance like sap from a tree) and is used to make soaps and with glycerol as a byproduct. Soaps have a slippery feel to them. Soaps can mix into water, unlike insoluble oils. Base solutions are often called **alkaline** solutions and their pH is said to be of an alkaline value greater than 7.

Clean drinking water with a pH within the range of 6.0 to 8.5 is considered safe to drink. The pH of human blood is typically about 7.4, hence it is and-or should be a mild base solution. Blood is typically 9g of salt / liter = 9g/1000mL = 9g/1000g = 0.009 = 0.9% vital salt, and this is just slightly less than 1% salt. Salt is generally needed by the body for various electro-chemical reactions and processes, and for digestion which is the breakdown or dissolving of food so as to extract the nutrients (nutrition - minerals [elements] and vitamins [A, B, C D, etc]) to then be more available to and absorbed by the body.

Water is a mild solvent when the pH is less than 7. If you put a substance such as a hard candy, salt, sugar, or maybe a hard vitamin into a glass of water, it will slowly dissolve (break apart to smaller pieces), but for a given volume of water, there is a maximum amount for each specific substance that can be dissolved by and into that amount of water. Solubility is a measure of how well a certain substance or material can be dissolved (dissociates, "breaks off or apart from") in a specific substance. The units of solubility is grams of the substance per milliLiter = g/mL at room temperature. 1mL = 1cc. The maximum solubility of sugar at room temperature water is:

(180g sugar/100cc water) = (180g/100mL), and from this, dividing both the numerator and denominator by 100, we get: (1.8g sugar) / (1 cc water) = 1.8g sugar / 1 mL water = the solubility of table sugar in water

When a solution can no longer dissolve any more of a certain substance placed into it, that solution is called a **saturated** ("fully filled") solution, and this determines the (max.) solubility of that specific substance in a specific liquid. Solubility can also be expressed as the number of mols per liter (L). The solubility of salt in fresh, pure water is 35g/1kg = 35g/liter = 35g/1000mL = 3.5g/100 mL. 35g/1000g = 0.0035 = a relative value of 3.5% maximum in any volume of water. Sea or ocean water has this value and is therefore already salt saturated. Adding more salt into the solution will not dissolve it, but rather the salt will be a **precipitate** (ie., "precipitation", "fall out from", can not dissolve further, and is then said as being insoluble) like rain coming from or falling out of the moist saturated clouds. This is due to that there is not enough chemical energy left in the solution to dissolve any more of the substance such as salt. Perhaps this is how large salt deposits were made, and of which are sometimes mined for table salt and-or for industrial usage.

If you add plain water into an acidic or base solution, you will dilute and weaken the concentration and ability of that acid or base solution. The pH of that solution will then change so that it is less of an acid or base and the pH of it will approach a value of 7 if enough water is added into the solution.

As the temperature of an acidic solution increases, it (ie., the atoms, ions, molecules) and any substance in that acid becomes more energized by the heat and become more chemically and-or electrically active. The protons and electrons become more mobile and potent with the higher kinetic energy, and can then dissolve things easier and faster as if more or a stronger acid was added into the solution. The heat energy can cause electrons to have enough kinetic energy to become free electrons, and or weaken the bonds of molecules so as the acid is more effective to dissolving them. Adding salt (a compound of sodium and chloride molecules) into a water solution is not known to change the pH of that water solution. Water will dissolve the salt into sodium and chloride ions. Please read the following cautionary note:

Many chemicals are potentially dangerous in some way. Please take the time to acquire and use eye protection and follow the recipe and steps that scientists have previously developed so as to avoid injury and other problems. Scientists and chemists have much proper equipment available, knowledge and training so as to make and do experiments more safely and avoid injuries and-or damages. Please label and store chemicals and other dangerous

[This space for book edits.]

1 / Rt = 1 / [(1/R1) + (1/R2) + . . .]   : parallel resistors   : this was derived previously in this book

For just 2 resistors in parallel, the formula simplifies to:

Rt = __R1R2__   : Total resistance of 2 resistances in parallel. The net or total resistance is always lower than
     R1 + R2   the least valued resistance. Two resistances in parallel reduces the total resistance of the
                      circuit path because there is more conduction area or "room" for current to flow. There is a
                      similar formula for two capacitors in series or two inductors in parallel.

A resistor can be thought of as having the qualities of an insulator and conductor, and which results in a low conductivity material that has a lower number of "free", "loose" or "unbound" electrons to help conduct current than a typical metal conductor has. Electrons passing through a resistor also have a higher chance of colliding with atoms and will then loose kinetic energy as heat energy. This wasted energy is therefore a power loss (joules/time = watts). In a certain way, a resistor is a type of semi-conductor.   Given a resistance with a certain voltage across it and current through it, to effectively force the same amount of current though a higher valued resistance will take a higher amount of energy, and that is a higher electric potential energy, potential energy difference or electromotive force called voltage.

For the **capacitance of a capacitor (C)**:

A **capacitor** is a relatively simple electronic device that can store an amount or capacity of electric charges, and that is why it is called a capacitor. Basically, the amount of charge it can store is mainly determined by the size of its two plates of metal or metallic-like substance. **DANGER**: A capacitor may already have energy stored in it which needs to be released, transferred or discharged through a relatively low resistance (ex. a 1ohm to 5 ohm power resistor) across its terminals before handling it.

A capacitor can store charge (ie., electrons and positive ions) on two parallel and separated metal plates (such as two pieces of aluminum foil separated by a sheet of thin paper, which could then be rolled up), much like a rechargeable battery.   One plate will hold electrons, and the other plate will hold positive ions. The charges are held in position by the applied voltage across those plates and-or electrostatic forces when the voltage is removed. Since the charges are stored, they will have electric potential energy and have a voltage across the plates. Capacitors therefore can accumulate and store energy much like a battery stores energy.

Charge is essentially forced or driven onto the plates with a voltage (electric, electromotive force, emf, applied to the charges giving them kinetic energy to move or flow). The energy of the charge will be stored or maintained in position by an electric field potential (emf), and capacitance is a measure of its storage ability and will have units called Farads. The stored charge will then have a voltage (emf = electromotive force, potential ability) between them, and this internal voltage can then send those charges (ie. electrons) as an electric current through an electric circuit so as to balance those charges to 0.

As a capacitor is charging up (with current, electrons) due to a supplied voltage source, the internal emf across its two plates will be equal to that supply voltage and will essentially prevent any more charges from accumulating on its plates. This is sometimes called a "back emf" or "(opposite measured polarity) reverse voltage". As this reverse voltage increases, it effectively decreases the net voltage applied to the capacitor, and therefore the current or charge to the capacitor will decrease. During charging, the repelling force due to the charge already on the capacitor is preventing more charge from being stored.

The time it takes a capacitor to charge up to the supply voltage is determined by the resistance in series with it, and which will essentially reduce the charging current or slow the rate of charge.   As a "rule of thumb" or very close approximation, a capacitor is considered as charged or discharged in 5 "time constants", where 1 **time constant** is equal to: (Rohms series resistance between the charging voltage and the capacitor) (Cfarads) with units of seconds of time. Time of complete charge or discharge = **Ts = RoCf**

In about 1745 a crude type of capacitor with charge storage ability was discovered by Ewald von Kleist in Germany.

Some **r**elative permittivity (E**r**) values of some various dielectric materials in a capacitor
that have a temperature of the "room" or "comfortable living" temperature of 72°F = 20°C :

material   relative permittivity = er = (permittivity of material) / (permittivity of vacuum) = e / eo

vacuum   1
air   1.00054 = ~ 1 : at standard pressure of 14.7lbs/in^2, value is slightly greater than a vacuum.
mineral oil   ~2   Though an air capacitor does not generally store much charge and has a
wax paper   2.5   low capacitance, it offers fine fine tuning of oscillators and radio receivers.
vegetable oil 2.5
mylar   ~3
paper   ~3.5 av
nylon   4.5 av
mica   4.75 av   : good
glass   7 av.
diamond   8 av.
aluminum oxide ~9   : high
zinc oxide ~10   : high
tantalum-oxide   ~26   : high
ceramic   30 av   : very high
niobium-oxide 45 av   : very high
glycerin   57 av.   : very high
titanium dioxide   115 av   : extremely high   : sometimes used in the dielectric of "super capacitors".

The permittivity of water has a wide range and varies much with temperature, hence it is possible
to use this type of capacitor as a thermometer sensor so as to measure a temperature such as air,
and comparing its measured capacitance to that capacitors corresponding temperature scale that
was pre-made, calibrated and known.

The energy stored in a capacitor and-or equivalent amount of work (W) needed to charge it up to that amount is:

W = (Cf)(V^2) / 2   : with units of joules   : this is similar to the kinetic energy formula KE = mv^2 / 2 , but here
                                                        mass has been replaced with capacitance and velocity (v) has been
                                                        replaced with voltage (V).   Note: joules of work = joules of energy

The Cf or farad rating of a capacitor is an indication of that capacitors charge holding ability or capacity, and not
specifically the amount of charge on its plates at any moment. The energy stored within a capacitor also depends on
the voltage applied across that capacitor. The maximum safe voltage rating of a capacitor is the dielectric breakdown
rating of a capacitor and will therefore determine the maximum safe energy storage in that capacitor.

Some other formulas for the energy in a capacitor are:

Since: Capacitance = Cf = amount of charge held on the plates per volt (emf) across the plates = Q / V:

$$W = \frac{CV^2}{2} = \frac{(Q) V^2}{(V)\ 2} = \frac{QV}{2} \ ,\ \text{joules}$$

Since: V = Q/Cf ,

$$W = \frac{QV}{2} = \frac{Q\,(Q)}{2\,(Cf)} = \frac{Q^2}{2Cf} \ ,\ \text{joules}$$

air presure

air pressure

decreasing capacity or volume

increasing capacity or volume

flow →

container1   pipe   container2

air pressure + water pressure at the bottom

In the above figure, each tank (with any maximum volume, size, shape or "capacity") represents a capacitor with a certain maximum capacity or volume. In this figure, container1 was filled up and container2 was initially empty. The amount or the volume of water present in each tank represent the total amount of electric charge in each capacitor. Note that a greater volume of water does not necessarily mean an increase in water pressure unless the height (ie., the depth of it) of that amount of water is also increased. Also, any surface of water on Earth will have the standard air pressure of 14.7psi applied to it, but if this is applied to both tanks of fluid in this system, these pressure values negate each other, adding to 0 net pressure to the actual fluid flow.

Initially, and when charging, at the bottom of container1, there is an higher pressure than at the bottom of container2 since the weight of the water also applies pressure to the bottom of the tank where the pipe is. There is a negligible difference in the water pressures of each tank due to the air pressure above, and it will not be considered further in this analysis, and in fact the two air pressures are of the same value and effectively cancel each other out mathematically in terms of net pressure and-or difference in pressure. Due to the volume and weight of the water in each tank, there is a difference in water pressure or force at each end and opening of the pipe. Here, the pressures corresponds to voltage and the difference in pressure or net effective pressure corresponds to a net or effective voltage difference. If there is a path to flow, the higher pressure (ie., a force) water will flow to a lower pressure region such as in container 2.

The volume of the tank represents the maximum amount of water that it can hold before overflowing and-or not allowing any more equal or lower pressure water into it if it was sealed with the filling pipe going into it, and this is analogous to the maximum amount of charge that a capacitor can hold or store before being damaged.

rate is relatively high.

Ex. Below is a figure to verify this discussion. Two empty plastic bottles were hot-glued together along their height side and with some extra supporting pieces of wood.   The actual volume does not matter for this initial experiment, but it can be used later so as to have some actual (rather than relative, portion or percent) values if needed. The bottles had a volume of about 2 liters each. Between the plastic bottles is a plastic tube with nearly 1/8" inside diameter. This tube was from a used spray bottle.   The tube ends were place close to the center of the bottom of each bottle, and also hot-glued into position between the bottles and to prevent water leakage. A bottle cap is initially needed on the second bottle so as to help keep the water in the first bottle from entering into the second bottle.   [FIG 240]

Initially, as a construction test, the cap was removed from the second bottle and it would fill up so as to be at pressure equilibrium (ie., the same pressure) with the first bottle.   A line was drawn on each bottle at this height. With a ruler, also measure half this height and mark that line on each bottle.   Use a timer and-or calculate the number of seconds until equilibrium is reach.   Be patient and try to wait till they are exactly equal for it will help in the accuracy of the calculation. Due to a theorized natural flow, increment, decrement of the values in this type of system being analyzed, let's set this number of seconds till equilibrium is reached equal to 5T = five time constants, and after this amount of time and changes become negligible and are practically 0, and $e^{(-t/T)} = 1/e^{(t/T)} = 1/e^{(5T/1T)} = 1/e^5) = 0.00674$. Dividing this 5T value by 5 we will have the time value of one time constant (T) for this particular system of natural forces or pressure, and whose formula for analysis will include the natural number (e).

During this experiment, equilibrium took 11minutes = 660seconds = 5T.   Ts = 1Ts = (660s / 5) = 132s

For the volume (V2) of the second bottle being filled from the water (Vt=total volume) in the first bottle:

$$V2 = Vt \ e^{(-t/T)} = \frac{Vt}{e^{(t/T)}}$$

not be any effective "reverse" or "back" pressure from another container and therefore, the flow rate does not decline as much. With this system, the water analogy of the capacitor is not present, but the flow rate still decreases from the reduced water pressure of the reduced volume of water as time increases. The water volume, pressure and size of the hole determines the output flow rate, and therefore the total time until the container is empty. Since there is no "back" pressure reducing the flow, the volume will drain faster as compared to the system noted above.   For a reasonable, but technically incorrect, linear-like estimate for the remaining volume left in the container, you can take the total time and divide it by 7 [6.6 to 7 is close] so as to have the time of the first quarter (1/4 = 0.25 relative value = 25%) of the volume drained, then double [2 is ok, and 1.9 is a better value] that time to find the time of the next quarter (50% of the volume drained off) and so on.   For example: The first or upper (highest) quarter of the volume took 18s to drain out , the second quarter will drain off after about (2)(18s) = 36s. The third quarter will drain off at about (2)(36s) = 72s, and the remaining fourth quarter will drain out at about (2)(72s) = 144s.   If the first quarter of the total volume took N seconds to drain out, the estimated total time till that total volume has drained out is: total time = (7)(Ns). A basic formula for an estimation of the percent of the volume left after time past is: % of volume remaining = 1 / 6.6^((1.2) time / total time). The formula is more accurate for the higher percentages of the volume remaining, but it can be adjusted after more experimentation. Solving for total time: (total time) = (2.264483$\underline{6}$) (time) / Ln (1 / percent remaining).

Another similar formula is: % of volume remaining = 1 / 9^(time / total time) , or:
%Vremaining = 1 / e^((2.4) time / total time) ,   or: %Vremaining = 1 - ((1.56) time / total time) which is a good approximation for up to 65% of the total time.

With the knowledge presented in this overall discussion, we can apply it to the flow of charge or current in a charging or discharging capacitor: With the above water analogy, volume two, (V2) will fill up to half of the initial value of the total volume (Vt). For a capacitor, the voltage across it, (Vc), after "filling up" or charging is equal to the supply voltage (Vt).

When an empty (of charge and voltage) capacitor is first charging, the flow rate (ie., amperes) of current is high, but then decreases due to that the effective voltage or potential difference (like a difference in pressure) causing that flow is decreasing. This constantly decreasing flow rate will then effectively increase the time it will take to charge up that capacitor to a certain amount of charge and-or voltage, hence charge up to an energy storage value.

The formula for the decreasing flow or charge rate (ie., current) to the capacitor will be one that expresses a natural-like decay with respect to time, and will cease at some certain amount of time, and this time is determined by the capacitor's capacity and the maximum charge rate initially to it. A higher capacity or capacitance means that at a certain initial or maximum charge (ie., current) rate value, it will take longer to charge that capacitor to a certain voltage. The formula for the increasing voltage of a capacitor will be one that expresses a natural-like increase in voltage over time and (practically, at 5T) ceases at some certain amount of time when the current ceases to a value of 0A due to a lack of voltage or pressure difference, and the capacitor has charged it up to the supply voltage.

Since (e) =~ 2.71828 is a natural-like value, it is typically used to express natural events such as growths. (e) is used to express the natural-like gravitational forces on a freely hanging chain from two supports at the same height above the ground, and which shape is called a catenary curve as discussed in this book.

$Vc = e^{(-t/T)} (Vs) = \dfrac{Vs}{e^{(t/T)}}$: formula for the increasing voltage across a charging capacitor. Even though the voltage increases over time, the rate of this increase actually decreases over time (t). T is related to the total time or period it will take to charge and-or discharge that capacitor charged up to a specific amount of voltage. T is discussed further below. The negative exponent of (e) indicates there will be a natural decay over time.

$Vo = e^{(-t/T)} Vc$:formula for the decreasing voltage across a discharging capacitor

$Ic = e^{(-t/T)} It = \dfrac{It}{e^{(t/T)}} = \dfrac{Ve}{R}$:formula for the decreasing charging or discharging current of a capacitor
It or Imax is the maximum current flow which is determined by the series resistance (R) to and-or from that capacitor. The higher the resistance, the

and the AC frequency applied. The units of resistance, impedance, and reactance are **ohms**. For a capacitor, the higher the applied AC frequency signal, the lower its effective resistance or impedance to current. A capacitor is said to more easily pass or allow higher frequency ac signals, hence it also rejects or does not easily pass lower frequency signals. For an inductor, the higher the applied AC frequency signal, the higher its effective resistance or impedance to current.

The reactance of a capacitor is how it reacts to an applied AC signal and is essentially a measure of its (perhaps unwanted, like a resistance) ability to impede (ie., resist) an AC (alternating current) signal. The specific value of its reactance is determined by both the capacity of the capacitor and the frequency of the applied AC signal. For example, consider that besides for an initial pulse of current to charge up a an empty (no charge on its plates) capacitor up to the input voltage, a capacitor will then block any direct current flow which has a frequency of 0hz. It becomes like an open circuit or very high resistance to low frequency AC signals trying to pass through it electrostaticly. This certain function of a capacitor is often used to isolate an AC input and signal from a connected devices DC (ie., a battery, etc.) power supply which may cause problems for both the input device and-or the signal efficiency.   High frequency signals will pass through a capacitor with ease since it will have a low impedance to high frequency AC signals, and the capacitor will be practically like a low resistance piece of wire, hence like a short-circuit, or closed switch. Because of this frequency dependent value of reactance to ac signals, a capacitor can be used as a frequency dependent filter to block and-or allow desired and-or undesired frequencies. For example, unwanted high frequency signals can be "grounded" or sent to the ground connection and away from another circuit. Unwanted bass or low frequency signals can be greatly reduced and-or blocked from entering a circuit. Basically, reactance is how the device (a capacitor or inductor) reacts to an applied AC signal (here, in particular its frequency), and reduces that signal like a resistance would.

$X_c = 1 / [2(pi)(F_{hz})(C_f)]$: **reactance of a capaacitor** in ohms units

Note for ex.: 1pF = 1 (10^{-12})F ,   120pF = 120 (10^{-12})F =
120pF = 1.2 (10^2) (10^{-12})F = 1.2 (10^{-10})F

$C_f = 1 / [2(pi)(F_{hz})(X_c)]$

As the input frequency to a capacitor increases, the charging and discharging time needed to reach or "follow" that input AC frequency and voltage signal is less, and the capacitor itself essentially reacts less with that signal and behaves more like a lower resistance to the signal. Remember that time=/frequency, so if the frequency is higher, the less time for each cycle of its wave. A smaller capacitor has a lower RC time constant and can quickly charge and discharge. A capacitor can be thought of as passing higher frequency signals easier with less reactance or impedance, and reducing lower frequency signals due to more reactance or impedance.

In general, the smaller the capacitor, the higher the frequency of the ac signals it will easily pass, and signals with frequencies below that high frequency will have more reactance and will be reduced as if a resistor was in series with that signal.   To pass a low ac frequency signal with least impedance, a large capacitor (ie., a higher farad rating) should be used, and these types of capacitors are often found in audio equipment so as to pass low frequency ("bass") signals or audio. Large capacitors are often used to temporarily store energy (electrons and their voltage) so as to maintain a steady voltage value such as needed in a DC power supply to a circuit.

Using the above derived formula for C_f, the capacity or Farad value of a capacitor can be calculated after taking some measurements in a test circuit. This test circuit will consist of a voltage divider consisting of a known value series resistor (ex. r=1000 ohms) and the capacitor in question.   An AC signal at a test frequency (ex. 2000 hz) is applied across the voltage divider circuit and the voltages (V_c and V_r) are measured with a (AC) volt meter. $X_c = V_c / I_c$. Note that: $I_c = I_r = V_r / r$. These calculated values can then be placed into the C_f formula. Except for the formula, a similar method can be used to calculate the value of an inductor.   There are also inexpensive electronic meters available to measure resistors, capacitors and inductors. These meters are sometimes called LCR meters. Individual meters are also available for each electronic component mentioned, and some DMM (Digital Multi-Meters) can measure some of these components such as resistors and small valued capacitors.

Another way to find the value of a capacitance is through the RC time constant: $T_s = RC = R_o C_f$

An inductor converts or transforms the (kinetic) energy of the input supplied current (ie., electrons) into a magnetic (flux) field (B) having potential energy, whereas a capacitor converts the energy of the input supplied current into an electric field and potential energy that can even be stored for a long duration of time without any more input energy applied.

The inductance (L) value of an inductor is a measure of its magnetic field creation and energy storage ability. Inductance is a measure of the magnetic field created by the applied current current. Inductance is a measure of the ability to create or induce a magnetic field. A transformer is a common example of an inductor(s), and the two coils, which are generally electrically unconnected are said to have mutual inductance because they affect each other, initially via (varying in strength) magnetic fields. Michael Faraday is credited to this understanding.

Inductance = $L = B / I$ = (magnetic flux) / (current)   : inductance has units of Henry's (H)

B has units of webbers (Wb) or the number of magnetic flux (force) lines, and this is determined by the applied current . Current = I has units of amperes. The units of inductance are: webbers / amp = Henry's

Ex.: 1H = 1 Wb per Amp.

1 Tesla = 1 webber per square meter = 1 (Wb / m^2)   : magnetic field strength, concentration or intensity
                                                                          of the magnetic "flux lines".

 A coil is a good version example of a device with self-inductance When current passes through a coil, it will induce or create a magnetic field about it which will essentially "cut" (through) that coil and create an (opposing) current and ("back emf" voltage, reverse polarity, opposing to current flow) voltage in the opposite direction to that voltage applied, and the result is that the current to that inductor will be reduced since there is less of a difference in voltage potential (ie., emf, the voltage) between the source voltage and the inductor. The inductance equation can be modified to include each wire turn (N) of a coil, which essentially sums up or effectively concentrates, adds to, and effectively magnifies the magnetic flux created or induced into a smaller area instead of a long length of wire, and therefor, the inductance of that coil will have a larger rating:

  $L = N (B / I) = NB / I$: L = inductance i Henry's , N = number of turns, B = magnetic flux lines from one turn or loop,
                      I = current in amperes (A)

Because an inductor reduces an alternating signal, it behaves much like a DC resistance that reduces or limits current, however, it will be called the reactance (Xl) of that inductor (and signal) since the effective AC resistance of inductor depends on the applied frequency of the AC or "(voltage) changing signal. Reactance has units of Ohms.

Passing a current through a length of wire produces a magnetic field that is curved perpendicularly around the center of that wire. A (opposite in polarity) voltage (emf, electric field) and corresponding current can then be induced (ie., like a simple transformer) in another or second wire that is parallel to that initial current carrying wire. The second wire, though parallel to the first wire, is actually then perpendicular to the direction of the magnetic field (flux or lines) direction of the first wire.

With the previous understanding, given just one wire that is made into a loop or coil (ie., circular and spiral shape, like a spring, and usually coated or painted with a thin electrical insulation layer of paint) along its length. The (invisible) magnetic flux or field lines and the strength of the magnetism can "self induce" a voltage in the nearby (essentially parallel wire) loops or windings of the coil. This voltage will cause a current which in turn produces more magnetism. The magnetism or magnetic flux produced by each loop or turn of the wire in the coil is essentially summed together to produce a stronger magnetic field for a given amount of current. The resulting magnetic field of the entire coil will have

its own power and generated electricity. In 1912 electric, high torque electric motors began to be used to start automobiles.

A basic form of one of the Maxwell equations is that EMF (electromagnetic force, voltage) induced is directly related to the rate of change (ie., speed) of the magnetic flux lines (B) with respect to time:

voltage induced = - $\frac{dB}{dt}$: The faster a magnetic field cuts across a loop of wire, or vice-versa, the higher the voltage induced.   Ex. The faster an electric generator spins, the higher the voltage produced. It is as if a higher force is in action to cause it. The higher the voltage generated, the higher the current generated. dB / dt is the instantaneous change in magnetic flux with respect to an instantaneous(ie., momentary, brief, "quick", short, small, low) amount of time, hence the voltage is also an instantaneous value, and particularly with (rotating) electric power generators.

This induced voltage will then cause an induced (ie., generated) current to flow which will also create a magnetic field which actually opposes the original magnetic field applied that lead to its creation, and the voltage induced by that magnetic field is of opposite polarity to that which initially caused it. If the magnetic and-or flux field does not change, there will be no voltage induced in the other (output) coil of a transformer, hence transformers required AC electricity or pulsed electricity (ie., current) applied to the (input) coil for it to function as a (power) transformer and-or circuit isolation.

Inductance can be increased by using more side by side loops or turns of the wire so as to effectively increase the magnetic field strength, inductance and generated voltage.

A common transformer which has two, non-connected, independent coils, functions by including another concept called mutual inductance. It may also be called as magnetic inductance. These coils are said as being only "magnetically (flux, energy) coupled", and that their (changing) magnetic fields (due to a changing or AC input signal) will influence each coil.

To create a simple **electricity generator** that creates a voltage and induced current, a single loop of wire is made to pass perpendicularly through a magnetic field, or vice-versa, and a small voltage and current will be generated. If there are N turns or loops in a coil of wire, the inductance and induced voltage will increase by that same factor of N and when the magnetic field effectively "cuts" or travels through only one half of the generators coil which has those turns of wire in the same direction, and so as the induced or generated voltage polarity and current will all have the same direction, and so as to not negate, reduce or cancel-out each other as a "back emf" or voltage would. When the direction of relative movement of the magnet and coil changes to the opposite direction, the induced voltage changes polarity, and the induced charge or current changes direction, and this is the basis of AC electric power generation.

When a current is created or induced in a wire due to a magnetic field changing strength, the electron flow in the wire will cause the creation of a magnetic field about (ie., perpendicular to) that wire, and the polarity of this field will be opposite to that of the magnetic field which caused it. These opposite magnetic fields will then create a physical repulsion force applied to both the wire and magnet. In a motor that is to spin or rotate, this magnetic repulsion force is what causes the torque (ie., rotation force) which then rotates the rotor (ie., rotating) part of it.

To convert the mechanical-like input energy and-or force, to electrical energy, this force can be used to cause a rotation in both the turbine (ie., wind or water kinetic energy and-or force collector blades) and the electric generator which is mechanically connected via a shaft and-or gear(s) to the turbine. Remember, a non-changing magnetic filed just passing through a non-moving coil, or vice-versa, will not induce any change in voltage and-or current such as from being 0V and 0A. There would be no motion and-or kinetic energy available in this system so as to even be available to create, transfer to, and-or produce electrical energy. Gears can also be used to increase the speed of the rotation of the generator's rotor part , either to change torque (force) and-or its rotational speed (ie., "angular velocity") so as to have a maximum and safe energy output. Another method to alter the output electrical power after it is created is with a transformer so as to change the voltage and-or current available. Power = (Voltage)(Current) = VI watts = joules of energy / second. If the voltage

We see that inductance is directly related to the number of turns of the coil, and the cross sectional area of the coil of which for a given current there is a greater magnetic flux and-or field created because the length of the wire and-or coil is longer and the resulting total magnetic field of the coil will then be greater.

For the total inductance of **inductors in series**:

Inductors in series is essentially creating a longer inductor or "coil" simply having more turns of wire.

Lt = L1 + L2 + . . .   : this formula is similar to that for series resistors

For the total inductance of **parallel inductors**:

1 / Lt = 1 / [(1/L1) + (1/L2) + . . .]   : this formula is similar to that for parallel resistors

For the **reactance (impedance or resistance to ac signals) of an inductor**:

The reactance of an inductor (ie., inductive reactance) is essentially its ability to impede (ie., resist) an AC (alternating current) signal. The value of its reactance is also determined by the frequency of the applied AC signal to it. The impedance of an inductor is opposite in nature to that of a capacitor. For an inductor, its impedance increases as frequency increases, and at some point it will be like an open circuit or very high resistance to high frequency AC signals. Like capacitors, inductors can also be used as (frequency dependent) AC signal filters and in "tuned" (resonant frequency) oscillators.

Xl = (2)(pi)(Fhz)(Lh)   : **reactance of an inductor**, **measured in ohm units**
                                      Lh = Xl / [(2)(pi)(Fhz)]

The higher the input frequency, the higher the rate of change of the current going to it, and therefore, the higher the (ac) reactance induced because the rate of change of the (invisible) magnetic flux (lines) created will also be higher, and therefore the back-emf or reverse voltage induced is higher, essentially or effectively acting like a higher resistance impeding even more current flow. The equations for Xc and Xl are essentially, reciprocals of each other.

For a common example of an inductor, a typical audio speaker is an electro-mechanical transducer (ie., an energy or power transformer of sorts) that transforms or converts the input electrical signal energy to (vibrating, or pulsed) force and motion so as to apply pressure to the nearby air that acts as a sound conductor and transmits or radiates the air pressure or sound (a force due to compressed or decompressed, like a spring, air molecules). Our ears are essentially mechanical to electrical (nerve signals) transducers, much like how an electric microphone converts the received (sound, air) pressure or force into an electric signal. An audio speaker usually consists of an internal coil which is essentially an inductor and which is standardized to be 8 ohms of impedance when a pure (AC) sine wave signal (voltage, current and frequency) having a frequency of 1000hz is applied through it.

Some other common inductors are solenoids and relays which are essentially electromagnets.

When a capacitor (C) and inductor (L) are in parallel to each other, such as in a (LC) tuner or "(LC) tank circuit", and each has its own reactance (X) to the applied input frequency, and the resulting or effective total reactance (Xt) and reactance formula is nearly identical to that of two resistors being in parallel (R$_{ll}$) because reactance essentially limits current just like resistance does:

Xt = $\underline{\text{Xc Xl}}$: formula for two parallel reactances, such as in a LC tuning circuit.
        Xc + Xl   At resonant frequency (Fr), Xc = Xl , and Xt or Zt is the highest possible.
                                  (Xc + Xl) effectively becomes: (Xc + Xc) and-or = (Xl + Xl) = 2Xl = 2Xc and
                                  Xt becomes: Zt = Xt = Xc Xl / (Xc + Xl) = XcXc / 2Xc = (Xc)^2 / 2Xc =
                                              = Xc / 2 = Xl / 2

leads around. An AC voltage or power source is much like taking the battery out of a DC circuit and switching the direction or polarity of it in the circuit, and very fast - either several to millions of times a second. This would be correctly called pulsed AC since the voltages are at maximum and not starting from 0 as in a typical AC signal such as a sine waveform.

The (dynamic) impedances or effective resistances of the charging and discharging capacitor and inductor are opposite or 180 degrees out of phase with each other and they can therefore effectively negate and-or nullify each other if they have the same magnitude or absolute value.   These impedances are also both 90° out of phase with a resistor in series with it. It may be helpful to here to note a somewhat similar example that the common Sine and Cosine waves (and each instantaneous event or value) are very similar to each other but are not mathematical opposites or 180 out of phase with each other, but are 90 degrees out of phase with each other. Their "phase (or instance) shift" is said as being 90 degrees. Due to the above reasons, the total effective impedance to an AC input supply and-or signal is not simply
 R + Xl, or R + Xc, or R + Xl + Xc.. The total impedance is rather said to be a complex impedance that is composed of several parts and each being out of phase with the other two. The phase angle of the impedance due to a capacitance is -90 out of phase with that of the resistor, and this is said as being "leading" (already happened or earlier). The phase angle of the impedance due to an inductor is +90 out of phase with that of the resistor, and is this said as being "lagging" (or later in time). We can consider the value of the resistance (R) as a real value along the x-axis, and the other leg of this triangle would be the signed number sum of the inductive and capacitive reactances. If the inductive reactance is greater than the capacitive reactance, the circuit is said to be "inductive". If the capacitive reactance is greater than the inductive reactance, the circuit is said to be "capacitive".   The effective resistance or total impedance (Zt) seen by the input signal is: [FIG 241B]

Total Impedance Of A Series RLC Circuit
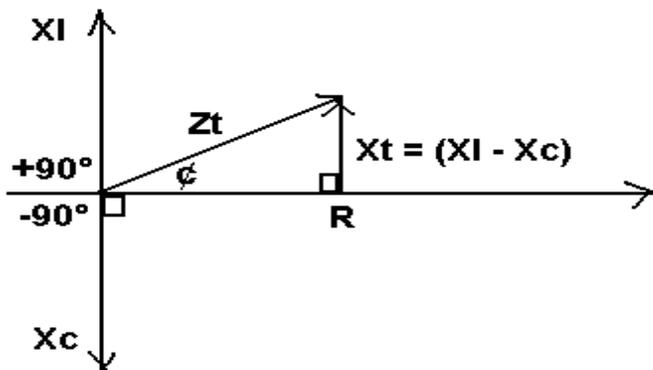
When Xl = Xc, the phase angle is 0° and the circuit looks as "resistive" to the applied AC input signal.

From the Pythagorean Theorem:

$$Zt = \sqrt{R^2 + Xt^2}$$

Ex. An antenna for a certain radio receiver might be affected by some stray, nearby capacitance, and the antenna is then said as being "capacitive", and is then not being as efficient as it could be. This capacitive reactance can effectively be canceled or negated by using an inductor of a certain value which has the same reactance with the same applied input frequency.

The concepts of Ohm's Law can be used for both DC and AC circuit analysis.

Voltage Divider Method To Find The Value Of An Inductor Or Capacitor

First, there are both capacitance and-or inductance meters available for purchasing, and as of the year (2022) they are widely available and relatively inexpensive. This note or discussion here has a forward reference to the general concept of the voltage divider that is shown after this one.

By having an AC voltage source and creating a voltage divider circuit with a resistor and either a capacitor or inductor in series with it, the value of that inductor or capacitor can be found from their reactance (ie., frequency related impedance or effective resistance) formulas:

From Ohms's Law: $I = Vr / R$ amps , we have: $R = Vr / I$ and $Xl = Vl / I$ and $Xc = Vc / I$

From: $Xl = 2$ (pi) Fhz Lh , ohms , we have: $Lh = Xl / [2$ (pi) Fhz $]$

From: $Xc = 1 / [2$ (pi) Fhz Cf$]$, ohms , we have: $Cf = 1 / [2$ (pi) Fhz Xc $]$

For an example AC, series circuit, the AC voltage source might be 10 volts, and having an arbitrary sinusoidal frequency of say 1000hz. The resistance (R) also helps reduce the current to a safe level, perhaps 50mA. Hence R would then be calculated at: $R = 10v/0.050A = 200ohms$. $Pr = (Ir)(R) = (0.50)^2 (200ohms) = 0.5Watts$, hence use a resistor that is rated to handle at least a half-watt (1/2W = 0.5W = 500mW) of power (ie., loss in it as heat energy). For better accuracy, you can experiment with the value of the input power supply's or AC signal frequency. The voltage drop across a capacitor or inductor is equal to its reactance times the current through it, and this is Ohm's Law. Since a certain frequency will cause an associated value of reactance in an inductor or capacitor, by using a stable or constant voltage source, it is also quite possible to also measure or have a good estimate of the input frequency by measuring the voltage across the inductor or capacitor. A calibrated scale or chart can be constructed beforehand and checked via experimenting with a variable range of input frequencies.

Besides an initial pulse of current, DC current will not flow through any branch of a circuit containing a capacitor.

For an AC applied signal, although an inductor or capacitor have a reactance to the applied AC voltage and-or signal and will reduce the circuit and-or branch current and available power, there is no theoretical power loss in them like a resistor would have. The inductor and capacitor will eventually give any energy storage back to the circuit and-or load resistance.

An inductor and capacitor in series can create a filter, and in particular, one that easily passes a certain frequency better than both the higher and lower frequencies. At this (frequency band and-or range) pass frequency where the impedance is effectively 0 ohms, a safety resistor can be used in series with the inductor and capacitor so as to keep the current in that wire or circuit branch to a safe value. A resistor and-or wire resistance will also reduce the available power output, and increase the charging and discharging times of both the capacitor and inductor.

The voltage drop (V) across a component in a DC or AC circuit is equal to the current (I) through that component times the effective impedance (Z) to the flow of that current. If two of these values are known, the third value can be calculated by using Ohms Law (I = V/R):

Vv = (Iamps) (Zohms)   : for two or more components are in series, the current is the same value through them

If the voltage is the same across each component in series, then their effective resistance is the same value.

Ex. Connected to an AC or DC supply voltage are two components (ex. resistor, capacitor, inductor) that are electrically connected in series with each other. It can be said that the (input) supply voltage is connected to and-or "across" those two components in series. Obviously, the sum of the voltages across each component cannot exceed the supply voltage, and in fact, it does equal the supply voltage. The actual voltage across each component will be less than the supply voltage:

Vsupply = Vcomponent1 + Vcomponent2 + . . .   : voltages across series connected components

The current through each component connected in series is obviously the same for each, and that is the total (It) or supply current. The value of this current is determined by the supply voltage (Vt) and the total impedance (Z) to the flow of the generated or induced current.

It = Vt / Rt   or= Vs / Zt   : It = Total current   and   Zt = Z1 + Z2 + . . . = tatal (current) impedance

I1 = I2 = . . . = It   Using Ohms Law, substitution, and considering just two components in series:

Vt =   V1   +   V2
Vt = (Z1 It)   +   (Z2 It)   Since (It) is a factor to each term, and factoring it out, we have:
Vt = It (Z1 + Z2) = It Zt   solving for (It):

It = _Vt_ = _V1_ = _V2_   : since the current (here, It) is the same value through series components
         Zt   Z1   Z2

Indicated by the above equation for (It), and that each component has the same amount of current through it, when given just two components in series, the ratio of voltage across each component to its impedance is the same for each component, and this same ratio value equals the same current (here, It). Mathematically:

 V1 = **_Z1_**   : **Equal ratio of voltages and impedances**. **Voltage Divider Equation** for two components
 V2   **Z2**   in series, and if they are resistors: R1=Z1, and   R2=Z2, and mathematically:

V1 = V2 (R1)
          (R2)

V2 = V1 (R2)
          (R1)

Vt = Vs = Vin = V1 + V2 = (It R1) + (It R2) =   It (R1 + R2) = (It)(Rt)

If the output voltage (Vout) is the voltage across R1, then R1=Rout:

Vout = V1 = V2 (R1 / R2)   = (It)(R1) = (Vin / (R1 + R2)) R1 = (Vin / Rt) R1 = (Vin / Rt) (Rout)

Vout = Vin _Rout   : A formula to calculate a series voltage across a component when the current
                Rt   is not known

needed to convert AC household voltages to (usually lower) DC voltages. The LM317 has a maximum input voltage of about 37Vdc Transformers are relatively expensive, and getting for them free or at a low cost from unwanted devices is ideal.

Because resistors, capacitors, and-or inductors have an impedance (Zohms) to AC current or signal, these components can be used to make (frequency dependent) AC voltage divider circuits.

Current can be divided when it goes through a parallel branch or circuit. A parallel branch is essentially a **current divider** circuit or path that is part of a larger circuit. The sum of current through each parallel branch (b) will be equal the total supply current to all those branches. According to Ohm's Law, the specific current through a certain parallel branch is determined by the total supply voltage (Vb) that is applied to all the branches in parallel, and the total impedance (Zb) of that certain branch being considered. If a branch has several components in series, then each will also have that same branch current through it, much like when a steam that varies in depth, wide and shallow, or narrow and deep, will have the same total flow of liquid past each location along it if no other energy is input into that system at some location.

Total current available to parallel circuit branches = (current in branch 1) + (current in branch 2) + . . .

It = Ib1 + Ib2 + . . .   For just two parallel branches containing just resistors:

$$It = \frac{Vb}{Rb1} + \frac{Vb}{Rb2}$$ Combining fractions:

We also know that:

It = Vt / Rt   = Vb / Rt   and for only two resistors in parallel, this becomes:

It = Vb / [(R1 R2) / (R1 + R2)] , and:

$$It = \frac{(R2)(Vb) + (R1)(Vb)}{(R1)(R2)} = \frac{Vb (R1 + R2)}{(R1)(R2)}$$

$$\frac{Ib1}{Ib2} = \frac{Vb / Rb1}{Vb / Rb2}$$

$$\frac{Ib1}{Ib2} = \frac{Rb2}{Rb1}$$: **Ratio of currents and resistance of parallel circuit branches.**
                This is an inverse type of ratio since current is inversely related to the resistance.
                The lower the resistance of a branch, the higher the current through it.

it is symbolic of one of the first diodes used and that is the point-contact diode, and where a pointed tip of a wire was placed on the surface material of the diode so as to find a good location where (diode) rectification happened. The banded indicated side or end of a diode is the negative or cathode end.

Each diode will have a maximum voltage, current and power rating. Each diode also has a maximum reverse or "inverse voltage" that can be applied before a diodes barrier electrically breaks down and starts to conduct current in the opposite direction.   This is a diodes breakdown voltage, and is usually much higher (maybe 25v to 1000v, check the data or information sheet or literature) than the "forward" or "turn-n" (conduction) voltage of the diode, such as 0.7v. These diode parameters, like any other circuit component or device, must be taken into consideration when making a circuit that has certain parameters (ie., values) and requirements.   Some galena, germanium, and Schottky diodes have a low forward voltage (Vf or Vd) of about 0.25V, but common silicon general purpose diodes, they have about a 0.6 to 0.7v drop across them needed to conduct well.

At lower applied voltage levels before a diodes barrier voltage (about 0.7v) is reached, a diode behaves like an open circuit or high resistance to the flow of electricity. An ideal diode would turn on and conduct electricity at exactly 0.7v and always behave like either an open or closed circuit depending on the applied voltage. In practice however, diodes are not ideal, and some forward current will flow depending on the applied voltage that is below the "turn on" voltage. With applied voltages of less than 0.7v, the diode will essentially behave like a voltage controlled resistor or resistance. As a thought experiment, it will change from say 100kohms at very low applied voltages to, about 1 ohm at 0.7v when the diodes barrier or junction voltage is reached and it will essentially "turn on" or "switch on" to conduct (forward, normal) current like a low value resistor, and ideally like a short circuit or switch.

The effective resistance of a diode is: (voltage across the diode) / (current through it) = Vd / Id

Diodes are often used in or for: radios, removing voltage transients ("spikes", quick and large voltage electric pulses, such as from static electricity, lightening and other (high voltage, high power) electronic noise such as from inductors and motors), voltage references, and in some voltage regulator circuits such as for a full wave AC rectifier (ie., wave inverter) or "bridge rectifier" in electronic power supplies or signal conditioning and processing so as to create a near DC signal. Many semiconductor devices, such as (BJT) transistors, essentially have diode junctions within them such as between the base and emitter (pn) semiconductor materials, and it is important to first have a basic understanding of diodes so as to help understand transistors and other "active (electronically controlled, operated) devices".   Sometimes diodes are used to protect electronic devices or components such as transistors (BJT's and FET's). For example, two reversed Zener diodes in series can prevent high voltage spikes from damaging the thin insulating gate of a FET by then conducting and shunting the energy to the ground and away from the FET transistor. The conduction voltage would be the (specific rated breakdown) voltage (Vz) of the Zener diode, plus the normal conduction voltage of a Zener diode (ie., Vd=0.7v). The "turn on" voltage of this arrangement would be: (Vz + Vd)

Because a radio wave (RF energy) received by an antenna is a very small amount, both in voltage and current, there is not enough voltage available to over come the conduction or forward voltage of a (0.7v) silicon diode, and rather a diode such a germanium with a forward voltage of about 0.25v is used. Still, even with a germanium diode, there will be signal power loss through the diode and this will reduce the audio output level.   Some people have even applied a low ("bias") voltage across (ie., parallel to) a diode so as to help it conduct, and so that the diode is not damaged by a high current,, a very high resistance (such as a fixed safety resistor, say 10k ohms in series with a variable 1Mohm variable resistor) is in series with the battery of say having 1.5v or possibly 3 small solar cells if there is enough sunlight.   It is also possible to amplify a weak radio signal before or after the diode by using a transistor, and in fact, a single transistor can be used as both a diode due to its base-emitter pn junction, and it can also be used to provide some signal amplification.   In this circuit, a BJT transistor and-or amplifier is biased by a supply voltage, but is biased to be barely in the ON or conduction state due to the very low DC current allowed into the base by using a fairly high resistance and-or impedance to the AC signal. The base-emitter diode junction will then have a high resistance also. The base bias resistance is say 100Kohms to 1M ohms when the voltage source is about 5v to 12v, and this also prevents "loading" or "dampening" of the oscillations in the LC "tank" or tuning circuit, and it helps ensure that a highest voltage of the received signal is applied to the input of the transistor. An FET transistor can be used also, and it was designed to already have a high input impedance. **As a reminder, when two resistances are in parallel, the effective or combined resistance is less than or equal to half**

Each transformer will have a maximum input and output power (voltage and current) rating. Even if the maximum output power drawn is not exceeded, if more current is drawn by the load device or circuit (ie., the resistance of the load circuit) than what that transformer is rated at, the coil can get hot and damaged. Fuses and-or circuit breakers with same current rating as that of the transformer can be used to protect the transformer and-or load circuit.

Electric power companies decided to transfer power to distant customers in the form of a very high voltage (thousands of volts) and low current. Placing a higher voltage on the wires effectively reduces power losses due to the high current resistance of very long transmission wires which would significantly reduce the available current and power (voltage x current) from the generator. This is because power available and-or power loss is related to the square of the current: P = I^2R.   Because this high voltage is impractical and dangerous to use for households (homes and small electrical devices), it must be reduced to a practical and safe level, and a voltage step-down transformer is usually used for that. By using a step-down transformer, household voltages supplied by the electric utility company and its "grid" (wires) system are reduced from a high voltage to be lower voltage - typically 240vac and 120vac so as to sufficient enough to be able to run household machinery and incandescent lamps.   Still, this voltage is even too high for many low power devices such as radios and TV's which require far less power, and the voltage to them is reduced by using another transformer - often placed directly inside the device, or an external (120vac to DC) plug in transformer. [FIG 244]
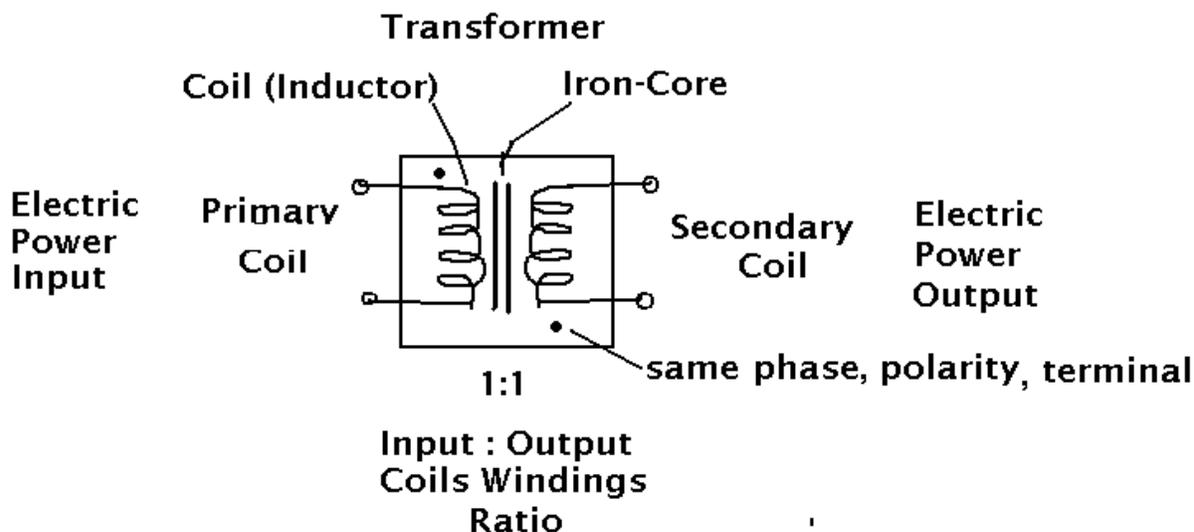
In the above drawing of a transformer, you can imagine a pulsating AC (ex., from a 120v, 60hz, household power outlet) input voltage and current signal applied to the primary coil and then creating a corresponding (ex., 60hz) pulsating magnetic field which will extend outwards around it and interact with the secondary coil and induce an (AC) voltage across it. If the transformer is a 120vac to 24vac voltage step-down transformer , that voltage ratio is 120vac / 24vac or= 5 / 1 or= 5 to 1, and should be indicated as 5:1 on the packaging and-or data sheet. The iron metal core helps conduct and transfer the magnetic field and its energy from one coil to the other, and it improves the efficiency of the transformer so that the power output from the transformer is nearly equal to the input power to that transformer.

A very useful, small and inexpensive circuit with a transistor (small NPN plastic type), a resistor (~1K ohms), a small homemade transformer using two (usually in parallel) strands of thin wire (about 30 gauge, such as from a junked speaker coil) wrapped (about 22 or slightly more turns) around a small (iron particles) ferrite ring or core, a 1.5V penlight battery, and a white light LED that you may wish to explore is popularly called a **Joule Thief** circuit. A more appropriate name and-or description would be a joule savior and-or energy retriever circuit so as most of the energy stored within a battery can be retrieved out from it. This circuit oscillates at several thousand hertz (cycles per second, here, pulsed DC) and boosts (ie., increases) a low voltage output of a 1.5v (or less) battery to about 3 or more volts necessary to make a LED give off a decent amount of light. Though voltages higher than about 3.5Vdc are not recommended for a white LED, but if those voltages are only briefly or temporarily applied as a pulse, there won't be as much of damage issue, and may

are the 2n1309 (pnp, gain of 90), 2n1307 (pnp, gain of 60). Because of the limited desire and availability of germanium transistors today (2022), they are currently several times higher in price than the mass produced silicon types. Germanium transistors generally have about half the power handling capability of similar sized silicon transistors. You can experiment if the value of Rb should be slightly higher or lower when a germanium transistor is used, and so as to conserve energy of the battery.

If you have an oscilloscope, you can visually observe the oscillations of this circuit when placing the oscilloscope's input or sensor probes across one or more of this circuits components such as the LED.

A voltage booster circuit was created and actually patented in about 1930, and this used a vacuum tube based amplifier and oscillator, and later, similar concept and patented versions by the early 1950's used the more fast and efficient - lower power loss, transistor. It should be of no surprise that if a transformer can boost an input oscillating (with a frequency, and changing in value) voltage, then an oscillator circuit could be used if the supply voltage is a DC supply voltage such as that from a portable battery, and so as to have a varying signal for the transformer to function. These concepts can be found in power inverters where a ("automobile") 12 volt or 24 volt, high load (ie., current) DC battery is converted to ("household") 120 Vac using an oscillator and transformer.   Though the voltage is boosted, say 10 times, the maximum current is relatively low, being reduced say 10 times by the transformer.   A 120W load on this type of inverter will require (from P=V/I, then: I = P/V = 120W/120V) = 1A of current.   A 1200W load will require 10A, and very thick wire must be used for safety.   Batteries save and-or create charge or current, and a high circuit current will deplete a battery relatively fast, and "deep cycle" batteries are preferred where they can be significantly depleted or drained without much negative effects, and are said to have a "longer life" or number of charging and discharging cycles.

The figure below shows the battery voltage of the above circuit when working with a load (here, a mainly the LED) with respect to time. [FIG 244B]

Wire diameter or "thickness" sizes can be expressed in mils (thousandths of an inch), or millimeters (mm, metric):

1 mil = 0.001 inch = a thousandth of an inch = 0.00254cm = 0.0254mm , mathematically, solving for 1mm:
1 mm = 39.37007874 mils = about 40 mils
1 cm = 393.7 mils =~ 394 mils
1/16 inch = (1/16)" = 0.0625 in = 62.5 mils
1 square-inch area = (1in)(1in) = 1in^2 = (1000mils)(1000mils) = (1000)(1000) mils^2 = 1,000,000 square mils
1 square mil = (1mil)^2 = (0.001 in)^2 = (0.001)^2 (in^2) = 0.000001 in^2 = a millionth of a square inch
         = (0.0254mm)^2 = (0.0254^2)(1 mm)^2 = 0.00064516 mm^2
1 circular mil (CM) area has a diameter of 1 mil = 0.0254mm, and its area is: A =(pi)(r^2)= (3.14159265)(0.0254mm/2)^2 =
         = 0.000506_1_mm^2 = (5.0671)(10^-4) mm^2

For the ratio of a square area to that of a circular area (Ac) having the same width or diameter, is to first consider or draw a circle in a square such that its diameter (d) of that circle equals the side (s) length of the square, say 1 unit long. Therefore s = d = 2r , where r=radius of the circle. The area of the circle will clearly be less than that of the surrounding square. The area of a circular mill (a thousandth of an inch) is less than that of a square mill, but by what factor?:

Ac / As = (pi) r^2 / (2r)^2 = (pi)r^2 / 4r^2 = (pi) / 4 = 3.14159265 / 4 = 0.7885_4_   (ie., about 79% or 80%)
We find that a circular mil is about (100% - 80%) = 20% less than that of the area of a square mil.

For a stranded wire containing several smaller strands of wire, the notation of its structure is usually expressed as:
     AWG #   "Number of strands / AWG gauge"

Some Common AWG and their corresponding approximate diameters, thickness or width:

Gauge Number Inches   Millimeters   (Solid copper wire is considered for the other parameters). 1in.= 2.54cm =25.4mm
1mm = 0.03937 in.
1mm =~ 0.04 in
~ four-hundredths of an inch

Gauge Number	Inches	Millimeters		
#0	0.325	8.26		
#2	0.258	6.55		
#4	0.2	5.2		
#6	0.162	4.11	=~	4mm
#8	0.1285	3.26	=~	(1/8)inch = 0.125in
#10	0.102	2.6	=~	(1/10)inch = 0.1 , resistance = 1ohm/1000ft
#12	0.081	2.05	=~	2mm
#14	0.064	1.63	=~	(1/16)inch = 0.0625in
#16	0.051	1.29		
#18	0.0403	1.024	=~	1mm ,
#20	0.0311	0.789	=~	(1/32)inch = 0.03125in , resistance = 10ohm/1000ft = 1ohm/100ft , A =~ 1cir.mil
#22	0.0253	0.6436		
#24	0.0201	0.510	=~	0.5mm = half a millimeter
#26	0.01594	0.4049	=~	(1/64)inch = 0.015625in
#28	0.0126	0.321		
#30	0.01	0.254	=	(1/100)in =~ 0.25mm , resistance = 100ohm/1000ft = 1ohm/10ft
#32	0.008	0.202		
#34	0.0063	0.160		
#36	0.005	0.127		
#38	0.004	0.1		
#40	0.0032	0.081		:resistance = 1000ohms/1000ft = 1ohm / 1ft
#50	0.001	0.0254	=	(1/1000)in = 1mil

As you can see in the above data, it is easier for most wire installers and users to use simple gauge numbers than trying to remember the very small differences between the diameters of each gauge of wire. There is a pattern that can be had from this data, and a basic formula was derived out from it so as to fit that data. The following formula is based on the

4   A safe recommended, maximum value is 100A, but the calculation below gives 120 A
6   A safe recommended, maximum value is 60A, but the calculation below gives 75.6 A
8   47.6 A = ~ 50A
10   30 A   : needed for some high current household devices and-or some 12v vehicle systems
12   18.8 A.   : the modern, common wire gauge required for many household (homes) and its electrical devices
14   11.8 A
16   7.4 A
18   4.6 A
20   2.96 A =~ 3A
22   1.87A =~ 2A   :this and AWG 24 are common ("hookup") wire gauges for many homemade electronic projects
24   1.18 A =~ 1A
26   0.74 A =~ 0.75A = 750mA
28   0.46 A =~ 0.5A
30   0.294 A =~0.3A   : this is a common, fine wire gauge for making homemade coils such as for a voltage booster
32   0.186 A

The formula used for the above data is based on AWG 10 of solid copper wire with a ampacity rating of 30A, and the fact that for every 3 gauge increase, the cross-sectional area and conductance is halved (1/2 = 0.5):

ampacity = (30A) $0.5^{((Gauge\# - 10) / 3)}$: **A basic ampacity formula for AWG solid copper wires**

This book does not have a table of powers of 0.5^x, but this can be used instead if needed:

From: ln 0.5 = -0.693147181 , we have: $e^{(-0.693147181)}$ = 5 , and raising each side to the (x) power, we have:

0.5^x = $e^{(-0.693147181)}$**^**x = $e^{(-0.693147181\ x)}$ = $10^{(-0.301029996\ x)}$: a form of: $n^x = (e^a)^x = e^{ax}$

ampacity = (30A) $0.25^{((Gauge\# - 10) / 6)}$: **Alternate basic ampacity formula for AWG solid copper wires**

          and : 0.25^x = $e^{(-1.386294361)}$^x = $e^{(-1.386294361x)}$ = $10^{(-0.602059991)}$^x

Because a wire has a maximum current rating, we can find the maximum voltage that can be applied to that wire:

Vmax = (Ampacity Rating) (Resistance of the wire)   : has the form of: Vmax = Imax R   or= V = I R

Because a wire has a maximum safe current or ampacity, and a voltage (Vmax) rating, there will be a maximum safe power loss in that wire = Pmax. The formula is based on: Power = Voltage x Current = V I = I^2 R = V^2 / R

Pmax = V I = (Vmax) (Ampacity Rating)   or:

Pmax = I^2 R = (Ampacity Rating)^2 (Resistance of the wire)   or:

Pmax = V^2 / R = (Vmax)^2 / (Resistance of the wire)

For increased safety and-or reduction of power losses in a wire, use a wire that has a greater current ability than what is actually needed for the total amount of amps needed and-or used for all the devices in series that are powered by the same power supply.   It is possible to have other electrical devices connected in parallel (ie., another current branch) to the battery, and of which thinner wires can be used if they are rated for the current needed. Be sure to consider if the total current drawn from the battery from all the powered devices is below the maximum amount that the battery is rated for. For safety, fuses and-or circuit breakers should be used on each current branch from the battery. To reduce power losses in a wire due to increased length and its increased resistance and voltage drop (ie., energy loss) across it, use a thicker gauge of wire.

can "break down", where the atoms are "ionized" (separated into atomic parts) and can then conduct charges (electricity, current) fairly well, and therefore may cause arcing and-or (a low resistance path) "shorting out" into and through nearby objects (particularly metal, trees and people) and into the ground. This amount of power is very dangerous and can pass through and cause serious life-threatening injury to people and-or fires.

POWER

Energy is the ability to do work such as applying a force to an object for a certain amount of time. Force is the application of energy, and then all that energy can be transferred to that object if there is no energy losses during the transfer of that energy. This transferred and gained energy in the object is usually transformed or converted to kinetic or motional energy of the object and giving it a faster motion, speed or velocity. The basic unit of energy is the joule (J). Power is a (average) measure of how much energy is available or used (ie.,transferred) during a certain amount of time or time period.

Watts are the units of power.   Power = (energy / time) watts = (joules / time) watts.

1 Watt of power = 1 Joule of energy used per or during 1 second of time = 1 Joule-second = 1W

1 W = 1 J / s .   The watt is named after James Watt who invented a newer, more efficient and more powerful
            steam engine for boats and other machinery.

For an example of electrical power: A 1 ohm resistor having a total (electric, electron) current flow or conduction of 1 Ampere = 1A constantly through it for 1 second, will gain a total amount of energy of 1watt, mostly in the form of (often wasted) heat energy. A formula is shown below. The source that supplied that energy will have correspondingly used and-or lost 1 watt of energy or power. Usually for resistors, the gain in energy is seen as a gain in heat or temperature, and the power it gained is usually called a power loss or power drop in or through it because its no longer available to the rest of the electrical circuit. Power losses should be avoided since it is usually wasted energy, time, money and resources.

For electricity and circuits: P = V I = (IR)I = I^2R = V (V/R) = V^2 / R   : with units of watts (W)

From Ohm's Law of: current flow = voltage (ie., emf) applied / resistance   = I = V / R = voltage / resistance.

A voltage (emf, electromotive, electrical or electronic force due to electrical charges attracting or repelling) of 1 volt applied to a circuit having 1 ohm of resistance will induce or create 1ampere of current to flow through that circuit.

Mathematically, from Ohm's Law:

V = I R   : Voltage can be thought of as the force or pressure to the available current and giving it kinetic energy.
                          Voltage is also called the electromotive force = "emf", and voltage is also called the potential
                          difference between two points since it will take a difference in this force or pressure so as to
                          have any current to flow, and at a value due to that net voltage value of the difference. For example,
                          If one point in a circuit is measured to be 302v and another point is at 300v, the potential difference
                          between those two points is (302v - 300v) is only 2 volts, and this 2 volts may be across a resistor,
                          and the current through that resistor is determined by that voltage, energy or (emf) difference, loss or
                          drop of and across it: I resistor = Ir = Vr / R ohms .
.
1V = (1A)(1ohm)   The greater the current (I=V / R), the more electrons (ie., matter, mass) with kinetic energy, and the
                          greater the total energy being transferred.   Power = (voltage)(current) = VI

When 1 joule (J) of energy is applied to 1 coulomb (c) of charge (Q, such as electrons), that charge will gain 1 joule of energy, and the energy density and-or electrical potential (ie., voltage, emf) ability of that stored (in electrons, as motion) energy is called 1 volt:

the nucleus of the atom are easier to break free because the electric force (ie., Coulomb force) that holds them in that orbit position (ie.,radius, distance) is less. It will take less applied energy or force to break them free from their atom. This amount of energy is sometimes called that electron's ionization energy or ionization potential. Dmitri Mendeleev who is generally credited to making the first practical periodic table of elements discovered the concept of ionization energy. When an atom has a larger number of protons, its nucleus is larger and it also has more electrons and with some at a farther distances from the nucleus, and the ionization energy required to free more distant electrons is then even less. A hydrogen atom which has only 1 electron relatively close to its nucleus will require a relatively high amount of electron volts to break that electron free from its close orbit with a high electric attracting force.

For hydrogen, the ionization energy is often noted at about: 13.6 eV   or =   1312 kJ / mol of hydrogen atoms.

An electron can also go to a higher orbit or "(electron) shell" after gaining a certain amount of energy that is less than the ionization energy for it to break free from the orbit of that atoms nucleus, and if this, or possibly another, electron falls back to a lower orbit, its gained energy is then released as some form of electromagnetic radiation (light, xrays, etc) energy.

The minimal ionization energy required to split a water molecule into its hydrogen and oxygen gas elemental components of each water molecule, is said to be about 1.23V. This electro-chemical process is usually called "**electrolysis**". The hotter the water, it and its electrons have more gained (thermal and kinetic) energy and the ionization energy will be less. Using a voltage higher than 1.23V is often said to be unnecessary and a waste of power, particularly with current passing through the water having a small amount of an electrolyte substance to allow the movement or conduction of current (ions and electrons, in opposite directions) between the two electrodes (+ pos   ,and - neg). The higher the temperature of the water solution is also an indication of wasted energy in the form of thermal energy, and is usually caused by applying too much power (voltage and-or current) and-or that the effective resistance of the water solution is high and power is being wasted through that resistance.

Extra, from: 1 eV = 1.602 (10^-19) J   , solving for 1J by dividing both sides by 1.602 (10^-19), we have:

$\qquad$ 1J =~ 6.2415 (10^18) eV   : = 1J = (1Qc)(1V) = (1 coulomb of electrons)(1 volt) electron volts ,

As a reminder and comparison to the above values, this was stated previously in this book:

$\qquad$ 1C of charged particles or electrons = (6.24)(10^18) charges or electrons,
$\qquad$ and if each side is divided by the factor of (6.24)(10^18), we find the
$\qquad$ charge (Qc) of 1 single single electron to be: 1.602564 (10^-19) C

$\qquad$ From: V = J / Q   , we have:
$\qquad\qquad$ J = Qc V   , and if the amount of charge is 1 electron, and V=1 volt:

$\qquad\qquad$ (1.602564)(10^-19)C (1V) J = (charge of an electron electron)(1V) J = 1ev

Below is an example circuit so as to have **electric power supply source** for electronic devices, circuits and experiments.

Adjustable Voltage Regulator Power Supply Example Circuit   This one uses the LM317IC . [FIG 245]

Vo = 1.25vcd to about 37vdc , The max. output voltage depends on the input voltage. This voltage is adjustable via a variable resistor (R2), but can be set or fixed at a desired voltage using a fixed resistor for R2.

Io = 1.5Amax. Higher output currents will be sensed by LM317 IC and it will stop the flow of current.

Transformer and Diodes =   Common AC household power can also be used if it is converted to a form of DC with a filter circuit. Usually this circuit consists of a transformer, and four, 2Amp power diodes (or a premade full-"bridge" rectifier) so as to make a full ac or sine wave rectification (to having one direction, hence [unregulated] dc) and eventually have a fairly steady [ie., regulated] dc voltage using a large capacitor. The output voltage of the transformer should be in the range of about 3vac to 30vac that the IC can regulate, and be able to deliver the amount of current needed (up to 1.5A max.). The transformer should be rated at about 2A or higher for safety if 1.5A will be needed. The maximum output voltage of this regulator circuit also depends on the maximum available voltage of the transformer, or DC input voltage such as from a battery. If possible, the circuit can be constructed to allow both house power and also have terminals for a DC input voltage supply.   If the voltage circuit is DC powered only, then a transformer and diode(s) are not needed.

IC Pins of the LM317 = External electrical contacts: Looking at the actual front or "top view" side of the 317IC chip, from left to right are the internal-external terminals or pins: 1=adjust , 2=output on center pin   , 3=input.
The pins shown in the figure or schematic are not numbered, but are just indicated as for what it is for, and to make the rest of the circuit drawing easier to understand.   Look at the IC packaging for the pin numbers and-or corresponding function of it. **These pins and or the IC chip itself can be thought of as equivalent to that of a high current or power transistor** with its collector pin being the input pin, the emitter pin being the output pin, and the base pin being the adjust pin for the amount of current passing through the transistor from the emitter to the collector. This concept is essentially varies the collector to emitter (CE or C-E) resistance of a transistor, and therefore it can vary the voltage drop across it and any other output resistors which for this IC are generally not in the output current circuit, but are relatively high in value and are rather used to provide a current reference, control or bias (electronic circuit settings) amount to the M317 IC so as to automatically adjust its output voltage to be a stable or constant desired value.

R1 = a value of 220 or 240 ohms ohms is typical

R2 = User selectable for a fixed output voltage such as by using using 1 fixed resistor, or a selectable resistor by using a user selectable or operated rotary switch (ex. single pole to 6 throw = 1 input pole or terminal, to 6 possible output poles or terminals), or use a 5k ohm linear variable resistor so as to have a variable output voltage. Here, the center terminal (pin 2) of the variable resistor should be connected to pin 1 or pin 3 of the variable resistor in such a manner that the output voltage increases as the variable resistor (potentiometer = "pot") is turned clockwise. This output voltage setting of the power supply can be indicated at the adjustment knob to this variable resistor, and can be done with simple pen lines on paper and set affixed with tape or glue during the calibration of those lines. An analog voltmeter is also typical on power supplies so as to quickly set a good rough value or approximation of a voltage. There are also inexpensive,small LED voltmeters that will normally display 3v or higher voltages.

The higher R2 is, the lower the "(IC chip) sensing current" and the lower the output voltage of the LM317 IC circuit. Though not often needed, an optional capacitor, say 10uF, across R2 can help improve the output voltage stability by helping to maintain a steady "sensing current" and therefore helping to reduce any output ripple voltage due to input and-or load impedance and current fluctuations.

C1: 0.1uF, usually for AC to DC conversion, and is optional if the input is from a DC source which may briefly vary in output power. Also use another capacitor of higher value, perhaps 1000uF or higher, if the input is from an AC source, such as from the output of a transformer and rectifier diodes, so as to have a more constant valued, consistent, "steady" or "smoother" DC voltage without much, or possibly no variation or "ripple" (rises and-or declines) in the value of the input and output voltage of the IC. Make sure the voltage rating of this capacitor is higher by at least 1.5 times the maximum voltage across it. Another optional high capacity capacitor (ex. 1000uF) is often used at the voltage output of the IC, but a smaller 10uF capacitor is typical and optional.

[FIG 246]

MAXIMUM POWER TRANSFER IS 50% FOR A RESITOR LOAD

Load in series with
its input power

Load in parallel with
its input power
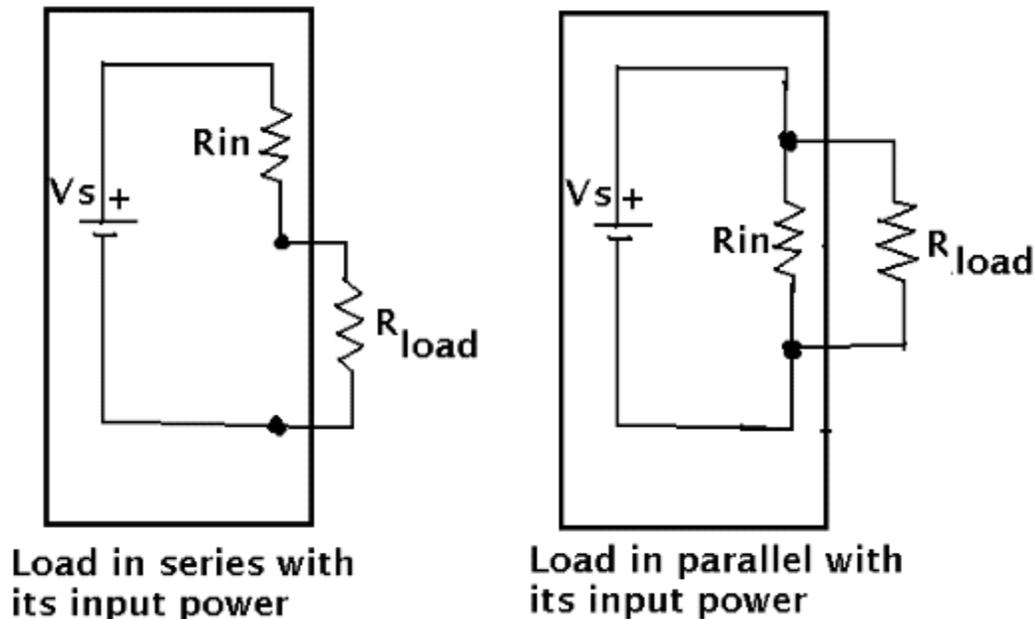
Given a power source or supply as indicated in the box in above drawing, the **maximum power that can be transferred to a resistive load** is 50% of the supply power. This may bring to mind the concept of an equal and opposite force. In the above figure, Rload is the load or external device being powered by the power supply. This also implies that 50% of the supply power to the load is lost and-or not available to the load. Rin represents all the internal resistance of the circuit of the source of energy. So as to have the maximum power condition to the load, the load impedance is the same (ie., "matches", "impedance matching") as the source impedance. Due to this power transfer concept, given a certain value load or impedance, a circuit is often designed with an internal or source impedance equal to this load impedance value so as to give or transfer maximum power to it.

In the left figure, the total internal resistance (Rin) is in series with the voltage source and Rload. This is essentially a voltage divider circuit. In the right figure, the total internal resistance is in parallel to the voltage source and internal resistance. This is essentially a current divider circuit.

In the above figure, the impedances shown are represented as resistances, but it could actually be composed of various resistors, capacitors and inductors which have a net impedance to the flow of AC signals (voltage, current, and frequency). Capacitors and inductors have a (AC, frequency dependent) reactive impedance which limit current much like a resistance would, but the electrical formula for maximum power transfer is more involved than that for resistors, and is beyond the scope of this book. Note that even a power source has a small internal (series) resistance which may be about only 1 ohm or less, but it could be significant for some circuits, especially if a battery's voltage is getting lower and its internal resistance is increasing and the available output power from that battery power source is greatly reduced by a growing internal resistance.

At maximum power transfer to the load:

    Pload = 50% Psource   and here:

than when it was charged to 1 volt.

Here in the image below is a fairly large capacity (300uF) non-polarized (NP, having no specific + or - terminals) capacitor that is typical of being used in audio circuits, power supplies, etc. and where relatively high amounts of current and-or power are involved. The capacitor is indicated as being made by the Culver company. The size of this capacitor in the image is about 4cm long, by 2cm wide or in diameter. It is rated at 300uF, with a tolerance of +,- 10%, and having a maximum voltage applied rating of 100Vdc. Due to the tolerance rating, the actual capacitance may be between 270uF and 330uF:

300 uF - (10%) of (300 uf) = 300 uF - (10%)(300 uF) = 300 uF - 30 uF = 270 uF   and:
300 uF + (10%) of (300 uf) = 300 uF + (10%)(300 uF) = 300uF + 30 uF = 330 uF   : a total range of:
330uF - 270uF) = 60uF

[FIG 246B]
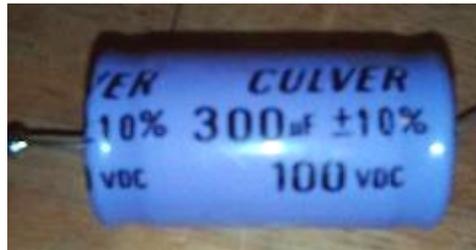
The "**energy density**" (EDc, or energy storage capacity) of a capacitor (c) is the amount of energy that can be stored in the volume or weight of the capacitor. This value is sometimes used for comparing various energy storage devices.   A high EDc is usually desired, and at a low cost.

EDc = energy per unit of weight = Jc / weight

As an indication of the amount of energy or power in a battery, a battery is given an **amp-hour (Ah) rating,** which is how much current (A) can be safely drawn per second for 1 hour.   A battery rated at 10Ah, can supply 10 amps (=10 Coulombs/s) of current for 1 hour before its considered drained or impractical to use for the rated voltage.   That same battery can also have less of a current drawn over a longer period of time such as for example. 5 amps for 2 hours is equal to the same amount of energy as 10 amps for 1 hour = 10Ah.   10Ah = (10amps)(1hour) = (5amps)(2hours). = (5)(2)(amp)(hours) = 10Ah. There will be some energy left in the battery, but its voltage will be less than its rated voltage. Deep cycle (charge and discharge ability) batteries have a relatively high amp-hour rating. We see that an amp-hour rating is really an indication of the amount of current that can be drawn and the total expected time of that. Note that since a battery has a small internal resistance, and if more current is drawn, the larger the voltage drop across its internal resistance, and the lower the available output voltage.   Batteries are often tested using an expected load of current such as for the headlights (having a resistance) of an automobile so as to determine the actual available output voltage and status of that battery.

The energy in a battery may also be rated in terms of milli-amp hours (mAh). 1000mAh = (1000)(10^-3)A = 1.0Ah

Ex. 500mAh = (500)(10^-3)A = 0.5A for 1 hour

Since 1A = 1 coulomb of charge per 1second = 1Qc/1s
1Ah = 1coulomb of charge per 1second, for 1 hour = 1 amp for 1 hour   ,   since 1 hr = 3600 seconds:
(1coulomb / 1s) x 3600 seconds = 3600 coulombs of charge in 1 hour

Since J = (Qc)(V)   , and if this specific battery is rated at 1.2V :

J = (3600coulombs)(1.2v)

(cost / 1000w) = (cost / 1Kw),
Since 100w is only a tenth of 1000w, its cost is also a tenth:
cost per 100Wh = ((cost per 1KWh) / 10) ,
and for the above example: (13.31 cents) / 10 = 1.331 cents.
To run a 100W appliance for 1 hour will cost 1.331 cents

The wire to this device or appliance must be rated to safely handle a current of 0.833A or more. It must have an ampacity (or maximum continuous current rating) rating of this value or greater.

The effective current resistance or impedance of this 100W device or appliance is:

Ro = Vv / Ia = 120v / 0.833A = 144.1 ohms

Another useful equation is:

Total Energy = E joules = (Power)(time) = ($\frac{J}{s}$) (s) = (V I) (s) joules   or=   VAs

Ex. A 12v , 50 aH rated battery can supply a rated and safe maximum power of:

P = (V)(I) = (12v)(50A) = 600W of power , and since this is rated for 1 hour, hence 600 wH = 600 watt-hours of total energy

Pin = Pout
wH in = wH out   : note that the number of watts and hours need not be the same, just as long as their product is the same. Considering a 300w device being powered by a 120Vac inverter:

First: From P=VI , I = P/V = 300w/120v = 2.5A

From wH = (watts)(hours) , hours = time = wH / watts = 600wH / 300w = 2h , max.
For the battery and its current out, another formula is: max. time = aH / A
And:

energy in = energy out   : Joules
600 wH in = 600 wH out
(12v)(50A)(1H) = (120v)(2.5A)(2H)   , note also that at 120v, the max safe, available current is: 600w/120v = 5A.
The voltage increased by 10, and the max. amperage decreased by 10.

600 watts for 1 hour = 300 watts for 2 hours   : the lower usage rate (energy/time) of the available energy, the longer the available time of usage of the available energy.

Ex. For a 48v , 50aH dc battery system, such as by connecting 4, 12V batteries in series connection:

For this battery, the maximum rated amount of power usage is: P = VI = (48v)(50A) = 2000w for 1 hour , and this corresponds to 2000wH of energy , Joules

P in = P out   considering the maximum rated value for this battery and-or system:
2000w = 2000w
(V)(I) = (V)(I)
(48v)(50A) = (120v)(16.67A)   : notice the high amperage from the battery into the 120Vac power inverter, and this requires very thick and-or short cables, typically 3 feet. Copper pipes flattened at each end with drilled holes can be used. Note that the voltage was increased by 4, and the current decreased by 4, still, to get that 16.67A

or device being powered. For this, you can connect and turn on the device to be powered and then measure the battery voltage. An alternative to using the specific devices to be actually used and powered, is to use a resistance such as from a resistor of perhaps in the range of 5 ohms to 10 ohms. An alternative load for a car would be to temporarily turn on its headlights or some other lights. When a battery is under load, its internal resistance can then be accounted for by being in the actual circuit, and this resistance will have a voltage drop or energy loss across and-or through it, and will result in the actual available and lower voltage and measured value or "reading". A battery may have a measured voltage of 12 volts without a load,and 10 volts with a load being powered, and this may be insufficient for it to function properly. An alternative to this problem is to use a voltage regulator designed for a device that was designed for and requires a lower input voltage, say 9v.

Resistance   ,   Set the meter on a high resistance scale and-or range. Do not place the terminals across an element having current through it, hence also having a voltage reading across it. This could damage the meter. Place the terminals across the (unpowered) resistance. Reduce the scale and-or range so as to find the resistance scale and-or range having a good precision and resistance display. Be wary of any other possible resistance in parallel and-or series with the resistance you are trying to measure, for it will then give an unintended measurement reading or displayed value on the meter. In some instances, you may need to unpower the circuit and cut or remove one of the two resistor leads so as to measure its resistance value.   Resistors are usually color coded so as to have color bands on them to indicate its rated value of resistance: Black (0), Brown (1), Red (2), Orange (3), Yellow (4), Green (5), Blue (6), Violet (7), Grey (8), White (9),   The third band is the power of 10 multiplier to the previous two bands. The fourth band is the "tolerance" code, but it is actually the possible, percent difference in error from the rated color coded value and actual true or measured value: Brown (+,- 1%) , Red (+,- 2%) , Gold (+,- 5%) , Silver (+, - 10%).

An ideal resistance meter has a high internal resistance, and since this resistance will be in parallel to the resistance being measured, it will have a low effect on the displayed value.

Amps = I = **Electric Current** , Set the meter on a high current scale and-or range. Place the probes in series with any circuit component or element in that circuit branch of which you want to directly find the current for. Reduce the scale and-or range so as to find the current reading having a good precision and display.

So as to not impede the circuit current being measured, a current meter will have a very low internal resistance, perhaps a tenth of an ohm. This resistance is actually a parallel resistance to the coil or sensor within the meter, and is often called a "shunt resistor", and is practically a short circuit to it. Shunt or a shunting resistor diverts ("shunts") the high current away from the sensitive current sensor so as to not damage it. An amp meter is rated at a maximum amount of current to measure, and if it is exceeded, it may blow an internal fuse, or cause damage to the meter, etc. Some amp meters have sensors that can measure the current flowing through a wire, and without the need to put the meter in series with the circuit.

Use Ohm's law for any calculations and-or checking needed: $I = V / R$, $V = IR$, $R = V / I$

$$P = \text{electrical power} = VI \text{ watts } = \text{joules / second}$$

$$P = V I\verb|^|2 = V\verb|^|2 / R = \text{watts } = J / s$$

Some meters and-or multi-meters can measure other electrical values such as: continuity or low resistance check, diode check, transistor check and-or gain, capacitance, inductance, temperature, frequency, and battery test.

Another type of meter is a **power meter** that can sense and measure both current and voltage in a circuit. This device is

```
-----------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "math.h"

/*-----------------------------------------------------------------------------------------*/
void main(void)
{
double increment=0.0000001;   /* increment a "millionth" = (one-millionth) = (one/millionth) = (1/million) = (1/1,000,000)   */
double angle=0.0;
double sine=0.0;
double sum=0.0;
unsigned long int n=0;

for(;;){

        sine=sin(angle);
        sum=sum+sine;
        angle=angle+increment;
        n=n+1;
        if(angle>=1.570796327){ break; };     /* 90 degrees , if this is doubled for 180 degrees, the average is still the
                                             same over 180 degrees since a sine waveform is symmetrical about 90°
                                             and will yield the same average value. */

    };


printf("\nThe average value of the sine waveform  is:  %lf",sum/n);    /*  Displayed is:  0.636620 */

printf("\n\nPress a key."); fflush(stdin);  getch();


};
/*-----------------------------------------------------------------------------------------*/

If the peak voltage of a sine waveform was 9 volts, the average value of that waveform and effective or equivalent dc
voltage is then:  (9v)(0.636620)  =  5.73v

/*-------------------------------------------------------------------------------------------------------------*/
/* Sine RMS Value.c

   RMS = Root Mean Square  =  Square-Root of the Average of the squared values.

   This program is based on sineaveragevalue.c that does not consider RMS, but the basic values of
   the curve such as for up to 90 degrees, and that average value is also the same value even if 180
   degrees is considered.  To consider the equivalent dc power of the entire wave, and so that the
   positive and negative halves do not cancel or negate each other to 0, the RMS method is done
   where squared values are considered.  Nonetheless, the RMS value of 0.707 is close in value to
   0.63 as found in the above program to find the average value of a sine waveform.

   (c) JPA Jan, 2021


-----------------------------------------------------------------------------------------*/

#include "stdio.h"
```

A general example of a car or automobile battery and (electrical power) inverter system.

The mechanical analogy of an electrical inverter system is a (mechanical) gear system that can alter and-or transfer mechanical (ie., direct, physical contact forces) power. Given a constant value of power = P = (voltage)(current) = VA or VI, the two variables: V and A will be altered by the **inverter**. The physical and mathematical relationship of those two factors of P is that they are **inversely** related. If one increases by (n), the other decreases by that same factor of (n) when the power value is a constant value.

A common 12V, non-deep cycle or drain, car or automobile battery has about a 50 amp-hour rating = 50 aH. This means it can supply 50A maximum for 1 hour, or mathematically 25A for 2h, and an equation for this is:

> **Rated Discharge Time = aH / a**   : expected discharge time until the car battery needs to be recharged.
>   aH = amp hour rating, a = A amps = current.

> **Rated Discharge Time = wH / w**   : **wH = watt-Hour rating = (V)(A)(H) = (V)(aH rating)**

After this rated discharge time, a car battery will usually have power still in it, but it has fallen below its rated 12V and is perhaps now just 10v or 11v, and which is enough to power circuits for emergency LED lights and-or other devices that can operate at lower voltages. You may which to then consider some type of voltage regulator and-or circuit for these. It is also of not that the current drawn from a battery during engine ignition is often higher than its aH rating, but its for just a brief amount of time.

Ex: If a device that uses 5A of current is connected directly to a fully charged 12V, 50aH car battery, the expected time of it to function using this battery is: time = 50 aH / 5a = 10H = 10 hours. For this to happen, the device also needs to operate at 12V.

Ex.: While using an inverter: Power in from the battery = Power out to a 20W device
>   20W = 20W
>   (12v)(1.67A) = (120vac)(0.167A)   : P=VI , I = P/V = 20 / 120v = 0.167A

In general, regardless if the system uses an inverter or not: A 12V, 50aH battery can supply a maximum power of:
Pmax = (V)(Imax) = (12v)(50A) = 600W, and this was tested and rated to last for 1 hour = 60 minutes of time. This can also be expressed as a total safe amount available as: (Power)(Time) = (watts)(Hours) = (V)(I)(H) = 600wH.

For a non-inverter system, using just the battery: A 20W device can be powered for 600w / 20w = 30 times longer than 1 hour, hence for 30 hours = (30h)(60min/h) = 1800 minutes of continuous or intermittent use. If the device is used only for a few minutes at a time, say for 5 minutes, the battery can be expected to provide this 20w of energy for: 1800min / 5min = 360 times. A battery array, either series and-or parallel, will increase the stored energy available and the amount of time a fixed or certain amount of power can be drawn. A 12v, 50aH rated battery would have a corresponding (maximum) watt-hour rating of: (12v)(50A) = 600wH, and therefore it can supply a maximum of 600w of power for 1 hour of time.
If the battery (mass-weighed) 18.1kg = 40lbs, its corresponding **energy density** or **specific energy** would be rated as: (wH / mass) = (600wH / 18.1 Kg) = 33.2 wH / 1 Kg . It is generally easier to weigh a battery than to calculate its volume, but an energy density could possibly consider the volume of the battery as: (wH / volume). The energy density rating of a battery is sometimes expressed in Joules of energy per mass. It was shown in this book that:

> **Energy = J = VAs = Ws = (Power)(time)**

>   1J = 1 Ws   , if the amount of seconds was 3600s = 1h , we have after multiplying each side by 3600:

>   (3600) J = (1 ws) (3600)

The maximum safe and-or available current for a 12V battery having a 50aH or (12v)(50aH) = 600w maximum rated power output is: (Note, the actually result is 600 watt-hours = 600wH.)

P max. in = P max. out   : Power = energy used and-or supplied / time   ,   P in = power from the battery
600w in = 600w out
(Vin)(I in) = (Vout)(I out)
(12v)(50A) = (120V)(I out)   , after solving for I out:

I out = 5A   : this value is one-tenth (1/10) the current drawn from the battery by the inverter. The ratio of
voltages equals the reverse ratio of currents: (120 Vout / 12 Vin) = (50 Ain / 5 Aout) = 10

Note that the time of this power usage, either input at the inverter and-or output on the AC line is still the same value for a certain max. safe wattage available. It is incorrect to think that the time of use will be: time = aH / A = 50 aH / 5 = 10h , but the max. time value is rather one-tenth of this value, and is 10h / 10 = 1h, and this corresponds to that from the battery to the inverter as used for this 600W of continuous use and where Pin = Pout: time =   50aH / 50 = 1h   .   To help overcome the maximum amount of time available at a give wattage, the current drawn can be significantly reduced if the battery system has a higher voltage. Its aH rating will also increase if the batteries are in parallel, and then the time of use of the available power will also increase.   If the output device used only half of the 600w max. available or 300w , and-or this could be calculated as 600w/2 = 300w, and-or 600w/300w = 2, and then the time of use would be twice as much or 2 (1h) = 2h. Also, 300w/600w = 0.5 and the time value will increase by the reciprocal of this ratio: 1/0.5 = 2 , As a check:

Energy in = Energy out   : (Power)(time) = total energy = total Joules
(Power)(Time) = watt-hours in = watt-hours out   : Power = joules / s = watts
: **(Power)(time)** = (J / s) (s) = (w)(s) = **J = Energy J**

3600J = 1J 3600s = 1 w for 1 hour = **1wH** = 1w (3600s) = 3600 ws = **3600J**
1s   s

1wH (1000) = 1000 wH = **1kwH** = 3600J (1000) = **3600000 J**

600wH in = (600w)(1hr) = (12V)(50A)(1hr) = (300w)(2 hr) = (120v)(2.5A)(2h) = 600wH out

To effectively double the amp hour rating of a battery system, you can connect two similar batteries in parallel. For two similar 50aH batteries connected in parallel, the total aH rating of this battery array and-or system is: (number of batteries in parallel) (aH rating) = (2 batteries)(50 aH / battery) = 100aH.   Similar voltage batteries connected in parallel will have the same output voltage as any one battery.   To be safe and not permit high currents to flow amongst batteries when they are connected together, please make sure they are at or near the same voltage level. Remember, it is a potential or voltage difference that will cause a current to flow

For **charging a battery**, many consider a fair and safe value of **one-tenth the amp value in the aH rating**, hence 1/10 the maximum output in current, or less. This reduces internal heat damage in the battery and gives more time for the chemical reactions to take place. For a 50aH battery: 50A / 10 = 5A of current.

A typical 100W, 12V rated solar panel can deliver about: from P = VI , we have I = P/V = (100w / 18max open circuit voltage) = about 5.5A maximum of current. The time it takes to charge a battery to its full rated capacity is determined by both the safe amount of recharging current, and how drained of energy the battery is and-or its current amount or capacity of energy.   When charging a battery, it is best to monitor its voltage if it is not performed by a charger or voltage regulator device.   To do this, temporarily stop charging the battery, and before and while measuring its voltage, temporarily draw an amp or so of current from it, such as for a light and-or other devices, and so as to have its true available output voltage value while being used. A battery itself has an internal resistance of which there will be a voltage drop or loss across and due to it, and this will not take place until it is in a circuit with current being drawn.

/*--*/

A COMPUTER PROGRAM TO FIND THE AVERAGE VALUE OF ANY CURVE

/* AverageValueOfAnEquation.c

   The example equation shown below is "hard coded" (ie., explicitly, constant), here as a parabolic or
   second-degree quadratic equation, and can be changed for any equation you need.

   This method will find the average of y=x^2, between the user settable upper and lower "bounds" or
   limits of x1 and x2.

   N sample values will be taken and summed together, and then divided by N.

   When the limits are between x=0 and x=1, the average value of y = x^2 is 0.33333**...**
   This is the case whether the curve is increasing from x=0 to x=1 or decreasing from x=1 to x=0.

   (c) JPA , April 16, 2021
   ---*/

#include "stdio.h"

```
goto start;
};
/*------------------------------------------------------------------------------------------------*/
```

BASIC CONCEPTS OF INTEGRATION

In the topic of calculus, there is a concept called integration. The basic meaning of integration is the summing of all the small parts so as to find the whole thing of which it came from. In calculus, the small parts are actually infinitesimally (as small as possible) small parts. This discussion and notation below is meant to be very basic so as the reader can gain at least a minimal understanding and to research the topic further if needed. If needed, you can review the topic of Understanding Basic Calculus shown previously in this book.

If given variable x, all the small bits of x are theoretically the infinitely small change or "differentials" (ie., a difference) of x = (dx), and will sum to that variable of x, regardless of any actually value of x. We can consider dx as like a very small first dimensional length amount and is equivalent to a very small change, piece or bit of the entire x or length that is infinitely long, and if each is identical and having a theoretical infinitely small size:

Sum of all the small bits of x = x1 + x2 + x3 + . . .   = dx + dx + dx + . . . = S dx = S 1 dx = x   :S means summation
                                                                                                    or integration

When the summation symbol does not have any indication of the bounds (boundary) or limits of the summation, it is considered an indefinite, undefined or a general summation and which basically means a complete, full, or infinite summation of all the small parts. When the limits of the summation are indicated, the integral is called a limited, definite

$$y = f(x) = A = S\ dA\ = S\ \underset{x=a}{\overset{x=b}{}} f'(x)\ dx\ = \int_{x=a}^{x=b} f'(x)\ dx$$

: The area will have the same units as those of (x) or some other variable being integrated. The term(s) being summed is formally called the integrand.

In the above notation, each bit of Area is an infinitesimally small rectangular area (dA), and each with a base or width of (dx) and a corresponding height of y(x) = f(x).

    Area = A = (base)(height) = Sum of each small rectangular, differential area = S dA

For each (dx) or small base length of the small portion of the Area, hence dA and which is not a constant value, (dx) is defined as an infinitesimally small constant value. but the specific, actual or corresponding value of (y) is not defined as a constant value, and is actually variable in value, and essentially is the corresponding value of the function using a specific value of (x), hence the (y) value of the function at that specific value of (x), hence y at f(x) = y(x) = f(x), and the result is the (variable) height of the curve above the (x) axis at that point.

The derivative of a function is the rate of change of (y) with respect to a change in (x) when the change in (x) is infinitesimally small in value, and therefore, the corresponding change in (y) will also be infinitesimally small and it is generally and symbolically not equal to the same value as the change in (x), and this is due to the specific functional equation and relationship of (x) and (y).

Given y=f(x) = "y equals a function of x" (ie., y, the output or result, depends directly upon the value of x and the equation it is in) below is some notation for: "the derivative of y with respect to x", and since the changes are infinitesimally small, unlike that when finding a general slope value, it is more properly spoken as: "the differential of y with respect to the differential of x":

$$\frac{d\ f(x)}{dx}\ =\ \frac{dy}{dx}\ =\ \frac{\text{change in } y \rightarrow 0}{\text{change in } x \rightarrow 0}\ =\ y\ '\ =\ f\ '\ (x)$$

: notation for the derivative of the function of (x) with respect to x, the independent variable. It is a constant or an expression for the point to point (on the curve of the specific equation, or instantaneous rate of change of (y)=f(x) with respect to (x). Mathematically:

$$d\ f(x) = \frac{dy\ dx}{dx} =\ dy\ = f'(x)\ dx$$: the product of the derivative of the function and the infinitesimally small change in (x) or dx is equal to a differential of the function. The sum of differentials, dy, of a function is equal to that entire function: f(x) If this sum is limited or bounded, the result is not equal to that function, but rather is essentially a defined, limited or bounded area.

    Notice that: dy = f'(x) dx   has the form of a linear or line equation:
          y = m x , and m = y / x = the slope or rate of change of
                          y with respect to a change in x, and:

    The sum of all the small bits of y will equal that entire function:

        S dy = y = f(x)   , and   S dy = y = f(x) = S (f'(x) dx)   : as shown above.

  dy = f'(x) dx = (derivative of the function) dx   Note : $\frac{dy}{dx} = \frac{d\ (f(x))}{dx} = f'(x) =$ derivative

  Given that: S dy = y = f(x) = S (f'(x) dx) , this clearly shows that to solve for the sum of all the differential bits of the function, we need to take the anti-derivative of the derivative, so as to have the original function = f(x) or= which may

Given a derivative of a function, this is noted as: y ' = f '(x) . When given a derivative, noted here as y ' or= f '(x) , we can find the original function (F) from which this derivative was derived out from.

Consider that F represents a function and its derivative is the function: y=f(x). F is the notation for the anti-derivative of y=f(x).   In more symbolic form: F is the anti-derivative of its, or a given, derivative.

The derivative of F is therefore equal to:

d F / dx = F '(x)

In short, the derivative of the anti-derivative of a given function is equal to that same given function.

The derivative of y=f(x) with respect to x is: y' = f'(x) = dy/dx . Given the derivative equation = dy/dx, the antiderivative of this is y=f(x) which is the original given function.

In the notation shown being summed or integrated, all the bits of the derivative of a function which is f '(x) = y '(x) will sum to Y = F(x) which is the "whole thing", the "whole function" and not just a "bit of the function", and it is also the anti-derivative function of the function being integrated.

The area (A) beneath both the bounded curves of a line and parabola will be developed below. [FIG 247]

In the graphing of the parabola curve on the right, the area (A) to be found is indicated as defined, limited or bounded by the lower bound value of (x=a) and the upper bound value of (x=b). An infinitesimally small "slice" or "differential area" (dA) bounded by dx which is a very small change in (x), and the height=y at that value of (x) is indicated within the total area being found.

Now considering the line graph.   The area being found is triangular and is easy to see that it is bounded by (x=0) and (x=2) as the triangles base, and having a height that is equal to: y(2) = f(2) = 3(x) = 3(2) = 6. The area of this triangle is:

At = (base)(height)   = (2)(6)   = 12 = 6 square units
           2   2   2

A slice = dA = (dx)(dy)   , dy = dA / dx = A'(x) = f'(x) ,   The antiderivative of A'(x) = A = y = f(x)

A = Area of all the "differential areas" = dA, and if A found by summing (S) all these (dA):
A = S (area of each slice or differential area) = S dA

y = f(x) = A(x) = S dA = S (dy)(dx) = S f'(x) (dx) = F(x) = Y or = y = f(x)   : F(x) is the antiderivative of
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ f'(x) = A'(x)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ F(x) = A

Let a limited value of being determined A = the summed area of all the area slices from x=(0 to b) =

$\quad$ b $\quad\ $ b $\qquad$ b
A $\ $ = S $\ $ dA = S f'(x) (dx) $\ $ = $\ $ A $\ $ - $\ $ A $\quad$ = $\ $ F(b) - F(a) $\qquad$: the Area is limited so as to have a definite
$\ $ a $\quad\ $ a $\qquad$ a $\qquad\qquad\qquad\qquad$ b $\quad$ a $\qquad\qquad\qquad\qquad\qquad\qquad\quad$ sum or integral, rather than an indefinite result
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ of just the function itself. (a) and (b) are the
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ limits or bounds of the summation of the
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ differential areas where each is dA.

Another way to consider or look at the derivation and meaning of all this:

$\quad$ y = f(x) = function1 $\quad$, $\quad$ d f(x) = dy $\quad$ = $\ $ (derivative of function1) = function2 $\quad$: function1 is the anti-derivative of
$\qquad\qquad\qquad\qquad\qquad\qquad$ dx $\qquad$ dx $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ of function2

$\quad$ y = f(x) , and $\quad$ dy / dx = d (f(x)) / dx $\ $ = f'(x) $\ $, and therefore, mathematically:

dy = d(f(x)) = f '(x) dx $\qquad$: f(x) is the anti-derivative of f '(x) $\quad$, after integrating (ie., S) each term, we find:

S dy = S d(f(x)) = S f '(x) (dx) $\qquad$, and these terms sum to:

$\quad$ **y = f(x) $\qquad$ = f(x)**

$\qquad\qquad\qquad\qquad\qquad\qquad$ x $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ x
$\ $ Area = F(x) = Y = S dy $\ $ = $\ $ S $\ $ (dx) (derivative of function1) $\ $ = $\ $ S $\ $ f ' (x) (dx)
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ 0 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 0

Note that the infinite sum of all the (bits of x) = S (dx) = (x). To help understand this, consider all the infinite, infinitesimally small line segment lengths or bits (or infinitely small line segments) of a line, and which when put, added, joined or combined together, they will sum up to or create that whole (infinite) line. This was shown previously in this topic.

If we consider the function = f =3x as a derivative of another function = F, then F is the anti-derivative of 3x which is equal to: F = (3x^2) / 2

$\quad$ b $\qquad\qquad\qquad\qquad$ b $\qquad\qquad\qquad$ b $\qquad\qquad\quad$ x=b
$\ $ S $\ $ f'(x) (dx) = $\quad$ S $\quad$ 3x dx $\quad$ = D $\ $ (3x)(x) $\ $ = $\quad$ D $\qquad$ 3x^2 $\quad$ = $\ $ 3(b^2) $\ $ - $\ $ 3(0^2) $\ $ = $\ $ 3(b^2) $\quad$ or= 1.5 b^2
$\ $ 0 $\qquad\qquad\qquad\qquad$ 0 $\qquad\qquad\qquad$ 0 $\quad$ 2 $\qquad\qquad$ x=0 $\ $ 2 $\qquad\quad$ 2 $\qquad\quad$ 2 $\qquad\qquad$ 2 $\qquad$: D = difference

For the example above when b=2, the area beneath the curve of y=3x, from x=0 to x=b=2 is:

$\quad$ Area = 3(2^2) $\ $ = $\ $ 12 $\ $ = 6 square units
$\qquad\qquad\quad$ 2 $\qquad\qquad$ 2

First note mathematically from the above expression:

dV = (rate) dT   = R dT   : extra, since dV and dT are instantaneous values, so is the (rate) or derivative.

S (dV)   = S (rate) (dT)

V = (rate) (T)

If the container had some initial volume; (+) of water or deficit; (-) of water, that can be included into the formula as an addition of this constant (C). This will only shift the curve of the volume equation in the vertical direction, and will not change the general shape (ie.. slopes and-or the rate of changes) of the curve This is also a more formal or complete way to write an equation. This initial constant has no affect on the flow rate, and only has an affect on the total volume at negative time vales before the start of time when it is considered as Th=0h:

V = (rate) (T) + C   : checking, at T=0h, V = 0 + C = C, the initial or starting amount of water volume

The formula for distance is very similar to the above formula for volume.

distance = (rate) (time) = (speed) (time) = (velocity) (time) =   D = V T

In the situation here, V = velocity, and not volume.

$\dfrac{dD}{dT} = \dfrac{V T^{(1-1)}}{1} = V T^0 = V(1) = V$ mathematically:

dD = (V) (dT)

S (d D) = S (V)(dT)

D = V T

[FIG 248]

A COMPUTER PROGRAM TO FIND THE AREA BENEATH A CURVE SEGMENT

```
/*************************************************************************************************************
   AreaUnderACurve.c

   (c) JPA , April 2020

   This is a program to find the area beneath the curve of a function y=f(x) and the x-axis, and has a width
   that is bounded by a lower limit of (x=a), and an upper limit of (x=b).  This program uses a successive
   approximation method of summing up many smaller areas or thin, narrow "slices" or segments of area,
   and so as to find the total area by summing smaller areas.  This is a nice program to understand, and is
   very useful for when standard methods of calculus are not being used for some reason.

   Adjust the program as needed.  For example, since the function is hard coded or set fixed as:
   y = f(x) = x^2,  and this is essentially a simple parabola curve and you will have to change it if you want
   to use other functions and their curves.

*************************************************************************************************************/

double findarea(double lowerlimit_a, double upperlimit_b);

/*********************************************************************************/
void main(void)
{

             /* findarea(), function use example program:  */

double area=0.0;
double a=0.0;     /* lower limit of integration */
double b=0.0;     /* upper limit of integration  */

printf("\n\nThe function currently used in this program is:  y = f(x) = x^2 ");
printf("\n");
printf("\nInput a = lower x limit of integration: ");  scanf("%lf",&a);
printf("\nInput b = upper x limit of integration: "); scanf("%lf",&b);

if(a>b){
        printf("\n\aPlease use an upper limit that is greater than the lower limit.");
        printf("\nPress A Key To End The Program");  getch();  exit(0);
     };

area=findarea(a,b);

printf("\n\nThe approximate summed area is:  %lf square units.",area);
printf("\n\nPress a key");  getch();

exit(0);
};
/*********************************************************************************/
```

A COMPUTER PROGRAM TO FIND THE ARC LENGTH OF A CURVE SEGMENT

```
/*******************************************************************************
  ArcLengthOfACurve.c

   This coding is mostly based on the algorithm found in the program
   AreaUnderACurve.c program that is shown in this book.

   (c) JPA,  Dec 11, 2021,

   This is a nice program to understand, and it is very useful for when standard
   methods  of calculus are not not being used and-or not known such as for
   when a difficult equation(s) is involved.      This is  very handy when the
   anti-derivative is difficult to find when using a standard calculus method.

    Basically, this is a form of integration by summing up many small distances
    along the entire (arc) length of the curve that was bound or delimited by limits.
    In the sample use program below, the function or curve is y=f(x) = mx, a linear
    or line equation,   and the limits or boundary section or definition, are from x=a
    to x=b.

*******************************************************************************/
#include "stdio.h"
#include "math.h"    /* : if needed, such as for sqrt(number) and pow(base, exponent */
#include "stdlib.h"   /* : for system() */

double findarclength(double lowerlimit_a, double upperlimit_b, double a);
/*  This program currently uses a hard coded, or constantly defined
    function, y = f(x), and of which needs to be changed for other equations.
    The specific function used is for a liner or line function. */

/*******************************************************************************/
void main(void)
{
/* Function use example program:  */

double a=0.0;   /* here the slope (m) of the line function:  y = f(x) =ax = mx   */

double L=0.0;     /* lower limit of integration */
double U=0.0;     /* upper limit of integration  */

double arclength=0.0;

char c=0;

for(;;){   /* main loop */

system("cls");

printf("\nA Program For:  Arc Length or Length Of A Curve Segment");

printf("\n\nThe function currently used in this program is the linear function:");
printf("\ny = f(x) = mx");
printf("\n");
```

```c
for(;;){
        x1=x;

        y1 = (a * x1);    /* : 1:  The hard coded function of the curve of which the
                                   area beneath it ant the (x) axis is being found.
                                   here: y = f(x) = ax + b ,   and is C language coded as:
                                   (a)( x) + 3   , for a linear equation, b shifts the entire curve
                                   vertically, and the arc length still remains the same, hence
                                   its not used here.  */

        /* : Now we have the first point's coordinates , p(x1,y1) */

                        /*  Three other example functions to consider, and the arclength() should be rewritten as needed,
                            in particular, for any arguments needed to be sent to this function:  */

                /* 2:  y1 = a * (pow(b,x1));     : such as for an exponential function:  y = (a)(b^x),
                            and the user could first input the base value (b) and multiplier (a).  */

                /* 3:  y1 =  sqrt((a * a) - (x1 * x1));     : such as for a circle function:  r = x^2 + y^2 ,
                            and the user could first input the radius value (r).   */

                /* 4:  For an ellipse:  This will give 1/4 of the perimeter depending on the limits, hence multiply by 4:
```

$$y = \pm\, b \sqrt{ 1 - \frac{x^2}{a^2} } \qquad \text{: Note that when x=0, y= +-b as should be expected, and}$$

```c
                y1 =  minoraxis * sqrt( 1 -  ((x1 * x1) / (majoraxis * majoraxis)) );   */

    x2 = x1 + dx;    /* : increment x so as to find the next close point and its coordinates */

    y2 = (a * x2);    /* 1: Now we have a nearby, second point's coordinates  p(x2,y2)  */

                /*  2:  y2 = a * (pow(b,x2)); */
                /*  3:  y2 = sqrt( (a * a) - (x2 * x2));  */
                /*  4:   y2 =  minoraxis * sqrt( 1 -  ((x2 * x1) / (majoraxis * majoraxis)) );   */

    /* For the distance between two close, successive points on the curve:
       This uses the standard distance equation, for the distance between two points on a plane:
       distance = sqrt ( (change in x)^2 + (change in y)^2 )
       In theory, for two very close points, this is an infinitely small distance value
       for that segment or portion of a curve between x1 and x2:                          */

    distance = sqrt( (x2 - x1) * (x2 - x1) + (y2 - y1) * (y2 - y1) );

    arclength =arclength + distance;  /* : add each distance value to the total distance called arclength */

     x = x + dx;    /* : incrementing x by the value of dx, so as to get the coordinates of the "next point" on the curve */

    if(x>=((upperlimit_b - dx))){ break; };
    };

return arclength;
};
/***************************************************************************************************************************/
```

The following drawing shows two common and fairly similar methods to visualize several component (or "dimensional") vectors and their corresponding effective or resultant vector. The one on the right is generally the methods used to

 visualize force vectors acting upon an object, and the resultant (R) force and direction is shown. The length of the vectors indicated by lines and directional arrows is proportional to the magnitude (size, value, amount) of that vector. This method of showing the resultant is often called the "head to tail" method, where one vector starts at the head or arrow of the previous one. The method on the left is very similar and is typically called the rectangular method and it produces the same resultant. Another vector summing method is called the "parallelogram" or "tail to tail" method which starts two vectors at the same position and considers these 2 sides of a parallelogram, and the complete 4 sided parallelogram which often looks like a slanted rectangle, is then drawn. The diagonal in the parallelogram from the starting point is the resultant vector. [FIG 249]
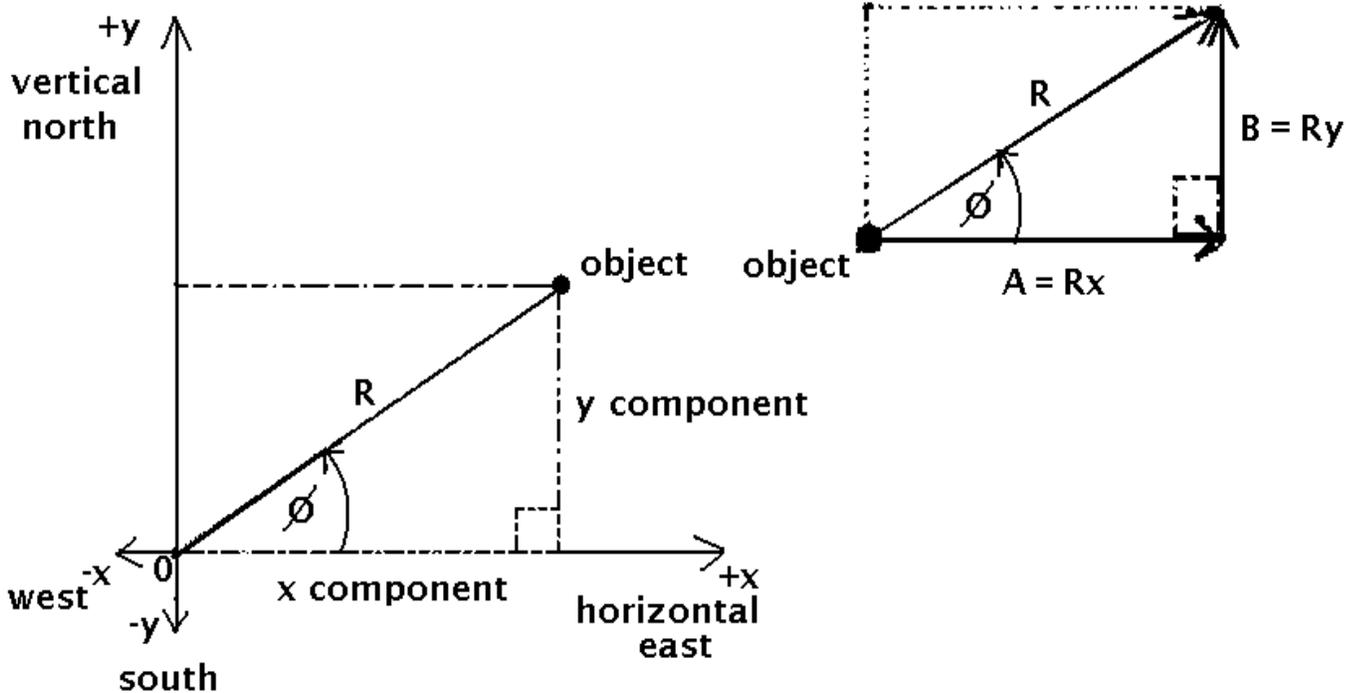
R=real or resultant vector

If A and B are vectors, the resultant (R) vector is the sum of those two vectors. In vector notation form this is expressed as:

R = A + B   : a special vector expression or notation form for the sum of two vectors. It is incorrect to sum their magnitudes for the resultant magnitude of the resultant "summed", combined, output vector or result.

From the study of trigonometry:

$\sin \phi = \dfrac{y}{R}$ and $\cos \phi = \dfrac{x}{R}$

$y = R \sin \phi$ $x = R \cos \phi$

A linear or non-angular version for a 3 dimensional system was shown previously as: P(x, y, z)

```c
system("cls");
fflush(stdin);

if(argc==2){ strcpy(filename1,argv[1]);       /*  arg[1] is the second real argument, and is the "input"
                                                dragged and dropped (with the mouse,) file. */

             printf("\n\nINPUT FILE: %s",filename1);   /* fflush(stdin); getch();    test ok   ex.  mydatafile.txt */
        };

printf("\n");

if(argc == 1){          /* : if no file entered on the command-line argument, and only the program name is an argument  */
           printf("\nPlease enter the text filename with the data values expressed as an ASCII text strings on each line: ");
           fflush(stdin);
           gets(filename1);
        };

/* This will  check if the file even exists */
file1 = fopen(filename1, "rt");      /* : open the file for access, here reading in text mode */
                                     /* text mode is similar to a viewable screen output mode, where numbers and text are
                                        read and-or written to as readable strings of text characters, and not generally as
                                        binary numeric values such as for numeric data.   Also consider for example that
                                        in text mode, a n indicated (as a string of ASCII text characters) value of 100 will take
                                        three bytes: "1 =49,  "0"=48, and "0"= 48 , however in byte mode it will require only 1
                                        byte and it will be: 100 expressed in binary numeric form as 8 binary bits:  01100100 */
        if(file1==NULL){
            printf("\n\aREAD ERROR --- OR FILE DOES NOT EXIST.");
            printf("\n\nPress A Key To Exit This Program >");  fflush(stdin); getch();  printf("\n"); exit(1);
          };

        for(;;){       /* : a loop to get each number (here, actually a string of ASCII text characters) or data element
                          that is stored in the file  */

               e = fscanf(file1,"%f",&x);

               /* fscanf() is similar to scanf() to get keyboard read text input, but is for reading computer files,
                  reads formatted text data. This is a simpler alternative to using fgets() and then having to
                  use a function to convert the text string to a floating point numeric value.  If values are
                  not stored in text or string character, ASCII form in the file, but binary numeric values, then
                  the file should be opened for accessing in binary (pure byte) mode, instead of text mode.
                  Here is a similar example of how to access coordinate values of points stored in a text file
                  and having the format:  x-value , y-value    , and for example:   5 , - 4.2
                  The coding for this would be:       e = fscanf(file1,"%g ,%g",&x,&y);                          */

               if(e==(-1)){  break; };     /* : The eof = end of file, is a C language defined constant, and is often -1 */

               printf("\n%g",x);             /* : opt. to display all the input data on the screen */

               datavalues[n]=(double)x;     /* : store the read values from the file into the numeric
                                               floating point values array */

            n=n+1;      /* : increment the index and-or counter to store the data into the next
                           array element and-or memory location  */
          };
```

/*--*/
/* Use example:

    Here, the example or test input (plain, ASCII, with no formatting symbols or commands from a word processor) text file,
    as shown below is called (ie., its filename) NumericData1.txt, which contains the string or ASCII text form of the
    numeric data to use in the program. In this way, a text file is being used as data storage that a program can use.
    A program can be made so as to input and-or create data files, and the data can be stored as numeric a values or
    readable (and-or easily editable or createable by anyone, even without the main program) ASCII text values which
    must then be converted to numeric values so as to be used by arithmetic in the (data) statistic program above.

-3
7.5
8.5
9.5

Program Output displayed on the screen:

INPUT FILE: NumericData1.txt

Number of data values = 4
The average value is: 5.625
The average deviation is: 5.75
The standard deviation is: 5.80768

Press A Key To Exit Program >

--*/

```c
for(;;){   system("cls");

        printf("\nRANDOM VALUES MENU");
        printf("\n");
        printf("\n1 Make New Data");
        printf("\n2 View Data");
        printf("\n3 Save Data");
        printf("\n4 Load Data");
        printf("\n5 Search Data");
        printf("\n0 Exit");

        printf("\n\nEnter Selection: ");
        fflush(stdin);
        scanf("%d",&c);

        if(c==1){ makedata(); };
        if(c==2){ viewdata();   };
        if(c==3){ savedata();  };
        if(c==4){ loaddata();   }
        if(c==5){ searchdata(); };

        if(c==0){ exit(0); };

      /*   if((c<0) || (c>5){  continue; };  optional, since will continue looping anyway, for other values entered */
    };
};
/*-------------------------------------------------------------------------------------------------------------------------------------------*/
void savedata(void)
{
FILE * file1;       /* declare a file pointer variable, so as to point to a specific  file and its data so as  to access it */

unsigned char filename1[256]={ "bytes.dat\0" };       /* the filename string or array or characters */
unsigned char c;
int n=0;
int e=0;   /* for file access (during reading or writing) error checking */

file1 = fopen(filename1, "rb");       /* open the file for reading for reading mode, so as to check to see if it already exists */

                /*---------------------------------------------------------------------------------------------------*/

    if(file1==NULL){               /* if the file does not yet exist */

            file1 = fopen(filename1, "wb");      /* open the file for (non-appending)  writing mode, filesize will be
                                          cleared to 0 bytes.   File data will essentially be overwritten */

            if(file1==NULL){       /* if there was  a problem opening or creating the file */
                printf("\n\aFILE CREATION, WRITE, OR ACCESS  ERROR 1.");
                printf("\n\nPress A Key To Continue");   getch(); return;
                };

            rewind(file1);       /* set the file pointer to the start of the file, where it should be already */
            goto WRITEDATA;
        };
            /*-----------------------------------------------------------------------------------------*/
```

```c
/*-----------------------------------------------------------------------------------------------------------*/
void makedata(void)
{
int n=0;
int m=0;
int z=0;
unsigned int r=0;              /* the returned random value from the random number generator */
int t=0;
unsigned char c=0;            /* will hold the random byte */
unsigned char found=0;       /* used as a "toggle" value, to determine if the byte value is already in the array */
unsigned char zerofound=0;  /* a toggle, used since all the bytes are set to 0 initially */

unsigned long M=0;           /* total number of random values generated, used or picked, even if it was already
genereated  previously */
unsigned long SUM=0;        /* for the sum of all the random values picked */

            /*---------------------------------------------------------------------------------------*/

srand(time(NULL));    /* seed, initialize or set  the computers random number generator with a starting value so as further
                        random numbers can be chosen.  The seed value can be any value in general, and not
                        necessarily the computers current time value in seconds as returned by time(NULL) which is
                        pseudo-random or near random in value enough to be used since it changes every second.
                        You can also use: time(0) for the argument to srand(), instead of time(NULL).
                        The "random number" generator will be seeded (ie., initialized with a starting value) using srand()
                        = "seed random" pseudo-random numbers using the current time as a pseudo-random value.
                        The values returned by rand() will be based on this seed value and a computer algorithm to
                        choose the next pseudo-random number.  srand() returns a two byte  integer value (0 to 65535).*/

system("cls");            /* the system() function can send commands to the computer system.  Here the command is: cls
                        which is the PC command to clear the standard output display screen.  */

/* Lets optionaly clear (set all the byte values to 0) the array just for some extra safety. */

for(n=0;n<=255;n++){
                        bytevaluesarray[n]=0;
                };

            /* ------------------ Fill the char or byte array with random byte values -----------------------*/

/* Lets fill each byte of the array with a random value, and where each byte is only  used once.  To accomplish this,
the array must be checked if each new random value was already used or not.  If it was already entered, the loop
will just continue without taking any further action, and it was not entered, it will be placed at the current position in
the loop, and the loop will then continue.  */

n=0;
for(;;){
        r=rand();                   /* get the next available generate random number, rand() returns an
                                    integer number data type, typically 2 bytes */

        c=(unsigned char)r;    /* we only need a 1 byte value to place into each element or position of the array */
        if(c==0){
                if(zerofound==1){   continue;  };
                if(zerofound==0){  zerofound=1;   };
            };
```

```
            fclose(file1);        /* formally, close the file */

printf("\n\nCompleted reading the data from the:  byte.dat  file.  Press a key.");  fflush(stdin); getch();


return;
};
/*----------------------------------------------------------------------------------------------------------------------------------*/
void viewdata(void)
{
int n=0;
int t=0;
unsigned long M=256L;
unsigned long SUM=0L;
                              /*-------------- Display each random byte value in the array. -------------------*/
system("cls");
                        /* Here,the first value is at offset 0 in the array which is no offset into it, and
                           is indicated as byte number 0, but it is actually the first (1) byte in the
                           array.  You may modify the code as to how you want to display these. */

printf("\nHere are the array offsets of the array elements, and the corresponding values");
printf("\nthey are now set to:");
printf("\n\n");

for(n=0,t=1;n<=255;n++,t++){
                        printf("%3d %3d    ",n,bytevaluesarray[n]);
                        SUM=SUM+bytevaluesarray[n];
                        if(t==5){  printf("\n");  t=0;  };
                  };

printf("\n\nThe average value of all the 256 data values shown above is: %lu",SUM/M);

printf("\n\nPress a key.");  fflush(stdin); getch();

return;
};
/*----------------------------------------------------------------------------------------------------------------------------------:*/
void searchdata(void)
{
int n=0;
unsigned char c=0;
int m=0;
int b=0;

for(;;){
        system("cls");
        printf("\n\n1 Find the logical (0 to 255) position, offset or address of a byte value.");
        printf("\n2 Find the byte value at a logical (0 to 255) byte number, position,");
        printf("\n  offset or address.");
        printf("\n0 Finished searching.");
        printf("\n\nInput Selection Number: ");
        fflush(stdin); scanf("%c",&c);    c=c-48;    if((c<0) || (c>2)){  printf("\a"); continue; };

        if(c==0){   return;  };
```

```
 95 103    96  41    97 125    98  58    99 179
100  95   101 208   102 111   103 233   104 104
105 212   106  67   107   7   108 181   109 244
110 255   111 116   112 157   113 161   114  89
115 176   116   5   117  42   118 184   119 201
120 135   121 120   122  68   123 139   124 195
125 130   126 151   127 121   128  90   129 177
130  40   131  62   132 152   133  37   134   8
135 232   136 100   137  46   138  34   139  65
140  80   141 160   142 140   143  14   144 146
145  18   146 219   147  83   148  10   149 207
150  45   151 119   152  20   153 200   154 234
155 245   156  86   157 251   158 238   159 216
160  13   161  12   162 247   163  97   164  35
165   6   166 183   167 211   168 210   169 128
170 192   171  36   172 178   173 129   174 241
175 214   176  57   177  87   178  22   179 137
180  69   181 191   182 227   183 254   184 222
185 110   186  17   187  75   188  48   189  30
190  64   191   0   192 163   193 162   194  73
195   2   196 237   197  19   198 108   199  60
200 136   201 155   202   1   203 144   204 171
205  56   206  16   207 187   208  32   209 114
210 158   211 134   212 182   213 248   214 218
215 166   216 175   217 253   218  98   219 196
220 141   221  99   222 127   223 164   224 246
225 150   226  51   227 225   228 230   229  38
230 213   231 154   232 236   233 198   234 132
235  47   236 105   237  52   238 115   239 188
240 240   241 131   242  50   243  84   244 194
245  27   246  31   247 220   248 217   249 170
250  39   251 243   252 172   253 250   254  79
255  54
```

The average value of all the 256 data values shown above is: 127

The above data is what is printed or displayed on the viewing screen of the computer used. The binary data was converted to ASCII text characters to be displayed on the screen, but in the computer file or memory, the data is stored as binary values. For example, a (text) screen display of the number 1 is the ASCII text character coded as a binary value of 49d = 0011001b, however the (numeric) binary value for 1 is 00000001b and this is what is saved on disk or memory if it was stored as a binary value and not ASCII or "text values". A file can be opened (ie., for read and write access) in binary mode or text mode. When a data file contains readable characters it may be best to access the file in text mode, but this mode does some extra things such as converting a carriage return (ASCII 13) character to two bytes of data: a carriage return (ASCII 13) and line-feed (ASCII 10) characters, hence it automatically adds a byte to the stream of data. If this is not desired, then open the file in (pure, strict) binary access mode so as to access only the (exact) saved bytes of the file. fprintf() is the usually function for "text printing" to a file. For example a text string (ie. with the %s format specifier) can be printed to a file with fprintf(). If you want to first "build" a string, such as containing multiple variables, etc, one function you can use for this is the **sprintf()** function which will catenate (ie., string catenation, appending or joining) text strings together rather than use the strcat() function and other numeric to text conversion functions, such as itoa() to build a string.   Once a string is built, it can then be displayed and-or saved to an opened text file, etc.   This function essentially prints or sends the resulting output to a string instead of the screen as printf() would.

Ex. sprintf(mystring,"The Filename is: %s, and the number is %d",string1,n);   printf("%s",mystring);

```
  printf("\nEnter Selection: \n:");
  fflush(stdin);  c=getch();  printf("%c ",c);
        if(c=='1'){ goto secondstimer;  };
        if(c=='2'){ goto countdowntimer;   };
        if(c=='0'){  exit(0); };
  };
                              /*-------------------------------------------------------------------------------------*/
secondstimer: ;

fflush(stdin);  printf("\n\nPress a key when you want to start the timing. "); getch();
printf("\nPress The ESC  Key When Finished Timing: ");
n=secondstimer();

printf("\nThe number of seconds is: %ld",n);

hours=n/3600;                    /* optional, display the number of hours,minutes, and seconds that have passed */
minutes=(n-(hours*3600))/60;     /* converts the number of seconds to its equivalent:  H:M:S    */
secs=n%60;
printf("\n   %ldh:%ldm:%lds", hours, minutes, secs);
printf("\nPress A Key"); getch();  goto start;


                        /*----------------------------------------------------------------------------------------*/
countdowntimer: ;                    /* sample program call, start countdowntimer()  */

printf("\n\nEnter the number of seconds to count down from to 0, for example, for 5 minutes");
printf("\nenter 300 since 60s/m x 5m = 300s > ");

fflush(stdin);  scanf("\n%ld",&s);  printf("Press a key when ready to begin: "); fflush(stdin); getch();

a=countdowntimer(s);

if(a==0){   printf("\nUser Terminated The Countdown Timer, Press A Key. "); getch(); goto start;   };
if(a==1){             /* beep, ring alarm bell , till user presses a key */

            fflush(stdin);

            for(;;){
                  if(beeps==0 ){
                     printf("\a");       /* bell, ascii, beep sound */
                     printf("\n********************* COMPLETED *********************\n");
                     printf("\nPress The ESC Key");
                     beeps=1;
                     };

                  b=b+1;
                  if(b==20000L){ beeps=0;  b=0L;  };

                  if(kbhit()){ c=getch(); if(c==27){  goto start;  };   };  /* esc key was pressed */
            };          /* kbhit() is non-ANSI C, however it common enough to be generally considered as in ANSI  C */
         };

return;
}
/*----------------------------------------------------------------------------------------------------------------*/
```

A COMPUTER PROGRAM FOR A STOPWATCH TIMER

This is somewhat similar to a count-up seconds timer as shown in the last program: timer.c, but with this program shown below, it allows the user to pause and continue the seconds timer.

```
/*----------------------------------------------------------------------------------------
stopwatch.c

   A timer with a user pause abilitry that is started at the press of a key,
   and ended at the press of a key, and then displays the elapsed time.

   (c) JPA September, 2021


   Extra:   A pseudo-type of code and-or steps so as to time a process
            such as a function, etc. is:

              1.  starttime=time(0);

              2.  Some process to be completed.

              3.  endtime=time(0);

              4.  total_seconds = endtime - starttime;

              5.  Display the number of seconds.

-----------------------------------------------------------------------------------------*/

#include "stdio.h"
#include "stdlib.h"

/*----------------------------------------------------------------------------------------*/
void main(void)
{
unsigned long starttime=0L;
unsigned long endtime=0L;
unsigned long seconds=0L;
int n=0;
int c=0;
int pausemode=0;
unsigned long pausetimestart=0L;
unsigned long pausetimeend=0L;
unsigned long pausedseconds=0L;

/*----------------------------------------------------------------------*/

system("cls");

printf("\nSECONDS STOPWATCH PROGRAM by JA, (c) 2021, From The Math eBook: Mathization.");

printf("\n\nThe seconds count displayed are approximate,");
printf("\nand within about 1 second of error.");
printf("\n\nPress A Key To Begin The Seconds Counter: ");
```

```
/*-------------------------------- display the results -----------------------------------------*/

printf("\a");

/* printf("\nTotal Seconds = %ld",endtime - starttime);  */  /* optional */

printf("\n\nPress A key To restart the Program, or ESC to end the program: ");

fflush(stdin); c=getch();

if(c==27){  return;  } else{ goto start; };


};
/*----------------------------------------------------------------------------------------------*/
```

```c
            if(c==75){ ;  };          /* left arrow ke y      */
            if(c==77){ ;  };          /* right arrow key      */

            printf("%c%d      ",13,n);    /* 13 = Carriage Return */    /* 8 = Backspace  */
                                             /* : display the current count of key presses */
            /* delay loop */
            d = delay *  6000000;
            for(;;){
                    d=d-1;
                    if(d==0){   break;  }
                };

            continue;
        };

      if(c==27){   printf("\a");  break;  };    /* user pressed the ESC key */

    };  /* ---------------------- end of main for loop -------------------------------*/

printf("\n\nPress A Key To Exit The Program:  ");  fflush(stdin);  getch();
return;
};
/*------------------------------------------------------------------------------------------------------------------------------*/
```

Rs is basically a "safety resistor" so that if you turn Rb all the way to a low value, excessive base (and-or collector) current current will usually damage the transistor. Rc is included to prevent the voltage across the laser from going over 4.5Vdc which would either prematurely age the laser and reduce its light output, and-or cause it to be burn out and be of no use.

To increase range, various light concentrators can be experimented with. The simplest is a funnel or cone shape having an inner surface made of a reflective material. Another simple method is to uses a magnifier so as to focus and concentrate the received light energy onto the solar panel. Mirrors can be used to direct a laser beam to a certain direction.

The circuit shown below in the next figure is essentially a preamp circuit to help amplify or "boost" the signal output from the solar panel before it is sent to the computer microphone input so as to be amplified. There are many things to experiment with these laser circuits, so be patient and keep notes of what works and what doesn't. There are also free oscilloscope programs available that work well with most audio signals, such as for setting the maximum level before "clipping" or signal distortion happens due to too much amplification. These programs use either the microphone and-or line input of the sound card for the input signal to be displayed on the screen. The laser circuit above generally works well without the need for the preamp shown below. [FIG 252]

Solar Panel Receiver Amplifier Circuit For Laser Communicator

+5Vdc usb

Solar-Panel is the type found in solar garden lamps.

Rs1   Rs2

Vout
Signal Output
to microphone
input on computer
sound card

Solar-Panel Receiver   Rb   Rc   C

Laser Light   C   c   b   Q   e

May need to connect a wire from center ring on stereo plug to the ground connection on the plug.

common circuit ground
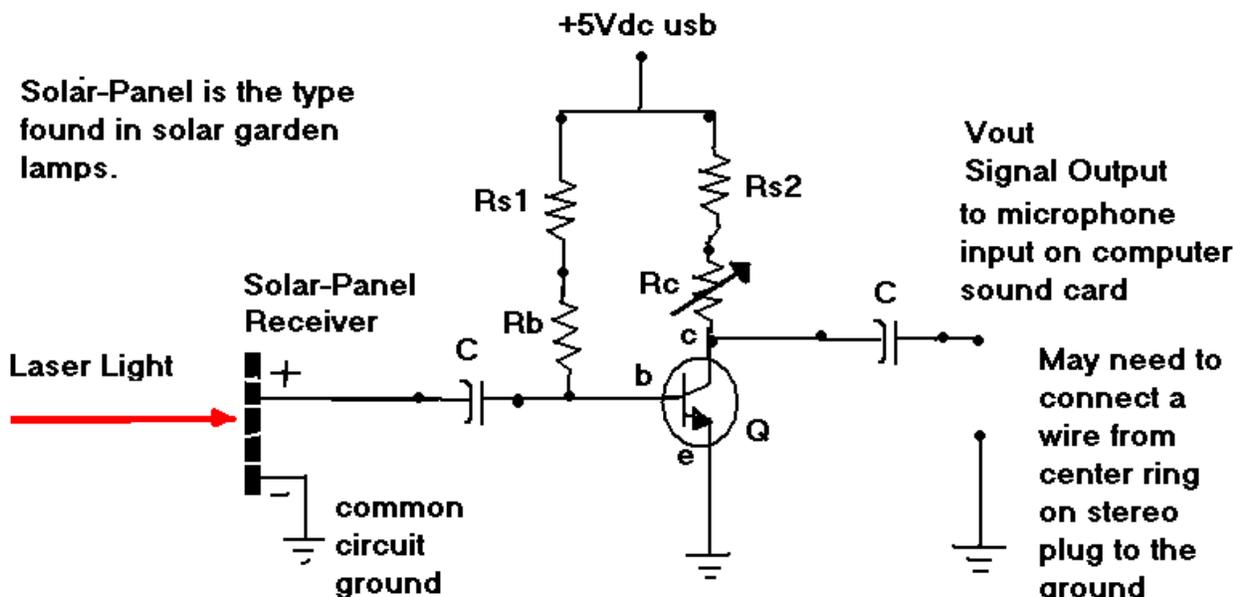
Q = 2n3904, NPN BJT Transistor (try what you have also)
C = 10uf (1uf or higher is ok)
Rs1 = 1K safety resistor,   Rb = 1M (fixed or variable)
Rs2 = 100 ohm safety resistor,   Rc = 5k Variable resistor
                                   Adjust Rc for best output

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Some solar garden panels from solar lights will work better than others. They type I used often had several smaller solar cells in it. If the circuit oscillates and produces noise or a tone, try putting about a 100pf capacitor across the transistors collector and emitter leads, or across Rc. The laser and-or preamp circuit is sometimes sensitive to the light frequency radiation or 50hz or 60hz noise from household lamps and some flashlights that boost their internal power supply voltage and frequency using some type of "voltage step-up" circuit. An **op-amp (operational amplifier)** on a IC chip is a small, very

waveform. Because of this, an op-amp can also function as a (input signal) difference amplifier if there are two inputs.

total memory needed =
   = (data size in bytes per element)(number of elements)

Here, the memory allocated is 129 bytes, with 128 being
used to hold a string, and 1 byte for numeric 0 which is
a byte value set to numeric 0,and not ASCII text character 0
which is assigned a code value of 48.

An unsigned numeric data type indicates that numeric values are to be
stored there as an unsigned numeric value, hence the sign is to be
discarded when storing a value there, and this allows twice as much
positive or sign-less values since the first or most significant bit
is essentially not set as the numeric sign indicator for the numeric value.

Be sure to always allow 1 extra byte or character of memory in this
character or byte array that will hold, store or contain a text string.
This extra byte will ensure there is a null or numeric 0 character to
be placed at the end of a (here) 128 byte text string. The scanf() will
automatically do this for you when you press the Enter key. The 0
or "null" byte will be placed in the next byte position past the last
text character of the string. Using scanf(), once the ENTER key has
been pressed, it essentially signals to the scanf function that you have
input the completed string of text characters or bytes. Scanf will then
"null terminate" that string, and so as to know when the end of the array
and-or string has been reached when displaying and-or processing that
string. There could be a system error if the user enters a string that
exceeds the total allocated size of the array.

When an array is declared, given a identifier or name, it is also possible
to also assign values to its elements.   */

```c
printf("\nEnter A String: ");
gets(string1);                    /* : A C function to input a string, and allows space characters.
                                     scanf() will terminate a string when the first space is entered.
                                     Ex.  scanf("%s",string1);
                                     scanf() does allow special modifications to its basic format
                                     specifiers, such as %s, so as to then allow certain characters
                                     such as spaces to be input.  */
 ;
printf("\n\nThe string you entered is:  %s", string1);    /* display the string  */

string1length=strlen(string1);    /* : A standard C function to calculate the length of a string up to, but  not including
                                       the null terminator character of 0 in this character count or length */

printf("\nThe length of the entered string is %d characters or bytes", string1length);

printf("\n\nPRESS THE ESC KEY or CTRL Break,TO EXIT THIS PROGRAM: ");


    for(;;){
            c=getch();                /* : A standard C function to input a single character from the keyboard. */
                                      /*   Using getch() is also a way to pause any program, such as to read the screen .*/
            if(c==27){  exit(0);   };  /* : Exit the program when the user enters ASCII 27, the ESC key.  */
        };
```

address assigned to that pointer, you can increment the pointer, and here fore example to the next
char or byte since that is the data type or memory size of this pointer : p = (p+1);   */

printf("\n%d",c);   /* : 66 will be displayed */
printf("\n%d,*p);   /* : 66 will be displayed since the pointer (p) is assigned to the computer memory address of variable
                (c). It could be said that (p) points to (c). *p is used to access the data that (p) is pointing to, hence
                it is accessing the data held by, at or assigned to variable (c)   */

Why use pointers if we can already use variables? When the amount of data and-or possible variables and their identifier
names become large, it is then difficult to assign, remember and use so may variable names and identifiers, and it is
easier to access data by using an assigned corresponding number or memory address, especially if that data is held in
successive memory and is repetitive in its data type and-or structure, such as having any data types and or elements.

A pointer with a FILE data type has been shown previously in this book. This type of pointer is used to access (read, and-
or write) computer (binary) files that were stored byte by byte on disk drives. For example: FILE * myfile;   When a FILE
pointer is first associated to a file and-or filename, that pointer is set to access the first byte of that file that is located on
the disk and its memory array of (digital, binary, 8 bit) bytes. When a FILE pointer is incremented by 1 in the program, it
will then be able to access the next byte of that file.   The C-language also has (ANSI) standard, common and-or required
functions for accessing files such as fread() and fwrite(), and these are analogous to those of accessing program, data
input from the keyboard such as with scanf(), and accessing program, data output to the display screen (computer
"monitor"). More technically, a FILE pointer points to a file data structure (a struct data type in the C language) assigned
to a file being accessed and that structure holds information about a desired file on data storage, such as its current
access or offset (from its start) location within it. This file data structure is usually "transparent" to the programmer since
the FILE pointer itself is used by the file functions that (transparently) access that file data structure, and the functions will
appear to be accessing the file directly.

The following methods are shown as portions of a larger program of which you can test and use them within, such as the
program shown previously to display a text string:

/* **Method 1**:   Displaying the string using string or **array indexing** (ie., numerical amount or offsetting from the start) each
                element of the array: */

/* Some initial program preparation before the loop, hence "loop preparation": */
printf("\n"); n=0;

for(;;){   /* : a loop to display the input string, (;;) is used to indicate an
                                            "infinite loop" that will keep running or executing until
                                            the loop is somehow terminated, and then the next
                                            statements in the program will be run.

                                            Can also use any loop preparation and or termination condition in the for-loop
                                            declaration:

                                                    Format: for(initialization , do while or until , adjustments)
                                                    The statements are separated by a semi-colon.
                                                     ex: for(n=0;c!=0;n=n+1){   */

        c=string1[n];   /* (n) is used as the index or offset from the start of the array
                                and-or its starting memory location. */

        if(c==0){ break; };   /* at the null terminator of the string, break or terminate the
                                loop (program repetition), otherwise: */
        printf("%c",c);   /* %c is the printf() format specifier to display a single

```
        n=n+1;
    };
```

Pointers can even be assigned as having the same address that another pointer is assigned or pointing to. For example, if p1 is a char pointer assigned to a memory location, and say p2 is NULL or 0, hence not assigned to a memory location yet, p2 can later be assigned to a memory address by setting it equal to another pointer.   An older saying or equivalent word for the word "NULL" is "NIL"..

Ex.

/* initial declarations: */

```
char c=65;
char * p1 = NULL;   /* NULL means an unassigned pointer, and or 0, or an address of 0 */
char * p2 = NULL;
```

/* an example of using the variables and or identifiers in the program: */

```
p1=(unsigned char *) &c;     /* : a (data) type cast of the address of a variable, here (c), is considered here,
                                    so as it can be properly assigned to an actual pointer variable. */

p2 = p1;           /* : setting pointer p2 equal to the same address that p1 is pointing to, or in reference to */

printf("\n%d",*p2);    /* the result displayed is 65  */
```

/*---*/

In the BASIC programming language, the default keywords to access a memory location are: PEEK and POKE. PEEK is used to read or get a byte stored in a memory location, and the general syntax format is:

```
    variable = PEEK (memory_address);
```

POKE is used to set (ie., write) a value into a memory location, and the general syntax format is:

```
    POKE (memory_address, value);
```

The above syntax looks similar to a function call in the C programming language. For comparison:

```
BASIC language:    value  =  PEEK (memory_address);     : PEEK is analogous to READ
     C language:    value = *pointer;

BASIC language:     POKE (memory_address, value);       : POKE is analogous to WRITE
     C language:     *pointer  =  value
```

The BASIC language was one of the first "user friendly" ways to create computer programs. Programs became fairly or relatively easy to understand, write and edit. There are many BASIC language interpreter programs. BASIC is called an interpreted language because the program is not compiled into a standalone executive program, but rather each line is executed at a time. Most of the BASIC interpreters use or require a similar syntax or format of the program or input language, and there has been an attempt to standardize BASIC and you can research this if you want. There are also some BASIC compilers now available so as to generate a standalone executive program for a specific computer system.

```c
void mygetstring(unsigned char * s, int maxchars, int display)          /* : a homemade getstring()  */
{
int n=0;                        /* : a counter, increment  */
unsigned char c=0;        /* : assigned to each new character entered */
unsigned char * p=s;     /* : a pointer to the start of the string sent to the function */
unsigned char * p1=p;    /* : to store the starting address of the string sent to the function */

/* A potential issue is that when backspacing, the cursor won't go back to the previous line on the display. */

    for(;;){
                fflush(stdin);
                c=(unsigned char) getch();
                /* To see each charcter value for testing then use:   printf("\n%d ",c);*/


            if(c==224){   c=(unsigned char)getch();     /* pressed a special key, such as the arrow keys */

                        if(c==72){  /* printf("\nUP "); */   };         /* UP ARROW key was pressed */
                        if(c==80){  ;  };         /* down arrow */
                        if(c==75){  ;  };         /* left arrow */
                        if(c==77){  ;  };         /* right arrow */

                     continue;
                };

            if(c==0){   c=(unsigned char)getch();     /* pressed a special key, such as the function keys */

                        if(c==59){ /* printf("\nF1 ") ; */  };         /* F1 key was pressed, then do something.  */
                        if(c==60){  ;  };         /* F2 */
                        if(c==61){  ;  };         /* F3 */
                        if(c==62){  ;  };         /* F4  */
                        if(c==63){  ;  };         /* F5  */
                        if(c==64){  ;  };         /* F6  */
                        if(c==65){  ;  };         /* F7  */
                        if(c==66){  ;  };         /* F8  */
                        if(c==67){  ;  };         /* F9  */
                        if(c==68){  ;  };         /* F10 */
                        if(c==133){  ; };         /* F11 */
                        if(c==134){  ; };         /* F12  */

                     continue;
                };


        if(c==27){   *p1=0;  break; };   /* : user pressed the ESC key, user quit this step, null terminate the
                                       string at the beginning of it   */
        if(c==13){   *p=0;  break; };   /* : user pressed the ENTER key null terminate the string */

        if(c=='\b'){                             /* : user pressed the BACKSPACE key */

                    if( (n>0) && (n<=128) ){          /* : if not at the start of the input string's first memory location */

                            printf("\b"); printf(" "); printf("\b");  /* : backspace, then overwrite with a space,
                                                      the backspace to re-position the text cursor */
```

LIST OF ASCII TEXT CHARACTER CODES

ASCII is the American Standard Code for Information Interchange (communication), and is used as a standard for machine and computer programming and display screens. These are essentially 1 byte values, hence 8 bits of which can be sent over 8 parallel wires, all at the same time. ASCII was standardized in the early 1960's by what would later become the ANSI which is the American National Standards Institute.

7 Bit ASCII Standard Text Character Codes: (0 to 127), and 8 Bit ASCII Extended Text Character Codes (128 to 255):

ASCII codes 0 through 31 are usually non-text display codes that are used to control (ie., send basic commands for various functions of) a machine such as a printer or computer display screen.

32	33 !	34 "	35 #	36 $	37 %	38 &	39 '	40 (	41)
42 *	43 +	44 ,	45 –	**46 .**	47 /	**48 0**	49 1	50 2	51 3
52 4	53 5	54 6	55 7	56 8	57 9	58 :	59 ;	60 <	61 =
62 >	63 ?	64 @	**65 A**	66 B	67 C	68 D	69 E	70 F	71 G
72 H	73 I	74 J	75 K	76 L	77 M	78 N	79 O	80 P	81 Q
82 R	83 S	84 T	85 U	86 V	87 W	88 X	89 Y	90 Z	91 [
92 \	93]	94 ^	95 _	96 `	**97 a**	98 b	99 c	100 d	101 e
102 f	103 g	104 h	105 i	106 j	107 k	108 l	109 m	110 n	111 o
112 p	113 q	114 r	115 s	116 t	117 u	118 v	119 w	120 x	121 y
122 z	123 {	124 |	125 }	126 ~	127 ¦	128 Ç	129 ü	130 é	131 â
132 ä	133 à	134 å	135 ç	136 ê	137 ë	138 è	139 ï	140 î	141 ì
142 Ä	143 Å	144 É	145 æ	146 Æ	147 ô	148 ö	149 ò	150 û	151 ù
152 ÿ	153 Ö	154 Ü	**155 ¢**	156 £	157 ¥	158 P	159 ƒ	160 á	161 í
162 ó	163 ú	164 ñ	165 Ñ	166 ª	167 º	168 ¿	169 ¬	170 ¬	171 ½
172 ¼	173 ¡	174 «	175 »	176 ¦	177 ¦	178 ¦	179 ¦	180 ¦	181 ¦
182 ¦	183 +	184 +	185 ¦	186 ¦	187 +	188 +	189 +	190 +	191 +
192 +	193 –	194 –	195 +	196 –	197 +	198 ¦	199 ¦	200 +	201 +
202 –	203 –	204 ¦	205 –	206 +	207 –	208 –	209 –	210 –	211 +
212 +	213 +	214 +	215 +	216 +	217 +	218 +	219 ¦	220 _	221 ¦
222 ¦	223 ¯	224 a	225 ß	226 G	227 p	228 S	229 s	230 µ	231 t
232 F	233 T	234 O	235 d	236 8	237 f	238 e	239 n	240 =	241 ±
242 =	243 =	244 (	245)	246 ÷	247 ~	**248 °**	249 ·	250 ·	251 v
252 n	253 ²	254 ¦	255						

Common ASCII machine control and-or formatting control characters and-or values:

```
0 = NULL
7 = A   = Alert and-or a bell sound
          In C, you can use the escape (from the normal text displaying, display and-or
          machine system control)character symbol followed by letter b so as to have
          access to, and signal the system to use the bell character, and from within a
          printf() statement.  These control characters are also called an "escape
          sequence (of characters)"  The backslash character is used first, so as to
          signal an escape sequence.  For example:
              printf("\a Did you hear a bell sound?");  This is equivalent to:
              printf("%c Did you hear a bell sound?,7);
          Note, to display a backslash character and-or the quote character, use for ex.
              printf("   \\    \"   ");
8  = BS  =  Backspace = Go one text position "back" or leftward on the current line.
```

USING COMMAND-LINE ARGUMENTS OR DATA WITH A COMPUTER PROGRAM

/*---

CommandLineArguments.c

This program shows an example of how to send optional and-or various argument data (such as text and-or numbers) to a program when it is first ran on a personal computer (PC) having a DOS (Disk Operating System) command line, and then how to access those arguments from within the program so as to be used within the program. The command-line is sometimes called the computer's operating system input and display (ie., typing, basic input and screen output) or (access or system) "terminal", and or the "DOS (Disk Operating System, of commands and access) command-line".

Format example: myprogram.exe argument1 argument2 . . .

Examples of the arguments: a word, sentences (ie., words with space between them) bounded by quotes (ie. "word word word"), numbers, a filename, etc., and these are input as ASCII text characters

Each argument should be separated by a space character. Extra spaces will be ignored unless they are within quotes such as for a string of text characters.

Using (mouse, left-click) "drag and drop" of an icon or filename onto the programs icon or filename, is also a way to send a filename and-or the file as an argument. The file can then be accessed within the program without asking the user to type in or select what file to use.   With the Windows operating system (technically a program itself), you can place a program in the SENDTO folder so as when you mouse-right-click on a filename, one of the menu options is "send to" so as to directly send that file to your program(s), and without needing to find and enter long directories, etc. To open the SENDTO folder or directory, press the START button the screen, select RUN, and then enter Sendto, and then drag and drop, or copy, your program to be in there.   To RUN or execute an .EXE format computer program, simply double click on it, or enter its name on the computer system's (DOS) command-line. To see a list of files in a folder or directory, enter DIR on the command-line. To see a list of DOS commands available for that computer system, enter: HELP, and for help with a specific command, enter: HELP command.   If possible, try to always make backup copies of your programs and-or data, for these devices tend to fail, can get damaged, hacked, lost or stolen. Put these in a safe and protected place.

The arguments are considered as null terminated (0), text strings, and you may then need to, for example, convert a ASCII text string to a binary numeric value that can be assigned to a variable, etc. One common C function for this is the atoi( ) which is the "ASCII text string to numeric integer" conversion function, and for converting an ASCII text string to a floating point value, use the atof( ) function. The return value is a double precision floating point number. Leading spaces will be ignored, a sign and decimal point can be used, and the last character accepted will be either a space or a text character other than a number. strtod( ) is a similar function. To convert a string to a long double floating point number, use strtold( ).

Since main( ) function is the default or required entry point of a C program, the arguments are passed or sent to argument holders described in the main(). Just like for any function with arguments, each argument to a function has a data type and an identifier, and here they are also specific C keywords and-or variables and their identifiers (identifying names) that need to be used so as to access that data. Still, you can always assign these values to your own variables used in the program.

The first argument is always the program name itself, and is identified with an offset of 0 as:
argv[0].   argv[ ] means "argument vectors" array which essentially means text strings for the arguments.
argc = argument count, will hold the number of total arguments sent to the program on the command line.

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To Learn To Program A Computer Device

To learn C programming, you can take a course or two and-or obtain some books about it. Some may desire to also take a course in computer science. For learning C, you will need a C source code compiler such as TinyC . This book considers the TinyC compiler which is generally an ANSI standard, C compiler with no specific computer system and-or its programming functions to consider. Other compilers may include many other specialized functions to be used for a certain or unique computer system, and such as for basic graphics to put a pixel, line or circle on the screen, or to make some sounds. These compilers must also be compatible with the specific computer system being programmed, but the ANSI C standard is generally compatible with any computer, and you generally will not have to learn or relearn C programming for general programming (creating computer programs) on any ANSI C compatible compiler for any computer system.

With programming, and much like math, just the basics of it can go a long way and be most useful or practical. Rather than make exquisite and specialized programs that take much knowledge and years to develop, it is sometimes better or more practical to make many relatively simple, useful and practical programs or even functions( ) that can calculate some numeric values of which others, such as at a factory, group or business, can utilize. Writing your own homemade functions is a good learning experience.   **Writing functions** which can be used in many of your programs and possibly by other programmers is a very useful way to become a good computer programmer. Strive to learn the ANSI C standard keywords and functions first, and before trying to write programs for a specific computer system that requires a specific and capable C compiler which also has some non-ANSI C standard functions for that computer system.

As a computer programmer, you are generally not expected to be initially proficient in the science and-or math necessary for any certain program or company, however you may need to research what is necessary and-or collaborate with other employees and those knowledgeable, such as other programmers, so as to make a specific and customized computer program. Please keep a written record so as you and-or others may need this data later. You may even be designated to do other things such as making the morning coffee and-or sweep the floors so as to be a multi-tasking employee with several useful and valued abilities. This book contains many programs for the common math functions found on a scientific calculator, and these are already part of the C standard and available as prewritten math functions, and hence you do not need to be a "math programmer" to do much general programming, but it would benefit any programmer if they know some math so as to make more advanced programs and-or to understand or edit programs written by others.

Many of the computer programs and-or functions shown in this book are relatively uncomplicated, and they may even be overlooked by many when scientific calculators and-or prewritten math computer functions are available in the C compiler. The set of ("scientific") fundamental math functions shown in this book, such as for those found on a scientific calculator or compiler, seem to be very rare, and should be of interest to computer programmers who are oriented to writing mathematical calculations. What is also important in these functions is how to create some specialized loops, such as for performing successive approximation to "zero in" or to get as close as possible to a numeric result.

It is best to make a study of the available C keywords, syntax and functions available in the plain ANSI C current standard. You will find that it is not too difficult, and due to the fact that certain functions like printf() and scanf() are used rather extensively, and that test programs and-or functions to see the results can be made relatively short. You can always make programs as complex as you want from just using the basics, interest and program testing and practice.

to use the function does not need to worry about the program coding to do so, and
can simply concentrate on what the function is supposed to do.

This program example below is to demonstrate using the following homemade functions:

btod()   : binary to decimal conversion of an ASCII text string containing 1's and 0's
       ASCII text characters, hence a string of text, and the return values is a
       numeric integer. Programmed to work with up to an 8 bit unsigned byte value
       only, however, you may wish to include a function parameter for the sign, say
       0 for unsigned, and 1 for signed.

       unsigned char btod(char * binary_text_string);

dtob() : decimal to binary ASCII text string
       Programmed to work with 8 bit byte values only.

       void dtob(int number, char * binary_text_string);

Three extra homemade bit functions are included:

rotateright()   : rotates the bits of a byte left, and does not loose any bits or data
rotateleft()   : rotates the bits of a byte right, and does not loose any bits or data
complement() : C already has a built in complement function, for example: ~byte , and
       this will invert or change the state of each bit from 0 to 1, or 1 to 0.
       Result Ex. 254d = complement(1d) , 1d = complement(254d) or=:
          11111110b = ~00000001b , 00000001b = ~11111110b

The C programming language has standard ANSI, C functions to shift the bits of a byte,
but either the LSB (Least Significant Bit) or MSB (Most Significant Bit) is lost.
(c) JPA 2021
--*/

```c
#include "stdio.h"
#include "string.h"

        /* Function (compiler) prototypes for error checking,and usage:  */

unsigned char btod(char * binary_text_string);         /* binary to decimal */
void  dtob(int number, char * binary_text_string);    /* decimal to binary */

unsigned char rotateright(unsigned char byte);        /* rotate the bits of a byte, righward */
unsigned char rotateleft(unsigned char byte);         /* rotate the bits of a byte, leftward */
unsigned char complement(unsigned char byte);  /* change the state of each bit of a byte */

/*--------------------------------------------------------------------------------------------------------------------*/
void main(void)
{
unsigned char string1[9];
unsigned char b=0;
unsigned int d1=0;
unsigned char d=0;
int L=0;
int a=0;
```

```
        b=rotateleft(b);
        dtob(b,string1);
        printf("\nThe rotate-left value of this is: %d = %s",b,string1);

        b=d;
        b=rotateright(b);
        dtob(b,string1);
        printf("\nThe rotate-right value of this is: %d = %s",b,string1);
    };

printf("\nPress A Key To Exit The Program: "); fflush(stdin); getch();

return;
};
/*----------------------------------------------------------------------------------------------------------*/
unsigned char btod(char * binary_text_string)
{
unsigned char b=0;
int n=0;
unsigned char c=0;
unsigned char binaryweight=1;

/* Let's find the end of the binary string so as to begin the conversion at the
   least significant bit = LSB: */

for(;;){
        c=binary_text_string[n];
        if(c != 0){ n=n+1;  };
        if(c==0){  break;  };
    };

/* A debugging test:   printf("\nbits = %d",n);  fflush(stdin); getch();  */

n=n-1;  /* rather than use [n-1] below, and since the first data element is at offset 0 */
for(;;){   /* Loop to convert the binary value to its equivalent decimal value. */

        if(binary_text_string[n] == '1'){   b=b+binaryweight; };    /* '1' = ASCII 50  */
        binaryweight = binaryweight * 2;
        n=n-1;
        if(n<0){ break; };
    };

return b;
};
/*----------------------------------------------------------------------------------------------------------*/
void dtob(int number, char * binary_text_string)
{
int n=0;
unsigned char binaryweight=128;

strcpy(binary_text_string,"00000000\0");   /* : ensures a starting value with a zero value */

            for(;;){
                    if(number>=binaryweight){
```

Enter A Decimal Value (1 to 255) or enter 0 to exit loop: 0

Enter A Decimal Value 1 to 255, or 0 to Exit Loop: 3

The corresponding binary value is: 3 = 00000011
The rotate-left value of this is: 6 = 00000110
The rotate-right value of this is: 129 = 10000001   : rotate is like a byte or bit location shift, but does not loose any digits

Press A Key To Exit The Program:

A homemade way to left shift a binary value is to multiply it by 2, and for a right shift, divide that binary value by 2. When this is done the most significant bit or the least significant bit will be lost if its value (ie., 1 to 0) or some indication, such as a toggle variable (ie., set to 1 or 0) was not stored before the mathematical operation.

Ex.   A computer byte or char value of 145d = 10010001b , multiplied by 2 is: 00100010b = 34d

In a similar manner to the above reasoning, for example, when working with single bytes or characters (char) values only, they will "roll over, to zero" like an automobiles mile counter when the highest count is reached, and that value is 11111111b = 255d. When 1 is added to this value, the leading carry is lost unless it was somehow saved first, and that binary value will then become 0000000b = 0d. So that this does not happen when you need to consider higher values, you can first define the variables as integers which are often 2 bites ("wide" or "width") and have a maximum value before rollover of: 11111111 11111111b = 65535d.   By having various data type or byte sizes available in a compiler and computer, it helps increase speed and saves memory for other things.

```c
selection0: ;  /*-------------------------- PROGRAM MENU  -------------------------------------------------------*/

for(;;){
        system("cls");
            fflush(stdin);

        printf("\nTHIS PROGRAM HAS TWO FUNCTIONS:");
        printf("\n\n1. Feet To Meters Conversion");
        printf("\n2. Meters to Feet Conversion");
            printf("\n\nESC To Exit Program");

        printf("\n\nEnter Your Selection To Begin: "); fflush(stdin);  ch=getch();
        if(ch=='1'){ goto selection1; };  /* : Using the goto command is discouraged and is mainly
                                            used as a quick and easy method to change program
                                            exection flow within a C function.  Using goto is popular
                                            in the BASIC language  */
        if(ch=='2'){ goto selection2; };
            if(ch==27){ exit(0); };    /* : user pressed the ESC key */
    };

selection1:  ;

for(;;){                        /*---------------- main loop to use FeetToMeters()  --------------------*/
        system("cls");
        fflush(stdin);

        printf("FEET TO METERS CONVERSION PROGRAM.  (c) 2021, JPA");

        result=0.0;  feet=0.0;  inches=0.0; eighths = 0.0, sixteenths = 0.0, thirtytwoseconds = 0.0;
        totalmeters=0.0; totalcentimeters=0L;  mm=0.0;

        printf("\n");

        printf("\nWith this program up to 5 units can be used: \nfeet, inches, eighths of 1in, sixteenths of 1in, and
thirtytwoseconds of 1in");
        printf("\n\nEnter a value of 0 if it is not used.");
        printf("\nUse a negative value such as -1  to return to the MENU. ");

        printf("\n\nFor some reference: 1 ft = 0.3048m , 1in = 2.54cm , 1mile = 5280ft = 1609.34m");
        printf("\n1m = 39.37in = 3.28084ft ,  1/8in = 0.125in ,  1/16in = 0.0625in");
        printf("\n1/32in = 0.03125in  , quarter-inch = 1/4 (1in) = 0.25in ");
        printf("\n1 half-inch = 1/2 (1in) = 0.5in ,  three-fourths inch = 3/4 (1in) = 0.75in");
        printf("\n1.234 meters = 1 meters + 23 centimeters + 4 millimeters");

        printf("\n");
        printf("\nEnter feet = ");  fflush(stdin);  scanf("%lf",&feet);
        if(feet < 0){  goto selection0; }; /* ie., if a negative value was entered */

        printf("Enter inches = ");  fflush(stdin);  scanf("%lf",&inches); if(inches<0){ goto selection0; };
        printf("Enter eighths = ");   fflush(stdin);   scanf("%lf",&eighths); if(eighths<0){ goto selection0; };
        printf("Enter sixteenths = ");  fflush(stdin);  scanf("%lf",&sixteenths); if(sixteenths<0){ goto selection0; };
        printf("Enter thirtytwoseconds = ");  fflush(stdin);  scanf("%lf",&thirtytwoseconds);
                if(thirtytwoseconds<0){ goto selection0; };
```

```c
                ninches =  totalinches;
                printf("%.14g inches = ",totalinches);

                printf("\n%ld Feet + %ld inches + %.14g 16ths of an inch = ",
                    f,(unsigned long)ninches,((double)ninches - ((long int)ninches)) / (1.0/16.0) );

                printf("\n\nPress A Key To Continue With This Function or ESC For The MENU: ");
                fflush(stdin); ch=getch();  if(ch==27){ goto selection0; };
        };

return;
};
/*-------------------------------------------------------------------------------------------------------------------------*/
double FeetToMeters(double feet, double inches, double eighths, double sixteenths, double thirtytwoseconds )
{
/*  1ft = 12in =  0.3048m = 30.48cm   ,  1inch = 2.54cm =  25.4mm                        */
/*  1in = 1ft / 12  =  0.083333333 ft                                                    */
/*  1eighth = 1in/8  =  0.125in   ,  1ft = (8/in x 12) = 96 eighths                      */
/*  1sixteenth = 1in/16  =  0.0625in  =  (1in/8)/2  ,  1 ft = 192 sixteenths             */
/*  1thirtytwoseconds = 1in/32  =  0.03125   ,  1ft =   384 thirtytwoseconds             */

double result=0.0;

/* For simplicity, let's convert each argument to feet, and then multiply once by 0.3048 to
   convert the number of feet units to its equivalent value of  meter units.  */

result = 0.3048 * ( feet + (inches * 0.083333333333333) + (eighths/96.0) + (sixteenths/192.0) +
(thirtytwoseconds/384.0) ) ;

return result;
};
/*-------------------------------------------------------------------------------------------------------------------------*/
double MetersToFeet(double meters, double centimeters, double millimeters)
{
double result=0.0;

/* For simplicitry, let's convert each argument to meters, and then multiply by 3.28084 to
   convert the number of meter units to its equivalent value of feet units. */

result = (meters + (centimeters / 100.0) + (millimeters / 1000.0)) * 3.28084;

return result;
};
/*-------------------------------------------------------------------------------------------------------------------------*/
```

1.011 meters = 3.31692924 feet

3 Feet + 3.80315088 inches =
3 Feet + 3 inches + 12.85041408 16ths of an inch =
3 Feet + 3 inches + 25.70082816 32nds of an inch

Press A Key To Continue With This Function or ESC For The MENU:

Also of note, for example, in the above program to convert feet to meters, you may also initially ask the user if they want to convert a decimal value of feet to meters, and-or if they want to convert the number of feet and-or fractional parts of feet to meters.

```
                        printf("\n\aPlease Use An Angle Between 0 and 90 degrees");
                        printf("\nPress A Key To Continue or ESC to exit");
                        fflush(stdin); ch=getch(); if(ch==27){ break; };
                        continue;
                    };

            printf("\nThe sine of %.14g degrees is %.14g",angle,dsin(angle));
            printf("\nThe cosine of %.14g degrees is %.14g",angle,dcos(angle));
            printf("\nThe tangent of %.14g degrees is %.14g",angle,dtan(angle));

            printf("\n");
            printf("\nPress A Key To Continue or ESC to exit: ");
            fflush(stdin);  ch=getch();  if(ch==27){  break; };
        };

return;
};
/*-------------------------------------------------------------------------------*/
```

/* Output of the above example function use program:

1 rad is 57.295779513082 degrees.
The arcsine of 0.7 is 44.427004000806 degrees.
The tangent 45 degrees is 1

Press A Key
```
-------------------------------------------------------------------------------*/
```


```
/* --------------------------------------------------------------------------------------------------

    HomemadeMathFunctions.h


    Some examples of homemade math functions which can be used in any program that
    needs them.  You may also explore the programming topic of object files, libraries,
    and (static and dynamic) linking files.  This file is mostly about the code for some
    functions, and does not contain a main( ) function since it is already within the program
   you are creating.  This header file will be saved as a plain ASCII text file with a   .h filename
    extension so that you know what it is for.

    (c) JPA 2021
-----------------------------------------------------------------------------------------------------------*/
```

#include "math.h" /* : included here incase it was not included elsewhere, and since these
                        homemade functions will (function) call and use the standard C math functions. */

/* These conversions can also be done "inline" or in a function. The ANSI C, standard math functions
    use and return radian angles, and these conversion functions are useful for easily using degree
    values which are often used and needed more commonly.
*/
                        /* Function Prototypes, and often found in a header file, and they help describe to
                            the compiler and-or programmer what to expect and use for the argument data. */

double RadiansToDegrees(double radians);   /* : converts a radian angle to a degrees angle */
double DegreesToRadians(double degrees);   /* : converts a degrees angle to a radian angle */

```
     the sin() requires as an argument */

return cos(degreesangle * 0.0174532925199433);  /* 0.0174532925199433 =  (PI) / 180.0) */
                                                  /* PI ~ 3.141592653589793  */
};
/*----------------------------------------------------------------------------------------------------------*/
double dacos(double cos)  /* returns the degrees angle = arccos (cos angle) for the cos of the angle */
{
double angle=0.0;

angle=acos(cos);    /* C's acos() returns a radian angle */

/* lets convert this radian angle value returned by the std. math function, to it's corresponding
   degrees angle  */

return (angle * 57.29577951308231);              /* about 57.29577951308232 degrees,
                                                    per 1 rad.  */
                                                 /* = 180.0 / (pi = 3.141592653589793)  */
};
/*----------------------------------------------------------------------------------------------------------*/
double dtan(double degreesangle)   /* returns the tangent of a degrees angle value */
{
 /* Lets first convert the degrees angle argument to it's equivalent radian value that
    the sin() requires as an argument */

return tan(degreesangle * 0.0174532925199433);  /* 0.0174532925199433 =  PI/180.0) */
                                                 /* PI ~ 3.141592653589793  */
};
/*----------------------------------------------------------------------------------------------------------*/
double datan(double tan)  /* returns the degrees angle = arctan (tan angle) for the sin of the angle */
{
double angle=0.0;

angle=atan(tan);      /* C's atan() returns a radian angle */

/* lets convert this radian angle value returned by the std. math function, to it's corresponding
   degrees angle  */

return (angle * 57.29577951308231);              /* about 57.29577951308232 degrees,
                                                    per 1 rad.  */
                                                 /* = 180.0 / (pi = 3.141592653589793)  */
};
/*----------------------------------------------------------------------------------------------------------*/
```

```
/* It is quite possible to use this function twice so as to construct a
    double float numeric data type.  Enter the string and break it into
    two parts, the whole part and the fractional part.  Send each string
    part to this similar function, say from within a function called
    stringtodouble(char * string) ,  but for the fractional part , the
    weights or powers of 10 will be divided by 10.  A double float
    variable being created will be the sum of both of the whole or
    integer part, and the fractional part of 1.
*/
/*------------------------------------------------------------------------------*/
void start(void)   /* example program to use the function */
{
char string[20];
long int n;

printf("\nEnter an integer value: ");
scanf("%s",string);

n=stringtolongint(string);
printf("\nThe integer part of the value you entered is: %ld",n);

printf("\nThe square root of %ld is %.15g",n,sqrt(n));

/* printf("\nOn this computer the size of a long integer data value is:  \n%d bytes",sizeof(long int)); /*
        /* usually 4 bytes , for testing */

printf("\n\nPress A Key");  fflush(stdin);  getch(); exit(0);

return;
};
/*------------------------------------------------------------------------------*/
long int stringtolongint(char * string)
{
char ch;
long int n=0;
int sign=0;
char digits[10]={ 0,0,0,0,0,0,0,0,0,0 };  /* :This is actually a string, and with each byte initialized to 0 */
unsigned long int powerof10=1;
int l=0;

while(1)           /* Lets get each numeric digit into an array for processing. */
{
    ch=*string;    /* : Read the char or byte value at the address that string contains and points to. */
```

COMPILING A C PROGRAM WITH A C-COMPILER APP AND-OR MAKING A PHONE APP

The following are some basic and helpful notes about compiling a C program on a phone system and-or making a phone app (ie., application, a [computer, phone program] as of March 2022. The operating system of the phone must be a version of the Android Operating System, and does not consider iOS (ie., the iphone operating system) and phones developed by the Apple Co.   There are several C and-or C++ compilers available and they usually can be found on the Google Play ("App Store") website. The author will only discuss two such apps available that he has tried in a minimal manner, and for the following minimal discussion. Be sure to use an APK extractor so as to save any apps that you may need to have again if he phone is damaged and-or needs to be reset or "cleared" with a factory reset or software re-installation selection in the phones settings, and if so, any data stored on it will be erased.

In your internet (ie., "web") browser such as the Chrome browser, search for Google Play and go to that website.

1. **For the Mobile-C Compiler App**

   This is a easy to use basic C compiler, and is similar to the TinyC compiler in terms of simplicity and speed, however Mobile-C does not make Apps. It will run a C source code program, and is great for learning and-or editing programs. There is a free version and a pay version available. There are other compilers available in the app menu, such as: C++, Javascript, Python and Lua.

   On the Google Play search bar or input, enter: Mobile-C
   Choose to install the Mobile-C App.
   Open the Mobile-C App.

   In the settings menu, there are several default compilers to use. New programmers should select the (basic) C compiler. This compiler seems to be a version of the TinyC compiler (TCC).

   Your C source files can be accessed from the Mobile-C folder (ie., directory) or the external storage directory which is often where most of the user files for the phone are placed. You can press on the filename to load (ie., move, place) that source file into the C program editor, or press Run so as to compile and run (ie, execute or perform the program code or instructions) the file without editing. If you choose to edit the file, there is a Run button in the editor.

   To allow the Mobile C app to access to the External Storage to the app, and which is the phones main storage location, directory or folder, select your phone settings, and then select the Mobile-C app, and in its permissions, allow access to the phones storage.

   This app can also edit a .txt (ie. a text) file much like editing a .c file. This app can also view .jpg images when you press on its filename.

   The reader may also read the C4droid section below for some relevant information.

2. **For the C4droid Compiler and App maker App**

   This is an easy to use App that can edit a C source code files and also make an APK file which can then be installed as an runable or "executable" App on your phone.

   Press the Google Play Store App/Link on phone. On the Google Play Store search bar or input, enter: C4droid   (ie., C For the Android operating system on the phone). Choose to install the C4droid App. The price of it was about $4USD as of March 2022, and is a good deal. This can be paid for using a Google Pay account having your bank account and debit card assigned to it. Open the C4droid App. You may also want to download and-or install from the Google Play Store (ie. Google's certified apps website location) the free GCC, and SDL (Simple Direct media Layer Software , and is sometimes called a Software or Graphics Defined Library). SDL contains premade graphics functions for a certain electronic device such as a computer or smartphone. SDL is used as a "plugin" or addition to

and not from the popular App sites such as Google Play Store which tries to verify all apps that they have available as being safe for the device and free of bad things like malware. You may also be asked about the App not requiring or not-requiring any special (phone, system) access such as file access permission. Install the App and, if no special device access such as files for media and-or photos is needed, then select "Deny" as a safety measure. The created app will usually be found in your main App folder, and you can then press and hold on its icon and then "drag and drop" it to one of your phone's screen pages for easy access. The APK file that was made by the C4Droid compiler can be shared, traded and-or possibly sold. C4Droid will access files in the main internal storage memory > android > .com.app_name > files directory. You can select and copy a file into this directory.

Here is a basic example of how to access a file called myfile in the main Internal Storage directory or folder of the phone:

```
FILE *fp;       /* Creating a pointer to a file */
.
.
fp=fopen("/storage/emulated/0/myfile.txt","rb");      /* Setting the file pointer identified as fp to a certain file. */
if(fp==NULL){  printf("\nFile Error , Press A Key");  fflush(stdout);
                   fflush(stdin);  getch();  exit(0);
                 }
```

 Often by pressing on a file and selecting "File Info" from the menu, you can see what directory it is in, for example:

    The phones main storage folder is: /storage/emulated/0/
    Camera photos are often in this folder: /storage/emulated/0/DCIM/Camera/
    Screenshots are often in this folder: /storage/emulated/0/Pictures/Screenshots/
    Downloads are often in this folder: /storage/emulated/0/download/

Often a phone will come with a Google App called Files By Google, and this can be used to view the files in your phones internal storage. Most phones using the Android operating system also have a method to access your phones internal files, and that is to select: Settings menu or button > Storage > Files , and you can select to display it with the Android's Files App.   Note that a "folder" is analogous to the older terminology of a "file directory" or location.

When running your app program, you can select Menu , Preferences, and (Shell) Terminal type. For simplicity, select either the "screen" or "screen-256color". The default is usually Linux.   You may also wish to select the Close Window On Exit, and this will fully terminate the app.

 There could be issues with some functions, etc, such as with the system() function. For example to clear the screen: system("cls") might not work in the resulting exportedl app. An alternative is to try using just this: clrscr(). Check the menu options and-or uses GCC instead of TCC. You could always print several blank lines to effectually or crudely clear the screen. Be sure to include all the header files needed by the functions in the program. Sometimes there are known solutions to try for certain issues, such as using fflush(0), for to process and clear all buffers, or fflush(1) instead of fflush(stdin) of which can be used before a user input such as: printf("\nPress A Key"); fflush(0); getch(). Also consider using fflush(stdout) so as to be sure text is displayed on the screen. Avoid using the ASCII bell sound character, such as in printf("\a"); ,because it may not work and-or cause issues with the Android phone operating system and cause the program to crash. Instead of using something like #include "stdio.h", rather use: #include <stdio.h> . Seek assistance from people and-or websites more knowledgeable if you need more help. Often, a solution is relatively simple.

The c4droid compiler wants the main() to have an integer return value, rather than void. So use something such as: int main(void)   ,   and at the end of the program use something like: return 0 , or some other integer value.

"DOS" or "terminal" programs.   Many of these older types of computers and-or systems also came with the BASIC language installed and-or on disk.   Today, we can download a BASIC language compiler so as to still use BASIC.   I believe the C computer programming language was created to overcome some limitations of the BASIC computer language which itself was a giant leap from the more difficult to read, edit, and-or program machine or assembly programming languages.   The DOS (Disk Operating [ie., accessing and control] System is user accessed by typing in special DOS commands which are technically a type of (system command and control) programming language or tool.   A BATCHFILE is simply a plain text file, much like a C language program, but with the .bat filename extension, and is essentially a list of DOS commands which may include some decision ability such as the Basic-like IF and GOTO commands. The commands are generally processed from the top of the file to the bottom, one by one just like regular computer programs. Modern forms of DOS seem to have less commands available to the user.   A batchfile is a program of DOS or system commands. To see a list of the DOS commands, enter: HELP on the command line.   To see the specific HELP data about a particular command, enter the commands name followed by a space character and-or then: /?

For much information about programming a device with the Android operating system, please visit: https://developer.android.com   There is also a link to this website and it is given on a page that is called something like: "Browse All Of Googles Produces And Services", and it was currently: https://about.google/products   , and then select the menu selection called: Android OS (ie., Android Operating System). There are also many websites and-or webpages throughout the internet that offer many tips and-or answered questions about programming any well known language and-or program.   Some other search words for Android programming are: Android App Development Kit, Android SDK (Software Development Kit) and Android Studio. There are also various websites that have associated programmers who can make apps for you for a fee, and just let them know what you are considering and negotiate a fee. You may need to first copyright your idea(s) or be able to somehow verify you were its creator, perhaps in a previous email to yourself and-or to a friend to save, or perhaps a published (ie., publicized, offered to the general public) video, and before letting others utilize them for their own control and benefit.   Even if you let others use your ideas or app and-or a version of it freely, you still need to have some type of copyright or verification so as others will not claim and-or independently create the same general type of idea, and then they may say that you and-others using it can no longer use their idea or need permission and-or a fee to use it.

After making the C source code for a program with C4Droid, I have noticed that this file was not displayed right in Windows, Notepad program. In particular, it was missing some new lines and was seen as jumbled together.   Notepad is supposed to be a pure/only ASCII text editor. I put this .c source code into the DOS EDIT.com text editor and it was displayed right, and could then be saved and then displayed in Windows Notepad. An option to correct the .c source code for proper display in Notepad is to make a copy of the file using a homemade program as shown in this book, and when a byte value of 10 (ASCII for LF = Line Feed) is read, then first write a byte value of 13 (ASCII for CR = Carriage Return), and then write the byte value of 10.
In short byte 10 will then effectively become a byte 13 and then 10 sequence or CR, LF sequence.

0.049	0.0008552113334772	0.0008552112292289	0.9999996343068	0.0008552115419739
0.050	0.0008726646259972	0.0008726645152352	0.9999996192282	0.0008726648475213
0.051	0.0008901179185171	0.0008901178009756	0.9999996038451	0.0008901181536003
0.052	0.0009075712110371	0.0009075710864448	0.9999995881573	0.0009075714602216
0.053	0.000925024503557	0.0009250243716377	0.9999995721649	0.0009250247673958
0.054	0.0009424777960769	0.0009424776565487	0.9999995558678	0.0009424780751335
0.055	0.0009599310885969	0.0009599309411726	0.9999995392662	0.0009599313834455
0.056	0.0009773843811168	0.0009773842255042	0.9999995223599	0.0009773846923423
0.057	0.0009948376736368	0.000994837509538	0.999999505149	0.0009948380018345
0.058	0.001012290966157	0.001012290793269	0.9999994876335	0.001012291311933
0.059	0.001029744258677	0.001029744076691	0.9999994698134	0.001029744622648
0.060	0.001047197551197	0.0010471973598	0.9999994516887	0.00104719793399
0.061	0.001064650843717	0.00106465064259	0.9999994332593	0.001064651245971
0.062	0.001082104136236	0.001082103925055	0.9999994145254	0.0010821045586
0.063	0.001099557428756	0.001099557207191	0.9999993954868	0.001099557871888
0.064	0.001117010721276	0.001117010488992	0.9999993761436	0.001117011185846
0.065	0.001134464013796	0.001134463770452	0.9999993564958	0.001134464500485
0.066	0.001151917306316	0.001151917051567	0.9999993365433	0.001151917815815
0.067	0.001169370598836	0.001169370332331	0.9999993162863	0.001169371131846
0.068	0.001186823891356	0.001186823612739	0.9999992957246	0.00118682444859
0.069	0.001204277183876	0.001204276892786	0.9999992748583	0.001204277766058
0.070	0.001221730476396	0.001221730172465	0.9999992536874	0.001221731084258
0.071	0.001239183768916	0.001239183451772	0.9999992322119	0.001239184403203
0.072	0.001256637061436	0.001256636730702	0.9999992104318	0.001256637722904
0.073	0.001274090353956	0.001274090009249	0.999999188347	0.001274091043369
0.074	0.001291543646476	0.001291543287408	0.9999991659576	0.001291544364611
0.075	0.001308996938996	0.001308996565174	0.9999991432636	0.00130899768664
0.076	0.001326450231516	0.001326449842541	0.999999120265	0.001326451009466
0.077	0.001343903524036	0.001343903119504	0.9999990969618	0.0013439043331
0.078	0.001361356816556	0.001361356396057	0.999999073354	0.001361357657554
0.079	0.001378810109076	0.001378809672196	0.9999990494415	0.001378810982836
0.080	0.001396263401595	0.001396262947914	0.9999990252244	0.001396264308959
0.081	0.001413716694115	0.001413716223208	0.9999990007027	0.001413717635932
0.082	0.001431169986635	0.00143116949807	0.9999989758764	0.001431170963766
0.083	0.001448623279155	0.001448622772497	0.9999989507455	0.001448624292473
0.084	0.001466076571675	0.001466076046483	0.9999989253099	0.001466077622062
0.085	0.001483529864195	0.001483529320021	0.9999988995698	0.001483530952544
0.086	0.001500983156715	0.001500982593108	0.999998873525	0.00150098428393
0.087	0.001518436449235	0.001518435865738	0.9999988471756	0.00151843761623
0.088	0.001535889741755	0.001535889137905	0.9999988205216	0.001535890949456
0.089	0.001553343034275	0.001553342409605	0.999998793563	0.001553344283617
0.090	0.001570796326795	0.001570795680831	0.9999987662997	0.001570797618724
0.091	0.001588249619315	0.001588248951579	0.9999987387318	0.001588250954789
0.092	0.001605702911835	0.001605702221842	0.9999987108594	0.001605704291821
0.093	0.001623156204355	0.001623155491617	0.9999986826823	0.0016231576298325
0.094	0.001640609496875	0.001640608760897	0.9999986542005	0.001640610968831
0.095	0.001658062789395	0.001658062029678	0.9999986254142	0.00165806430883
0.096	0.001675516081915	0.001675515297953	0.9999985963233	0.001675517649839
0.097	0.001692969374435	0.001692968565719	0.9999985669277	0.001692970991868
0.098	0.001710422666954	0.001710421832968	0.9999985372275	0.00171042433493
0.099	0.001727875959474	0.001727875099696	0.9999985072227	0.001727877679033

0.62	0.01082104136236	0.01082083018208	0.9999414531032	0.0108214637452
0.63	0.01099557428756	0.01099535272322	0.9999395492821	0.01099601744037
0.64	0.01117010721276	0.01116987492942	0.9999376150011	0.01117057180553
0.65	0.01134464013796	0.01134439679537	0.9999356502602	0.01134512685133
0.66	0.01151917306316	0.01151891831575	0.9999336550596	0.01151968258841
0.67	0.01169370598836	0.01169343948525	0.9999316293992	0.01169423902739
0.68	0.01186823891356	0.01186796029854	0.9999295732792	0.01186879617893
0.69	0.01204277183876	0.01204248075031	0.9999274866996	0.01204335405366
0.70	0.01221730476396	0.01221700083525	0.9999253696605	0.01221791266222
0.71	0.01239183768916	0.01239152054803	0.9999232221618	0.01239247201524
0.72	0.01256637061436	0.01256603988335	0.9999210442038	0.01256703212338
0.73	0.01274090353956	0.01274055883589	0.9999188357865	0.01274159299726
0.74	0.01291543646476	0.01291507740032	0.9999165969098	0.01291615464753
0.75	0.01308996938996	0.01308959557134	0.999914327574	0.01309071708484
0.76	0.01326450231516	0.01326411334363	0.999912027779	0.0132652803198
0.77	0.01343903524036	0.01343863071187	0.999909697525	0.01343984436308
0.78	0.01361356816556	0.01361314767075	0.999907336812	0.01361440922531
0.79	0.01378810109076	0.01378766421495	0.9999049456401	0.01378897491713
0.80	0.01396263401595	0.01396218033915	0.9999025240093	0.01396354144918
0.81	0.01413716694115	0.01413669603803	0.9999000719198	0.01413810883211
0.82	0.01431169986635	0.01431121130629	0.9998975893715	0.01431267707655
0.83	0.01448623279155	0.01448572613861	0.9998950763646	0.01448724619314
0.84	0.01466076571675	0.01466024052966	0.9998925328992	0.014661181619254
0.85	0.01483529864195	0.01483475447414	0.9998899589753	0.01483638708538
0.86	0.01500983156715	0.01500926796672	0.9998873545931	0.0150109588823
0.87	0.01518436449235	0.0151837810021	0.9998847197525	0.01518553159394
0.88	0.01535889741755	0.01535829357495	0.9998820544537	0.01536010523096
0.89	0.01553343034275	0.01553280567996	0.9998793586967	0.01553467980398
0.90	0.01570796326795	0.01570731731182	0.9998766324817	0.01570925532366
0.91	0.01588249619315	0.0158818284652	0.9998738758086	0.01588383180065
0.92	0.01605702911835	0.0160563391348	0.9998710886777	0.01605840924557
0.93	0.01623156204355	0.01623084931529	0.999868271089	0.01623298766908
0.94	0.01640609496875	0.01640535900136	0.9998654230425	0.01640756708182
0.95	0.01658062789395	0.0165798681877	0.9998625445384	0.01658214749444
0.96	0.01675516081915	0.01675437686898	0.9998596355768	0.01675672891757
0.97	0.01692969374435	0.0169288850399	0.9998566961577	0.01693131136187
0.98	0.01710422666954	0.01710339269513	0.9998537262812	0.01710589483798
0.99	0.01727875959474	0.01727789982936	0.9998507259474	0.01728047935655
1.0	0.01745329251994	0.01745240643728	0.9998476951564	0.01745506492822
1.1	0.01919862177194	0.01919744239969	0.9998157121216	0.01920098090772
1.2	0.02094395102393	0.02094241988336	0.9997806834748	0.02094701390966
1.3	0.02268928027593	0.02268733357278	0.9997426093227	0.02269317458436
1.4	0.02443460952792	0.02443217815265	0.9997014897812	0.02443947358526
1.5	0.02617993877991	0.02617694830787	0.9996573249756	0.02618592156919
1.6	0.02792526803191	0.02792163872357	0.9996101150404	0.02793252919659
1.7	0.0296705972839	0.02966624408511	0.9995598601194	0.02967930713181
1.8	0.0314159265359	0.03141075907813	0.9995065603657	0.03142626604335
1.9	0.03316125578789	0.03315517838853	0.9994502159418	0.03317341660413
2.0	0.03490658503989	0.0348994967025	0.9993908270191	0.034920769491175
2.1	0.03665191429188	0.03664370870656	0.9993283937787	0.03666833538873
2.2	0.03839724354388	0.03838780908752	0.9992629164106	0.03841612498281
2.3	0.04014257279587	0.04013179253256	0.9991943951144	0.04016414896719

7.7	0.1343903524036	0.1339861854183	0.9909831997148	0.1352053046478
7.8	0.1361356816556	0.1357155724343	0.9907478404714	0.1369829606388
7.9	0.1378810109076	0.1374445460371	0.9905094632383	0.1387614668393
8.0	0.1396263401595	0.1391731009601	0.9902680687416	0.1405408347024
8.1	0.1413716694115	0.1409012319376	0.9900236577166	0.1423210757029
8.2	0.1431169986635	0.1426289337055	0.9897762309078	0.1441022013377
8.3	0.1448623279155	0.144356201001	0.989525789069	0.1458842231255
8.4	0.1466076571675	0.1460830285624	0.989272332963	0.1476671526079
8.5	0.1483529864195	0.1478094111296	0.9890158633619	0.1494510013491
8.6	0.1500983156715	0.1495353434437	0.988756381047	0.1512357809366
8.7	0.1518436449235	0.1512608202472	0.9884938868087	0.1530215029812
8.8	0.1535889741755	0.152985836284	0.9882283814466	0.1548081791176
8.9	0.1553343034275	0.1547103862995	0.9879598657694	0.1565958210043
9.0	0.1570796326795	0.1564344650402	0.9876883405951	0.1583844403245
9.1	0.1588249619315	0.1581580672545	0.9874138067509	0.160174048786
9.2	0.1605702911835	0.1598811876918	0.987136265073	0.1619646581215
9.3	0.1623156204355	0.1616038211034	0.9868557164068	0.163756280089
9.4	0.1640609496875	0.1633259622416	0.986572161607	0.1655489264724
9.5	0.1658062789395	0.1650476058607	0.9862856015372	0.1673426090814
9.6	0.1675516081915	0.1667687467161	0.9859960370705	0.169137339752
9.7	0.1692969374434	0.168489379565	0.9857034690889	0.1709331303467
9.8	0.1710422666954	0.170209499166	0.9854078984835	0.1727299927553
9.9	0.1727875959474	0.1719291002794	0.9851093261548	0.1745279388944
10.0	0.1745329251994	0.1736481776669	0.9848077530122	0.1763269807085
10.1	0.1762782544514	0.175366726092	0.9845031799744	0.1781271301699
10.2	0.1780235837034	0.1770847403196	0.9841956079692	0.1799283992793
10.3	0.1797689129554	0.1788022151163	0.9838850379335	0.1817308000657
10.4	0.1815142422074	0.1805191452506	0.9835714708134	0.1835343445874
10.5	0.1832595714594	0.1822355254921	0.983254907564	0.1853390449315
10.6	0.1850049007114	0.1839513506127	0.9829353491496	0.1871449132152
10.7	0.1867502299634	0.1856666153856	0.9826127965436	0.1889519615851
10.8	0.1884955592154	0.1873813145857	0.9822872507287	0.1907602022186
10.9	0.1902408884674	0.1890954429899	0.9819587126964	0.1925696473232
11.0	0.1919862177194	0.1908089953765	0.9816271834477	0.1943803091377
11.1	0.1937315469714	0.1925219665259	0.9812926639922	0.1961921999321
11.2	0.1954768762234	0.19423435122	0.9809551553492	0.1980053320081
11.3	0.1972222054754	0.1959461442425	0.9806146585466	0.1998197176992
11.4	0.1989675347274	0.1976573403791	0.9802711746217	0.2016353693715
11.5	0.2007128639793	0.1993679344172	0.9799247046208	0.2034522994237
11.6	0.2024581932313	0.201077921146	0.9795752495993	0.2052705202874
11.7	0.2042035224833	0.2027872953565	0.9792228106218	0.2070900444279
11.8	0.2059488517353	0.2044960518418	0.9788673887617	0.2089108843441
11.9	0.2076941809873	0.2062041853966	0.9785089851018	0.2107330525689
12.0	0.2094395102393	0.2079116908178	0.9781476007338	0.21255656167
12.1	0.2111848394913	0.2096185629038	0.9777832367586	0.2143814242497
12.2	0.2129301687433	0.2113247964554	0.9774158942861	0.2162076529457
12.3	0.2146754979953	0.213030386275	0.9770455744353	0.218035260431
12.4	0.2164208272473	0.2147353271671	0.9766722783342	0.219864259415
12.5	0.2181661564993	0.2164396139381	0.9762960071199	0.2216946626429
12.6	0.2199114857513	0.2181432413965	0.9759167619387	0.2235264828971
12.7	0.2216568150033	0.2198462043528	0.9755345439459	0.2253597329968
12.8	0.2234021442553	0.2215484976195	0.9751493543056	0.2271944257987

18.2	0.317649923863	0.3123349185122	0.9499720515247	0.3287832710562
18.3	0.319395253115	0.3139924559674	0.9494254776419	0.3307183800747
18.4	0.321140582367	0.3156490369471	0.9488760116445	0.3326557243238
18.5	0.322885911619	0.3173046564051	0.9483236552062	0.3345953195021
18.6	0.3246312408709	0.3189593092981	0.9477684100096	0.336537181372
18.7	0.3263765701229	0.3206129905857	0.947210277746	0.3384813257607
18.8	0.3281218993749	0.3222656952305	0.9466492601157	0.3404277685604
18.9	0.3298672286269	0.3239174181981	0.9460853588275	0.3423765257287
19.0	0.3316125578789	0.3255681544572	0.9455185755993	0.3443276132897
19.1	0.3333578871309	0.3272178989791	0.9449489121575	0.3462810473341
19.2	0.3351032163829	0.3288666467386	0.9443763702375	0.34823684402
19.3	0.3368485456349	0.3305143927132	0.9438009515832	0.3501950195735
19.4	0.3385938748869	0.3321611318837	0.9432226579476	0.3521555902892
19.5	0.3403392041389	0.3338068592338	0.9426414910922	0.3541185725307
19.6	0.3420845333909	0.3354515697503	0.9420574527873	0.3560839827314
19.7	0.3438298626429	0.3370952584231	0.941470544812	0.3580518373949
19.8	0.3455751918949	0.3387379202453	0.9408807689542	0.3600221530958
19.9	0.3473205211469	0.3403795502131	0.9402881270104	0.3619949464801
20.0	0.3490658503989	0.3420201433257	0.9396926207859	0.3639702342662
20.1	0.3508111796509	0.3436596945856	0.9390942520947	0.3659480332449
20.2	0.3525565089029	0.3452981989985	0.9384930227596	0.3679283602804
20.3	0.3543018381548	0.3469356515733	0.9378889346119	0.3699112323111
20.4	0.3560471674068	0.3485720473218	0.9372819894919	0.3718966663499
20.5	0.3577924966588	0.3502073812595	0.9366721892484	0.3738846794848
20.6	0.3595378259108	0.3518416484047	0.936059535739	0.3758752888799
20.7	0.3612831551628	0.3534748437793	0.9354440308299	0.3778685117758
20.8	0.3630284844148	0.3551069624081	0.934825676396	0.3798643654902
20.9	0.3647738136668	0.3567379993196	0.934204474321	0.3818628674187
21.0	0.3665191429188	0.3583679495453	0.9335804264972	0.3838640350354
21.1	0.3682644721708	0.3599968081201	0.9329535348255	0.3858678858936
21.2	0.3700098014228	0.3616245700821	0.9323238012155	0.3878744376263
21.3	0.3717551306748	0.363251230473	0.9316912275855	0.3898837079472
21.4	0.3735004599268	0.3648767843376	0.9310558158625	0.3918957146512
21.5	0.3752457891788	0.3665012267243	0.930417567982	0.3939104756149
21.6	0.3769911184308	0.3681245526847	0.9297764858883	0.3959280087977
21.7	0.3787364476828	0.3697467572738	0.9291325715341	0.3979483322421
21.8	0.3804817769348	0.3713678355502	0.9284858268809	0.3999714640748
21.9	0.3822271061868	0.3729877825758	0.9278362538989	0.4019974225069
22.0	0.3839724354388	0.3746065934159	0.9271838545668	0.4040262258352
22.1	0.3857177646907	0.3762242631394	0.9265286308718	0.4060578924424
22.2	0.3874630939427	0.3778407868185	0.92587058481	0.4080924407983
22.3	0.3892084231947	0.379456159529	0.9252097183858	0.4101298894601
22.4	0.3909537524467	0.3810703763503	0.9245460336123	0.4121702570735
22.5	0.3926990816987	0.3826834323651	0.9238795325113	0.4142135623731
22.6	0.3944444109507	0.3842953226598	0.923210217113	0.4162598241834
22.7	0.3961897402027	0.3859060423243	0.9225380894562	0.4183090614197
22.8	0.3979350694547	0.3875155864521	0.9218631515885	0.4203612930882
22.9	0.3996803987067	0.3891239501402	0.9211854055657	0.4224165382877
23.0	0.4014257279587	0.3907311284893	0.9205048534524	0.4244748162096
23.1	0.4031710572107	0.3923371166036	0.9198214973217	0.4265361461392
23.2	0.4049163864627	0.393941909591	0.9191353392552	0.428600547456
23.3	0.4066617157147	0.395545502563	0.9184463813431	0.4306680396351
23.4	0.4084070449667	0.3971478906348	0.917754625684	0.4327386422474

28.8	0.5026548245744	0.4817536741017	0.8763066800439	0.5497546521928
28.9	0.5044001538264	0.483282383255	0.875464527	0.552029657799
29.0	0.5061454830784	0.4848096202463	0.8746197071394	0.5543090514528
29.1	0.5078908123304	0.4863353804235	0.8737722230355	0.556592859789
29.2	0.5096361415823	0.4878596591387	0.8729220772698	0.5588811095998
29.3	0.5113814708343	0.4893824517488	0.8720692724321	0.561173827836
29.4	0.5131268000863	0.4909037536151	0.8712138111202	0.563471041608
29.5	0.5148721293383	0.4924235601035	0.8703556959399	0.5657727781878
29.6	0.5166174585903	0.4939418665842	0.8694949295052	0.56807906501
29.7	0.5183627878423	0.4954586684324	0.8686315144382	0.5703899296733
29.8	0.5201081170943	0.4969739610276	0.8677654533689	0.5727053999421
29.9	0.5218534463463	0.4984877397538	0.8668967489356	0.5750255037476
30.0	**0.5235987755983**	**0.5**	**0.8660254037844**	**0.5773502691896**
30.1	0.5253441048503	0.5015107371595	0.8651514205697	0.5796797245379
30.2	0.5270894341023	0.5030199466302	0.8642748019537	0.5820138982337
30.3	0.5288347633543	0.504527623815	0.8633955506068	0.584352818891
30.4	0.5305800926063	0.5060337641212	0.8625136692073	0.5866965152984
30.5	0.5323254218583	0.5075383629607	0.8616291604415	0.5890450164206
30.6	0.5340707511103	0.5090414157504	0.8607420270039	0.5913983513995
30.7	0.5358160803623	0.5105429179116	0.8598522715969	0.5937565495564
30.8	0.5375614096143	0.5120428648706	0.8589598969307	0.5961196403933
30.9	0.5393067388663	0.5135412520582	0.8580649057236	0.5984876535943
31.0	0.5410520681182	0.5150380749101	0.8571673007021	0.6008606190276
31.1	0.5427973973702	0.5165333288666	0.8562670846003	0.6032385667467
31.2	0.5445427266222	0.5180270093731	0.8553642601605	0.6056215269924
31.3	0.5462880558742	0.5195191118795	0.8544588301328	0.6080095301944
31.4	0.5480333851262	0.5210096318406	0.8535507972753	0.6104026069728
31.5	0.5497787143782	0.522498564716	0.8526401643541	0.6128007881399
31.6	0.5515240436302	0.5239859059701	0.851726934143	0.615204104702
31.7	0.5532693728822	0.5254716510723	0.850811109424	0.617612587861
31.8	0.5550147021342	0.5269557954967	0.8498926929869	0.6200262690161
31.9	0.5567600313862	0.5284383347223	0.8489716876291	0.6224451797657
32.0	0.5585053606382	0.5299192642332	0.8480480961564	0.6248693519093
32.1	0.5602506898902	0.5313985795181	0.8471219213821	0.6272988174489
32.2	0.5619960191422	0.5328762760707	0.8461931661276	0.6297336085912
32.3	0.5637413483942	0.5343523493898	0.8452618332219	0.6321737577492
32.4	0.5654866776462	0.535826794979	0.844327925502	0.6346192975442
32.5	0.5672320068982	0.5372996083468	0.8433914458129	0.6370702608075
32.6	0.5689773361502	0.5387707850069	0.8424523970071	0.6395266805826
32.7	0.5707226654021	0.5402403204777	0.8415107819453	0.6419885901269
32.8	0.5724679946541	0.5417082102827	0.8405666034957	0.6444560229135
32.9	0.5742133239061	0.5431744499507	0.8396198645344	0.6469290126334
33.0	0.5759586531581	0.544639035015	0.8386705679454	0.6494075931975
33.1	0.5777039824101	0.5461019610144	0.8377187166204	0.6518917987383
33.2	0.5794493116621	0.5475632234926	0.836764313459	0.6543816636122
33.3	0.5811946409141	0.5490228179981	0.8358073613683	0.6568772224013
33.4	0.5829399701661	0.550480740085	0.8348478632634	0.6593785099158
33.5	0.5846852994181	0.5519369853121	0.8338858220672	0.6618855611957
33.6	0.5864306286701	0.5533915492433	0.8329212407101	0.6643984115131
33.7	0.5881759579221	0.554844427448	0.8319541221305	0.6669170963744
33.8	0.5899212871741	0.5562956155003	0.8309844692743	0.6694416515222
33.9	0.5916666164261	0.5577451089797	0.8300122850954	0.6719721129376
34.0	0.5934119456781	0.5591929034707	0.829037572555	0.6745085168424

39.4	0.6876597252858	0.6347305132023	0.7727335734973	0.8214092605418
39.5	0.6894050545378	0.6360782202778	0.7716245833877	0.8243363858175
39.6	0.6911503837898	0.6374239897487	0.7705132427758	0.8272719459725
39.7	0.6928957130418	0.6387678175156	0.7693995550469	0.8302159954806
39.8	0.6946410422937	0.640109699485	0.7682835235935	0.8331685892351
39.9	0.6963863715457	0.6414496315692	0.7671651518153	0.8361297825525
40.0	0.6981317007977	0.6427876096865	0.766044443119	0.8390996311773
40.1	0.6998770300497	0.6441236297614	0.7649214009184	0.8420781912861
40.2	0.7016223593017	0.645457687724	0.7637960286346	0.8450655194919
40.3	0.7033676885537	0.6467897795105	0.7626683296957	0.8480616728487
40.4	0.7051130178057	0.6481199010631	0.7615383075367	0.8510667088561
40.5	0.7068583470577	0.6494480483302	0.7604059656	0.8540806854635
40.6	0.7086036763097	0.6507742172659	0.7592713073349	0.8571036610749
40.7	0.7103490055617	0.6520984038304	0.7581343361976	0.8601356945537
40.8	0.7120943348137	0.6534206039901	0.7569950556518	0.8631768452273
40.9	0.7138396640657	0.6547408137173	0.7558534691676	0.8662271728915
41.0	0.7155849933177	0.6560590289905	0.7547095802228	0.8692867378162
41.1	0.7173303225697	0.6573752457941	0.7535633923016	0.8723556007495
41.2	0.7190756518217	0.6586894601187	0.7524149088957	0.8754338229228
41.3	0.7208209810737	0.6600016679609	0.7512641335035	0.8785214660562
41.4	0.7225663103257	0.6613118653237	0.7501110696305	0.8816185923632
41.5	0.7243116395777	0.6626200482157	0.748955720789	0.884725264556
41.6	0.7260569688296	0.6639262126522	0.7477980904985	0.8878415458505
41.7	0.7278022980816	0.6652303546544	0.7466381822854	0.8909674999719
41.8	0.7295476273336	0.6665324702495	0.7454759996829	0.8941031911598
41.9	0.7312929565856	0.6678325554711	0.7443115462312	0.8972486841735
42.0	0.7330382858376	0.6691306063589	0.7431448254774	0.9004040442979
42.1	0.7347836150896	0.6704266189588	0.7419758409756	0.9035693373483
42.2	0.7365289443416	0.671720589323	0.7408045962867	0.9067446296769
42.3	0.7382742735936	0.6730125135098	0.7396310949786	0.9099299881777
42.4	0.7400196028456	0.6743023875837	0.7384553406259	0.9131254802927
42.5	0.7417649320976	0.6755902076157	0.7372773368101	0.9163311740174
42.6	0.7435102613496	0.6768759696827	0.7360970871197	0.9195471379071
42.7	0.7452555906016	0.6781596698681	0.73491459515	0.9227734410822
42.8	0.7470009198536	0.6794413042615	0.7337298645029	0.9260101532352
42.9	0.7487462491056	0.6807208689589	0.7325428987874	0.9292573446357
43.0	0.7504915783576	0.6819983600625	0.7313537016192	0.9325150861377
43.1	0.7522369076096	0.6832737736808	0.7301622766207	0.9357834491848
43.2	0.7539822368616	0.6845471059287	0.7289686274214	0.9390625058175
43.3	0.7557275661136	0.6858183529274	0.7277727576572	0.942352328679
43.4	0.7574728953655	0.6870875108044	0.726574670971	0.9456529910219
43.5	0.7592182246175	0.6883545756938	0.7253743710123	0.9489645667149
43.6	0.7609635538695	0.6896195437357	0.7241718614375	0.9522871302494
43.7	0.7627088831215	0.6908824110769	0.7229671459096	0.9556207567464
43.8	0.7644542123735	0.6921431738704	0.7217602280984	0.9589655219629
43.9	0.7661995416255	0.6934018282758	0.7205511116803	0.9623215022995
44.0	0.7679448708775	0.694658370459	0.7193398003386	0.9656887748071
44.1	0.7696902001295	0.6959127965923	0.7181262977632	0.9690674171938
44.2	0.7714355293815	0.6971651028546	0.7169106076505	0.9724575078326
44.3	0.7731808586335	0.698415285431	0.7156927337037	0.9758591257685
44.4	0.7749261878855	0.6996633405134	0.7144726796328	0.9792723507258
44.5	0.7766715171375	0.7009092642999	0.7132504491542	0.9826972631157
44.6	0.7784168463895	0.7021530529952	0.712026045991	0.986133944044

50.0	0.8726646259972	0.766044443119	0.6427876096865	1.191753592594
50.1	0.8744099552492	0.7671651518153	0.6414496315692	1.195986581111
50.2	0.8761552845012	0.7682835235935	0.6401096994849	1.20023727841
50.3	0.8779006137532	0.7693995550469	0.6387678175156	1.204505821911
50.4	0.8796459430051	0.7705132427758	0.6374239897487	1.20879235041
50.5	0.8813912722571	0.7716245833877	0.6360782202778	1.213097004093
50.6	0.8831366015091	0.7727335734974	0.6347305132023	1.217419924558
50.7	0.8848819307611	0.7738402097265	0.6333808726275	1.22176125483
50.8	0.8866272600131	0.7749444887042	0.6320293026648	1.226121139379
50.9	0.8883725892651	0.7760464070666	0.6306758074313	1.230499724141
51.0	0.8901179185171	0.777145961457	0.6293203910498	1.234897156535
51.1	0.8918632477691	0.778243148526	0.6279630576493	1.239313585483
51.2	0.8936085770211	0.7793379649315	0.6266038113645	1.243749161427
51.3	0.8953539062731	0.7804304073383	0.6252426563357	1.248204036353
51.4	0.8970992355251	0.7815204724188	0.6238795967094	1.252678363808
51.5	0.8988445647771	0.7826081568524	0.6225146366376	1.257172298919
51.6	0.9005898940291	0.7836934573258	0.6211477802783	1.261685998418
51.7	0.9023352232811	0.7847763705331	0.6197790317951	1.266219620661
51.8	0.9040805525331	0.7858568931754	0.6184083953575	1.270773325645
51.9	0.9058258817851	0.7869350219613	0.6170358751407	1.275347275038
52.0	0.9075712110371	0.7880107536067	0.6156614753257	1.279941632193
52.1	0.9093165402891	0.7890840848347	0.6142852000989	1.284556562176
52.2	0.911061869541	0.7901550123757	0.612907053653	1.289192231785
52.3	0.912807198793	0.7912235329675	0.6115270401858	1.293848809575
52.4	0.914552528045	0.7922896433552	0.6101451639013	1.298526465881
52.5	0.916297857297	0.7933533402912	0.6087614290087	1.303225372841
52.6	0.918043186549	0.7944146205354	0.6073758397233	1.307945704421
52.7	0.919788515801	0.7954734808549	0.6059884002657	1.31268763644
52.8	0.921533845053	0.7965299180242	0.6045991148624	1.317451346593
52.9	0.923279174305	0.7975839288252	0.6032079877453	1.322237014477
53.0	0.925024503557	0.7986355100473	0.601815023152	1.32704482162
53.1	0.926769832809	0.7996846584871	0.6004202253259	1.331874951503
53.2	0.928515162061	0.8007313709487	0.5990235985156	1.336727589586
53.3	0.930260491313	0.8017756442438	0.5976251469755	1.34160292334
53.4	0.932005820565	0.8028174751911	0.5962248749656	1.34650114227
53.5	0.933751149817	0.8038568606172	0.5948227867513	1.351422437946
53.6	0.935496479069	0.8048937973559	0.5934188866037	1.356367004027
53.7	0.937241808321	0.8059282822485	0.5920131787992	1.361335036296
53.8	0.938987137573	0.8069603121438	0.5906056676199	1.366326732684
53.9	0.940732466825	0.807989883898	0.5891963573533	1.371342293302
54.0	0.9424777960769	0.809016994375	0.5877852522925	1.376381920471
54.1	0.9442231253289	0.8100416404458	0.5863723567358	1.381445818754
54.2	0.9459684545809	0.8110638189893	0.5849576749872	1.386534194986
54.3	0.9477137838329	0.8120835268918	0.5835412113561	1.391647258305
54.4	0.9494591130849	0.813100761047	0.5821229701573	1.396785220187
54.5	0.9512044423369	0.8141155183563	0.5807029557109	1.401948294476
54.6	0.9529497715889	0.8151277957286	0.5792811723427	1.407136697421
54.7	0.9546951008409	0.8161375900802	0.5778576243835	1.412350647706
54.8	0.9564404300929	0.8171448983351	0.5764323161698	1.417590366489
54.9	0.9581857593449	0.818149717425	0.5750052520433	1.422856077432
55.0	0.9599310885969	0.819152044289	0.573576436351	1.428148006742
55.1	0.9616764178489	0.8201518758738	0.5721458734455	1.433466383205
55.2	0.9634217471009	0.8211492091337	0.5707135676844	1.438811438224

60.6	1.057669526709	0.8712138111202	0.4909037536151	1.774714095593
60.7	1.059414855961	0.8720692724321	0.4893824517488	1.781979041782
60.8	1.061160185213	0.8729220772698	0.4878596591387	1.789289319004
60.9	1.062905514465	0.8737722230355	0.4863353804235	1.796645397821
61.0	1.064650843717	0.8746197071394	0.4848096202463	1.804047755271
61.1	1.066396172969	0.875464527	0.483282383255	1.811496874981
61.2	1.068141502221	0.8763066800439	0.4817536741017	1.818993247281
61.3	1.069886831473	0.8771461637056	0.4802234974432	1.82653736932
61.4	1.071632160725	0.877982975428	0.4786918579406	1.834129745188
61.5	1.073377489977	0.878817112662	0.4771587602596	1.841770886034
61.6	1.075122819229	0.8796485728666	0.4756242090703	1.849461310193
61.7	1.076868148481	0.8804773535092	0.4740882090471	1.857201543314
61.8	1.078613477733	0.881303452065	0.472550764869	1.864992118485
61.9	1.080358806985	0.8821268660177	0.4710118812194	1.872833576372
62.0	1.082104136236	0.8829475928589	0.4694715627859	1.880726465346
62.1	1.083849465488	0.8837656300887	0.4679298142606	1.888671341631
62.2	1.08559479474	0.8845809752151	0.4663866403399	1.896668769437
62.3	1.087340123992	0.8853936257544	0.4648420457246	1.904719321107
62.4	1.089085453244	0.8862035792312	0.4632960351199	1.912823577266
62.5	1.090830782496	0.8870108331782	0.461748613235	1.920982126971
62.6	1.092576111748	0.8878153851364	0.4601997847838	1.929195567863
62.7	1.094321441	0.888617232655	0.4586495544843	1.937464506325
62.8	1.096066770252	0.8894163732913	0.4570979270587	1.945789557644
62.9	1.097812099504	0.8902128046111	0.4555449072335	1.954171346174
63.0	1.099557428756	0.8910065241884	0.4539904997395	1.962610505505
63.1	1.101302758008	0.8917975296052	0.4524347093118	1.971107678635
63.2	1.10304808726	0.8925858184521	0.4508775406894	1.979663518141
63.3	1.104793416512	0.8933713883278	0.4493189986159	1.988278686367
63.4	1.106538745764	0.8941542368394	0.4477590878388	1.9969538556
63.5	1.108284075016	0.894934361602	0.4461978131098	2.005689708259
63.6	1.110029404268	0.8957117602394	0.4446351791849	2.014486937092
63.7	1.11177473352	0.8964864303834	0.4430711908242	2.023346245365
63.8	1.113520062772	0.8972583696743	0.4415058527917	2.032268347069
63.9	1.115265392024	0.8980275757606	0.4399391698559	2.041253967122
64.0	1.117010721276	0.8987940462992	0.4383711467891	2.050303841579
64.1	1.118756050528	0.8995577789552	0.4368017883677	2.059418717851
64.2	1.12050137978	0.9003187714022	0.4352310993723	2.068599354919
64.3	1.122246709032	0.9010770213221	0.4336590845875	2.077846523564
64.4	1.123992038284	0.9018325264051	0.432085748802	2.087161006596
64.5	1.125737367536	0.9025852843499	0.4305110968083	2.096543599088
64.6	1.127482696788	0.9033352928633	0.4289351334031	2.105995108622
64.7	1.12922802604	0.9040825496608	0.4273578633872	2.115516355532
64.8	1.130973355292	0.904827052466	0.4257792915651	2.125108173157
64.9	1.132718684544	0.9055687990111	0.4241994227454	2.134771408104
65.0	1.134464013796	0.9063077870367	0.4226182617407	2.14450692051
65.1	1.136209343048	0.9070440142915	0.4210358133675	2.154315584313
65.2	1.1379546723	0.9077774785329	0.4194520824462	2.164198287535
65.3	1.139700001552	0.9085081775267	0.4178670738011	2.174155932562
65.4	1.141445330804	0.9092361090471	0.4162807922604	2.184189436438
65.5	1.143190660056	0.9099612708765	0.4146932426562	2.194299731165
65.6	1.144935989308	0.9106836608062	0.4131044298245	2.204487764009
65.7	1.14668131856	0.9114032766354	0.4115143586051	2.214754497813
65.8	1.148426647812	0.9121201161723	0.4099230338416	2.225100911321

71.2	1.24267442742	0.9466492601157	0.3222656952305	2.937480700323
71.3	1.244419756672	0.947210277746	0.3206129905857	2.954372734604
71.4	1.246165085924	0.9477684100096	0.3189593092981	2.971439874557
71.5	1.247910415176	0.9483236552062	0.3173046564051	2.988684962743
71.6	1.249655744428	0.9488760116445	0.3156490369471	3.006110903495
71.7	1.25140107368	0.9494254776419	0.3139924559674	3.023720664615
71.8	1.253146402932	0.9499720515247	0.3123349185122	3.041517279111
71.9	1.254891732184	0.9505157316278	0.3106764296307	3.059503847001
72.0	1.256637061436	0.9510565162952	0.3090169943749	3.077683537175
72.1	1.258382390688	0.9515944038794	0.3073566177998	3.096059589318
72.2	1.26012771994	0.9521293927421	0.3056953049631	3.114635315898
72.3	1.261873049192	0.9526614812536	0.3040330609255	3.133414104222
72.4	1.263618378444	0.9531906677929	0.3023698907504	3.152399418564
72.5	1.265363707696	0.9537169507482	0.3007057995043	3.171594802363
72.6	1.267109036948	0.9542403285163	0.2990407922561	3.191003880498
72.7	1.2688543662	0.9547607995028	0.2973748740778	3.210630361639
72.8	1.270599695452	0.9552783621223	0.295708050044	3.230478040689
72.9	1.272345024704	0.9557930147983	0.2940403252323	3.2505508013
73.0	1.274090353956	0.956304755963	0.2923717047227	3.270852618484
73.1	1.275835683208	0.9568135840576	0.2907021935983	3.29138756132
73.2	1.27758101246	0.9573194975321	0.2890317969445	3.312159795747
73.3	1.279326341712	0.9578224948453	0.2873605198497	3.333173587472
73.4	1.281071670964	0.9583225744651	0.285688367405	3.354433304968
73.5	1.282817000216	0.9588197348682	0.2840153447039	3.375943422591
73.6	1.284562329468	0.9593139745401	0.2823414568429	3.39770852381
73.7	1.28630765872	0.9598052919752	0.2806667089208	3.419733304551
73.8	1.288052987972	0.9602936856769	0.2789911060392	3.442022576669
73.9	1.289798317224	0.9607791541576	0.2773146533024	3.464581271549
74.0	1.291543646476	0.9612616959383	0.275637355817	3.487414443841
74.1	1.293288975728	0.9617413095492	0.2739592186924	3.510527275335
74.2	1.29503430498	0.9622179935293	0.2722802470406	3.533925078986
74.3	1.296779634232	0.9626917464265	0.2706004459759	3.557613303092
74.4	1.298524963484	0.9631625667977	0.2689198206153	3.58159753563
74.5	1.300270292736	0.9636304532086	0.2672383760783	3.605883508761
74.6	1.302015621988	0.9640954042341	0.2655561174868	3.630477103515
74.7	1.30376095124	0.9645574184578	0.2638730499654	3.655384354652
74.8	1.305506280492	0.9650164944723	0.2621891786409	3.680611455724
74.9	1.307251609744	0.9654726308792	0.2605045086426	3.706164764325
75.0	1.308996938996	0.9659258262891	0.2588190451025	3.732050807569
75.1	1.310742268248	0.9663760793213	0.2571327931547	3.75827628777
75.2	1.3124875975	0.9668233886045	0.2554457579358	3.784848088366
75.3	1.314232926752	0.9672677527759	0.2537579445848	3.811773280078
75.4	1.315978256004	0.967709170482	0.2520693582431	3.839059127324
75.5	1.317723585256	0.9681476403781	0.2503800040544	3.866713094899
75.6	1.319468914508	0.9685831611286	0.2486898871649	3.89474285493
75.7	1.32121424376	0.9690157314069	0.2469990127227	3.923156294129
75.8	1.322959573012	0.9694453498951	0.2453073858788	3.951961521347
75.9	1.324704902264	0.9698720152847	0.243615011786	3.981166875449
76.0	1.326450231516	0.970295726276	0.2419218955997	4.010780933536
76.1	1.328195560768	0.9707164815782	0.2402280424773	4.040812519505
76.2	1.32994089002	0.9711342799096	0.2385334575786	4.071270712997
76.3	1.331686219272	0.9715491199976	0.2368381460656	4.102164858732
76.4	1.333431548524	0.9719610005785	0.2351421131026	4.13350457625

81.8	1.427679328131	0.9897762309078	0.1426289337055	6.939519248958
81.9	1.429424657383	0.9900236577166	0.1409012319376	7.026366229041
82.0	1.431169986635	0.9902680687416	0.1391731009601	7.115369722384
82.1	1.432915315887	0.9905094632383	0.1374445460372	7.206611624812
82.2	1.434660645139	0.9907478404714	0.1357155724343	7.300178031898
82.3	1.436405974391	0.9909831997148	0.1339861854183	7.396159511677
82.4	1.438151303643	0.9912155402515	0.1322563902571	7.494651398881
82.5	1.439896632895	0.9914448613738	0.1305261922201	7.595754112725
82.6	1.441641962147	0.9916711623831	0.1287955965776	7.699573500448
82.7	1.443387291399	0.99189444259	0.1270646086014	7.806221209101
82.8	1.445132620651	0.9921147013145	0.1253332335643	7.915815088305
82.9	1.446877949903	0.9923319378855	0.1236014767405	8.028479627059
83.0	1.448623279155	0.9925461516413	0.1218693434052	8.144346427974
83.1	1.450368608407	0.9927573419294	0.1201368388347	8.263554722759
83.2	1.452113937659	0.9929655081065	0.1184039683065	8.386251933179
83.3	1.453859266911	0.9931706495385	0.1166707370993	8.512594282256
83.4	1.455604596163	0.9933727656004	0.1149371504929	8.642747461031
83.5	1.457349925415	0.9935718556766	0.1132032137679	8.776887356869
83.6	1.459095254667	0.9937679191606	0.1114689322063	8.915200850055
83.7	1.460840583919	0.9939609554552	0.1097343110911	9.057886686238
83.8	1.462585913171	0.9941509639723	0.107999355706	9.205156433325
83.9	1.464331242423	0.9943379441332	0.1062640713362	9.357235532477
84.0	1.466076571675	0.9945218953683	0.1045284632677	9.514364454222
84.1	1.467821900927	0.9947028171172	0.1027925367873	9.676799972121
84.2	1.469567230179	0.9948807088288	0.101056297183	9.844816568211
84.3	1.471312559431	0.9950555699612	0.09931974974365	10.01870798637
84.4	1.473057888683	0.9952273999818	0.09758289975916	10.19878895214
84.5	1.474803217935	0.9953961983672	0.09584575252023	10.38539708014
84.6	1.476548547187	0.9955619646031	0.09410831331852	10.5788949934
84.7	1.478293876439	0.9957246981846	0.09237058744657	10.77967268272
84.8	1.480039205691	0.995884398616	0.09063258019779	10.98815013809
84.9	1.481784534943	0.9960410654108	0.08889429686645	11.20478028987
85.0	1.483529864195	0.9961946980917	0.08715574274767	11.43005230276
85.1	1.485275193447	0.9963452961909	0.08541692313738	11.66449527324
85.2	1.487020522699	0.9964928592495	0.08367784333233	11.9086823891
85.3	1.488765851951	0.996637386818	0.08193850863005	12.16323561999
85.4	1.490511181203	0.9967788784562	0.08019892432887	12.42883101984
85.5	1.492256510455	0.9969173337331	0.07845909572786	12.70620473617
85.6	1.494001839707	0.9970527522269	0.07671902812683	12.99615983897
85.7	1.495747168959	0.9971851335251	0.07497872682634	13.29957410233
85.8	1.497492498211	0.9973144772245	0.07323819712764	13.6174088978
85.9	1.499237827463	0.9974407829309	0.0714974443327	13.95071938921
86.0	1.500983156715	0.9975640502598	0.06975647374414	14.30066625671
86.1	1.502728485967	0.9976842788356	0.06801529066526	14.66852922449
86.2	1.504473815219	0.997801468292	0.06627390040001	15.05572272448
86.3	1.506219144471	0.9979156182722	0.06453230825297	15.46381410007
86.4	1.507964473723	0.9980267284283	0.06279051952932	15.89454484386
86.5	1.509709802975	0.9981347984219	0.06104853953487	16.3498554761
86.6	1.511455132227	0.9982398279238	0.05930637357597	16.83191481342
86.7	1.513200461479	0.998341816614	0.05756402695958	17.34315455927
86.8	1.514945790731	0.998440764182	0.05582150499318	17.88631037992
86.9	1.516691119983	0.9985366703262	0.05407881298479	18.46447093073

Here is a slightly better linear interpolation method that uses the "point slope" form of a line which will consider
a more correct slope value for those close points or area of the (sin) curve, here, points close to the known sin 37°
Since the slope value of a line is a strict (ratio) numeric value, it is unitless, therefore, use radian (unitless, strict
numeric) angle values.

m = $\frac{(y2 - y1)}{(x2 - x1)}$: slope of a line, between two points (locations) on it

(y2 - y1) = m (x2 - x1)   : "points and slope" for m of a linear (line) equation

 The slope of a tangent line (ie. between two close points on the curve) at a point on the SIN curve is not a
 constant value as for lines, since the SIN curve is non-algebraic and non-linear. The slope value on a line or
 curve is equal to the derivative of the curve at that point or location. The derivative of the SIN curve at any point,
 hence also the entire curve, is equal to the COS curve.

 Here, the (x) values are the angle values, and the (y) values are the corresponding SIN values:

        (y2 - y1)   =   m   (x2 - x1)

(SIN 37.5° - SIN 37°) = COS 37° (37.5° - 37°)   After some simplification and transposing:

SIN 37.5° =   COS 37° (0.5°) + SIN 37°   0.5° = 0.008872664625997165 (radians):

SIN 37.5° =   0.79863551004729 (0.008872664625997165) + 0.6018150231520484

SIN 37.5° = 0.6087844327388832   : a calculated ("close" mathematical estimate-approximation) interpolated value

 This calculated value differs from the true value by only: 0.000230037301623964, or about:: +0.0002

This above table, and many others in the book, were made with the assistance of a computer which is essentially a fancy
automatic calculator machine with a typing (letters and numbers)) keyboard input screen. This electronic machine can
process what are called programs which are composed of smaller steps of the desired calculation, and for its input and
output such as on the screen and-or printer. In the old days, tables such as this were calculated by hand and pen, usually
with many people involved, and some to compare or check the work of the others. But once it was done, it was treasured
and distributed in books to be used for many years and even till this day.

0.200 1.2214027581602   0.201 1.2226247718233   0.202 1.2238480081114   0.203 1.2250724682475
0.204 1.2262981534562   0.205 1.2275250649632   0.206 1.2287532039953   0.207 1.2299825717808
0.208 1.2312131695489   0.209 1.2324449985303   0.210 1.2336780599567   0.211 1.2349123550614
0.212 1.2361478850785   0.213 1.2373846512436   0.214 1.2386226547935   0.215 1.2398618969661
0.216 1.2411023790007   0.217 1.2423441021378   0.218 1.2435870676191   0.219 1.2448312766875
0.220 1.2460767305874   0.221 1.2473234305641   0.222 1.2485713778643   0.223 1.249820573736
0.224 1.2510710194284   0.225 1.2523227161919   0.226 1.2535756652782   0.227 1.2548298679403
0.228 1.2560853254323   0.229 1.2573420390098   0.230 1.2586000099295   0.231 1.2598592394492
0.232 1.2611197288283   0.233 1.2623814793273   0.234 1.2636444922078   0.235 1.2649087687329
0.236 1.2661743101669   0.237 1.2674411177753   0.238 1.2687091928249   0.239 1.2699785365838
0.240 1.2712491503214   0.241 1.2725210353082   0.242 1.2737941928162   0.243 1.2750686241185
0.244 1.2763443304895   0.245 1.2776213132049   0.246 1.2788995735417   0.247 1.2801791127783
0.248 1.281459932194   0.249 1.2827420330698   0.250 1.2840254166877   0.251 1.2853100843312
0.252 1.2865960372848   0.253 1.2878832768346   0.254 1.2891718042678   0.255 1.2904616208729
0.256 1.2917527279397   0.257 1.2930451267594   0.258 1.2943388186242   0.259 1.295633804828
0.260 1.2969300866658   0.261 1.2982276654337   0.262 1.2995265424294   0.263 1.3008267189517
0.264 1.3021281963009   0.265 1.3034309757784   0.266 1.3047350586869   0.267 1.3060404463307
0.268 1.3073471400149   0.269 1.3086551410465   0.270 1.3099644507332   0.271 1.3112750703846
0.272 1.3125870013111   0.273 1.3139002448247   0.274 1.3152148022387   0.275 1.3165306748676
0.276 1.3178478640273   0.277 1.319166371035   0.278 1.3204861972091   0.279 1.3218073438695
0.280 1.3231298123374   0.281 1.3244536039353   0.282 1.3257787199868   0.283 1.3271051618172
0.284 1.3284329307528   0.285 1.3297620281215   0.286 1.3310924552523   0.287 1.3324242134757
0.288 1.3337573041234   0.289 1.3350917285285   0.290 1.3364274880255   0.291 1.33776458395
0.292 1.3391030176393   0.293 1.3404427904317   0.294 1.341783903667   0.295 1.3431263586863
0.296 1.3444701568321   0.297 1.3458152994481   0.298 1.3471617878796   0.299 1.3485096234729
0.300 1.349858807576   0.301 1.351209341538   0.302 1.3525612267095   0.303 1.3539144644423
0.304 1.3552690560897   0.305 1.3566250030062   0.306 1.3579823065479   0.307 1.359340968072
0.308 1.3607009889372   0.309 1.3620623705034   0.310 1.3634251141322   0.311 1.3647892211862
0.312 1.3661546930295   0.313 1.3675215310276   0.314 1.3688897365474   0.315 1.370259310957
0.316 1.371630255626   0.317 1.3730025719255   0.318 1.3743762612276   0.319 1.375751324906
0.320 1.377127764336   0.321 1.3785055808938   0.322 1.3798847759572   0.323 1.3812653509056
0.324 1.3826473071195   0.325 1.3840306459808   0.326 1.3854153688728   0.327 1.3868014771803
0.328 1.3881889722894   0.329 1.3895778555876   0.330 1.3909681284638   0.331 1.3923597923082
0.332 1.3937528485125   0.333 1.3951472984698   0.334 1.3965431435745   0.335 1.3979403852225
0.336 1.3993390248109   0.337 1.4007390637385   0.338 1.4021405034053   0.339 1.4035433452127
0.340 1.4049475905636   0.341 1.4063532408622   0.342 1.4077602975141   0.343 1.4091687619265
0.344 1.4105786355077   0.345 1.4119899196677   0.346 1.4134026158177   0.347 1.4148167253704
0.348 1.41623224974   0.349 1.417649190342   0.350 1.4190675485933   0.351 1.4204873259122
0.352 1.4219085237186   0.353 1.4233311434336   0.354 1.4247551864799   0.355 1.4261806542815
0.356 1.4276075482638   0.357 1.4290358698539   0.358 1.4304656204799   0.359 1.4318968015717
0.360 1.4333294145603   0.361 1.4347634608786   0.362 1.4361989419604   0.363 1.4376358592412
0.364 1.439074214158   0.365 1.4405140081492   0.366 1.4419552426545   0.367 1.4433979191152
0.368 1.4448420389739   0.369 1.4462876036747   0.370 1.4477346146633   0.371 1.4491830733866
0.372 1.4506329812932   0.373 1.4520843398328   0.374 1.4535371504569   0.375 1.4549914146182
0.376 1.4564471337711   0.377 1.4579043093712   0.378 1.4593629428758   0.379 1.4608230357434
0.380 1.4622845894342   0.381 1.4637476054097   0.382 1.465212085133   0.383 1.4666780300684
0.384 1.468145441682   0.385 1.4696143214411   0.386 1.4710846708147   0.387 1.4725564912731
0.388 1.4740297842881   0.389 1.4755045513331   0.390 1.4769807938826   0.391 1.4784585134131
0.392 1.4799377114023   0.393 1.4814183893293   0.394 1.4829005486748   0.395 1.4843841909209
0.396 1.4858693175514   0.397 1.4873559300513   0.398 1.4888440299073   0.399 1.4903336186074
0.400 1.4918246976413   0.401 1.4933172685   0.402 1.494811332676   0.403 1.4963068916636
0.404 1.4978039469581   0.405 1.4993025000568   0.406 1.500802552458   0.407 1.502304105662
0.408 1.5038071611701   0.409 1.5053117204856   0.410 1.5068177851129   0.411 1.5083253565581

0.624 1.8663786452865   0.625 1.8682459574322   0.626 1.8701151378241   0.627 1.8719861883312
0.628 1.8738591108247   0.629 1.8757339071775   0.630 1.8776105792643   0.631 1.8794891289619
0.632 1.8813695581488   0.633 1.8832518687053   0.634 1.8851360625139   0.635 1.8870221414587
0.636 1.8889101074259   0.637 1.8907999623032   0.638 1.8926917079807   0.639 1.8945853463501
0.640 1.896480879305   0.641 1.8983783087409   0.642 1.9002776365552   0.643 1.9021788646474
0.644 1.9040819949186   0.645 1.9059870292719   0.646 1.9078939696125   0.647 1.9098028178471
0.648 1.9117135758847   0.649 1.9136262456361   0.650 1.9155408290139   0.651 1.9174573279327
0.652 1.9193757443089   0.653 1.9212960800611   0.654 1.9232183371095   0.655 1.9251425173764
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0.668 1.950332751865   0.669 1.9522840601083   0.670 1.9542373206359   0.671 1.956192535401
0.672 1.9581497063588   0.673 1.9601088354665   0.674 1.9620699246831   0.675 1.9640329759698
0.676 1.9659979912897   0.677 1.9679649726078   0.678 1.9699339218909   0.679 1.9719048411082
0.680 1.9738777322304   0.681 1.9758525972306   0.682 1.9778294380835   0.683 1.9798082567661
0.684 1.981789055257   0.685 1.9837718355372   0.686 1.9857565995893   0.687 1.9877433493983
0.688 1.9897320869507   0.689 1.9917228142354   0.690 1.99371553324231 0.691 1.9957102459665
0.692 1.9977069544003   0.693 1.9997056605412   0.694 2.0017063663879   0.695 2.0037090739412
0.696 2.0057137852037   0.697 2.0077205021802   0.698 2.0097292268773   0.699 2.0117399613038
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0.716 2.0462318913749   0.717 2.0482791467234   0.718 2.0503284503511   0.719 2.0523798043075
0.720 2.0544332106439   0.721 2.0564886714136   0.722 2.0585461886722   0.723 2.0606057644772
0.724 2.062667400888   0.725 2.0647310999665   0.726 2.0667968637762   0.727 2.068864694383
0.728 2.0709345938546   0.729 2.073006564261   0.730 2.0750806076741   0.731 2.077156726168
0.732 2.0792349218188   0.733 2.0813151967048   0.734 2.083397552906   0.735 2.085481992505
0.736 2.0875685175862   0.737 2.0896571302361   0.738 2.0917478325432   0.739 2.0938406265984
0.740 2.0959355144944   0.741 2.098032498326   0.742 2.1001315801904   0.743 2.1022327621865
0.744 2.1043360464155   0.745 2.1064414349807   0.746 2.1085489299876   0.747 2.1106585335436
0.748 2.1127702477582   0.749 2.1148840747433   0.750 2.1170000166127   0.751 2.1191180754822
0.752 2.12123825347   0.753 2.1233605526962   0.754 2.1254849752832   0.755 2.1276115233553
0.756 2.1297401990391   0.757 2.1318710044633   0.758 2.1340039417587   0.759 2.1361390130581
0.760 2.1382762204968   0.761 2.1404155662119   0.762 2.1425570523427   0.763 2.1447006810308
0.764 2.1468464544197   0.765 2.1489943746552   0.766 2.1511444438853   0.767 2.15329666426
0.768 2.1554510379316   0.769 2.1576075670544   0.770 2.1597662537849   0.771 2.1619271002819
0.772 2.1640901087061   0.773 2.1662552812207   0.774 2.1684226199906   0.775 2.1705921271834
0.776 2.1727638049685   0.777 2.1749376555176   0.778 2.1771136810046   0.779 2.1792918836053
0.780 2.1814722654982   0.781 2.1836548288635   0.782 2.1858395758838   0.783 2.1880265087439
0.784 2.1902156296306   0.785 2.1924069407332   0.786 2.1946004442429   0.787 2.1967961423532
0.788 2.1989940372599   0.789 2.2011941311608   0.790 2.2033964262559   0.791 2.2056009247477
0.792 2.2078076288406   0.793 2.2100165407413   0.794 2.2122276626588   0.795 2.2144409968041
0.796 2.2166565453905   0.797 2.2188743106337   0.798 2.2210942947514   0.799 2.2233164999636
0.800 2.2255409284925   0.801 2.2277675825624   0.802 2.2299964644002   0.803 2.2322275762346
0.804 2.2344609202967   0.805 2.23669649882   0.806 2.2389343140399   0.807 2.2411743681944
0.808 2.2434166635234   0.809 2.2456612022692   0.810 2.2479079866765   0.811 2.2501570189919
0.812 2.2524083014645   0.813 2.2546618363456   0.814 2.2569176258888   0.815 2.2591756723497
0.816 2.2614359779865   0.817 2.2636985450595   0.818 2.2659633758312   0.819 2.2682304725665
0.820 2.2704998375324   0.821 2.2727714729984   0.822 2.275045381236   0.823 2.2773215645192
0.824 2.2796000251241   0.825 2.2818807653293   0.826 2.2841637874154   0.827 2.2864490936655
0.828 2.2887366863649   0.829 2.2910265678012   0.830 2.2933187402642   0.831 2.2956132060461
0.832 2.2979099674415   0.833 2.300209026747   0.834 2.3025103862617   0.835 2.304814048287

5.0 148.41315910258 5.1 164.0219072999   5.2 181.27224187515 5.3 200.33680997479
5.4 221.40641620419 5.5 244.69193226422 5.6 270.42640742615 5.7 298.86740096706
5.8 330.29955990965 5.9 365.03746786533 6.0 403.42879349273 6.1 445.85777008251
6.2 492.74904109325 6.3 544.57191012593 6.4 601.84503787208 6.5 665.14163304436
6.6 735.09518924197 6.7 812.40582516754 6.8 897.84729165041 6.9 992.27471560502
7.0 1096.6331584284 7.1 1211.9670744926 7.2 1339.4307643944 7.3 1480.2999275845
7.4 1635.9844299959 7.5 1808.042414456   7.6 1998.1958951041 7.7 2208.3479918872
7.8 2440.6019776245 7.9 2697.2823282685 8.0 2980.9579870417 8.1 3294.4680752838
8.2 3640.9503073323 8.3 4023.8723938223 8.4 4447.0667476998 8.5 4914.7688402991
8.6 5431.6595913629 8.7 6002.9122172609 8.8 6634.2440062778 8.9 7331.9735391559
9.0 8103.0839275753 9.1 8955.2927034824 9.2 9897.1290587438 9.3 10938.019208165
9.4 12088.380730217 9.5 13359.726829662 9.6 14764.781565577 9.7 16317.607198015
9.8 18033.744927828 9.9 19930.37043823   10.0 22026.465794806

To calculate a value that is not listed in the table above, we can use a method that is similar to that previously described above for calculating a value that is not listed in the SIN x table.

Since the slope of the e^x curve is equal to e^x we will use:

        (y2 - y1) = m (x2 - x1)

        (y2 - y1) = e^x (x2 - x1)

Here are two entries from the table, and we need to find the value of e^0.5405:

        e^0.540 = 1.7160068621849   and   e^0.541 = 1.7177237273365

The result should be a value that is someplace between these to values, of about 1.7165.

    (e^0.5405 - e^0.540) = e^x (0.5405 - 0.540)   After some simplification and transposing:

    e^0.5405 = e^0.540 (0.0005) + e^0.540

    e^0.5405 = 0.000858003 + 1.7160068621849

    e^0.5405 = 1.7168648651849   : a calculated ("close" mathematical estimate-approximation) interpolated value

This calculated value differs from the true value by -0.00085822, or about:: -0.0008,   which is less than a thousandth.

0.204 -1.5896352851379   0.205 -1.5847452998437   0.206 -1.5798791101926   0.207 -1.5750364857168
0.208 -1.5702171992808   0.209 -1.5654210270173   0.210 -1.5606477482647   0.211 -1.5558971455061
0.212 -1.5511690043101   0.213 -1.5464631132727   0.214 -1.5417792639603   0.215 -1.5371172508545
0.216 -1.532476871298   0.217 -1.5278579254417   0.218 -1.523260216193   0.219 -1.5186835491656
0.220 -1.5141277326298   0.221 -1.5095925774644   0.222 -1.5050778971099   0.223 -1.500583507522
0.224 -1.4961092271271   0.225 -1.4916548767777   0.226 -1.4872202797099   0.227 -1.4828052615007
0.228 -1.4784096500277   0.229 -1.4740332754279   0.230 -1.4696759700589   0.231 -1.4653375684603
0.232 -1.4610179073158   0.233 -1.4567168254164   0.234 -1.4524341636244   0.235 -1.448169764838
0.236 -1.4439234739565   0.237 -1.439695137847   0.238 -1.4354846053107   0.239 -1.4312917270506
0.240 -1.4271163556401   0.241 -1.4229583454915   0.242 -1.4188175528254   0.243 -1.4146938356416
0.244 -1.4105870536889   0.245 -1.4064970684374   0.246 -1.4024237430498   0.247 -1.3983669423542
0.248 -1.3943265328172   0.249 -1.3903023825174   0.250 -1.3862943611199   0.251 -1.3823023398504
0.252 -1.3783261914707   0.253 -1.3743657902546   0.254 -1.3704210119636   0.255 -1.3664917338237
0.256 -1.3625778345026   0.257 -1.3586791940869   0.258 -1.3547956940605   0.259 -1.3509272172826
0.260 -1.3470736479666   0.261 -1.3432348716594   0.262 -1.339410775221   0.263 -1.3356012468044
0.264 -1.3318061758358   0.265 -1.3280254529959   0.266 -1.3242589702004   0.267 -1.3205066205819
0.268 -1.3167682984713   0.269 -1.3130438993803   0.270 -1.3093333199838   0.271 -1.3056364581024
0.272 -1.3019532126861   0.273 -1.2982834837972   0.274 -1.2946271725941   0.275 -1.2909841813156
0.276 -1.287354413265   0.277 -1.2837377727948   0.278 -1.2801341652915   0.279 -1.2765434971608
0.280 -1.2729656758129   0.281 -1.2694006096484   0.282 -1.265848208044   0.283 -1.2623083813389
0.284 -1.2587810408209   0.285 -1.2552660987135   0.286 -1.2517634681623   0.287 -1.2482730632225
0.288 -1.2447947988462   0.289 -1.2413285908697   0.290 -1.2378743560016   0.291 -1.2344320118106
0.292 -1.2310014767139   0.293 -1.2275826699651   0.294 -1.2241755116435   0.295 -1.2207799226423
0.296 -1.2173958246581   0.297 -1.2140231401794   0.298 -1.2106617924767   0.299 -1.2073117055915
0.300 -1.2039728043259   0.301 -1.2006450142333   0.302 -1.1973282616073   0.303 -1.1940224734728
0.304 -1.1907275775759   0.305 -1.1874435023747   0.306 -1.1841701770298   0.307 -1.1809075313949
0.308 -1.1776554960086   0.309 -1.1744140020844   0.310 -1.1711829815029   0.311 -1.1679623668029
0.312 -1.1647520911727   0.313 -1.161552088442   0.314 -1.1583622930739   0.315 -1.1551826401565
0.316 -1.1520130653952   0.317 -1.1488535051049   0.318 -1.145703896202   0.319 -1.1425641761973
0.320 -1.1394342831884   0.321 -1.1363141558521   0.322 -1.1332037334377   0.323 -1.1301029557595
0.324 -1.1270117631898   0.325 -1.1239300966524   0.326 -1.1208578976154   0.327 -1.1177951080849
0.328 -1.114741670598   0.329 -1.1116975282168   0.330 -1.1086626245216   0.331 -1.1056369036051
0.332 -1.1026203100656   0.333 -1.0996127890017   0.334 -1.0966142860054   0.335 -1.0936247471571
0.336 -1.0906441190189   0.337 -1.0876723486298   0.338 -1.0847093834991   0.339 -1.0817551716017
0.340 -1.0788096613719   0.341 -1.0758728016986   0.342 -1.0729445419195   0.343 -1.0700248318162
0.344 -1.0671136216087   0.345 -1.0642108619508   0.346 -1.0613165039244   0.347 -1.0584304990353
0.348 -1.0555527992077   0.349 -1.0526833567797   0.350 -1.0498221244987   0.351 -1.0469690555163
0.352 -1.044124103384   0.353 -1.0412872220488   0.354 -1.0384583658484   0.355 -1.0356374895067
0.356 -1.0328245481301   0.357 -1.0300194972025   0.358 -1.0272222925814   0.359 -1.0244328904939
0.360 -1.021651247532   0.361 -1.0188773206493   0.362 -1.0161110671564   0.363 -1.0133524447173
0.364 -1.0106014113454   0.365 -1.0078579253996   0.366 -1.0051219455808   0.367 -1.0023934309276
0.368 -0.99967234081321   0.369 -0.99695863494161   0.370 -0.99425227334387   0.371 -0.9915532163747
0.372 -0.98886142470899   0.373 -0.98617685933832   0.374 -0.9834994815676   0.375 -0.98082925301173
0.376 -0.97816613559224   0.377 -0.97551009153413   0.378 -0.97286108336255   0.379 -0.97021907389971
0.380 -0.9675840262617   0.381 -0.96495590385544   0.382 -0.96233467037556   0.383 -0.95972028980149
0.384 -0.95711272639441   0.385 -0.95451194469435   0.386 -0.95191790951731   0.387 -0.94933058595235
0.388 -0.94674993935886   0.389 -0.94417593536369   0.390 -0.94160853985844   0.391 -0.93904771899677
0.392 -0.93649343919167   0.393 -0.93394566711288   0.394 -0.9314043696842   0.395 -0.92886951408101
0.396 -0.92634106772766   0.397 -0.92381899829495   0.398 -0.9213032736977   0.399 -0.91879386209227
0.400 -0.91629073187415   0.401 -0.91379385167557   0.402 -0.91130319036312   0.403 -0.90881871703545
0.404 -0.90634040102099   0.405 -0.9038682118756   0.406 -0.9014021193804   0.407 -0.89894209353954
0.408 -0.89648810457797   0.409 -0.89404012293933   0.410 -0.89159811928378   0.411 -0.8891620644859
0.412 -0.88673192963261   0.413 -0.8843076860211   0.414 -0.88188930515682   0.415 -0.87947675875144

0.628 -0.46521511251394   0.629 -0.4636240222817   0.630 -0.46203545959656   0.631 -0.46044941644092
0.632 -0.45886588483528   0.633 -0.45728485683796   0.634 -0.45570632454491   0.635 -0.45413028008944
0.636 -0.45255671564201   0.637 -0.45098562340997   0.638 -0.44941699563735   0.639 -0.4478508246046
0.640 -0.44628710262842   0.641 -0.44472582206147   0.642 -0.44316697529218   0.643 -0.44161055474452
0.644 -0.44005655287778   0.645 -0.43850496218636   0.646 -0.43695577519953   0.647 -0.43540898448124
0.648 -0.43386458262986   0.649 -0.43232256227805   0.650 -0.43078291609245   0.651 -0.42924563677357
0.652 -0.42771071705548   0.653 -0.42617814970571   0.654 -0.42464792752494   0.655 -0.42312004334688
0.656 -0.42159449003805   0.657 -0.42007126049753   0.658 -0.41855034765682   0.659 -0.41703174447963
0.660 -0.41551544396167   0.661 -0.41400143913045   0.662 -0.41248972304513   0.663 -0.41098028879627
0.664 -0.4094731295057   0.665 -0.40796823832628   0.666 -0.40646560844175   0.667 -0.40496523306651
0.668 -0.40346710544549   0.669 -0.40197121885391   0.670 -0.40047756659712   0.671 -0.39898614201045
0.672 -0.39749693845899   0.673 -0.39600994933741   0.674 -0.39452516806983   0.675 -0.39304258810961
0.676 -0.39156220293917   0.677 -0.39008400606986   0.678 -0.38860799104174   0.679 -0.38713415142344
0.680 -0.38566248081198   0.681 -0.38419297283262   0.682 -0.38272562113867   0.683 -0.38126041941135
0.684 -0.37979736135959   0.685 -0.37833644071991   0.686 -0.37687765125625   0.687 -0.37542098675979
0.688 -0.37396644104879   0.689 -0.37251400796848   0.690 -0.37106368139083   0.691 -0.36961545521447
0.692 -0.36816932336447   0.693 -0.36672527979223   0.694 -0.36528331847533   0.695 -0.36384343341734
0.696 -0.36240561864772   0.697 -0.36096986822161   0.698 -0.35953617621976   0.699 -0.35810453674833
0.700 -0.35667494393873   0.701 -0.35524739194755   0.702 -0.35382187495633   0.703 -0.35239838717147
0.704 -0.35097692282409   0.705 -0.34955747616987   0.706 -0.34814004148889   0.707 -0.34672461308556
0.708 -0.34531118528842   0.709 -0.34389975245001   0.710 -0.34249030894678   0.711 -0.3410828491789
0.712 -0.33967736757016   0.713 -0.33827385856784   0.714 -0.33687231664255   0.715 -0.33547273628813
0.716 -0.33407511202149   0.717 -0.33267943838252   0.718 -0.33128570993391   0.719 -0.32989392126109
0.720 -0.32850406697204   0.721 -0.32711614169719   0.722 -0.32573014008931   0.723 -0.32434605682337
0.724 -0.32296388659642   0.725 -0.32158362412746   0.726 -0.32020526415734   0.727 -0.31882880144862
0.728 -0.31745423078545   0.729 -0.31608154697348   0.730 -0.3147107448397   0.731 -0.31334181923236
0.732 -0.31197476502082   0.733 -0.31060957709548   0.734 -0.30924625036762   0.735 -0.3078847797693
0.736 -0.30652516025326   0.737 -0.3051673867928   0.738 -0.30381145438166   0.739 -0.30245735803393
0.740 -0.30110509278392   0.741 -0.29975465368605   0.742 -0.29840603581476   0.743 -0.29705923426438
0.744 -0.29571424414904   0.745 -0.29437106060258   0.746 -0.29302967877838   0.747 -0.29169009384932
0.748 -0.29035230100766   0.749 -0.28901629546492   0.750 -0.28768207245178   0.751 -0.286349627218
0.752 -0.2850189550323   0.753 -0.28369005118224   0.754 -0.28236291097418   0.755 -0.28103752973311
0.756 -0.2797139028026   0.757 -0.27839202554469   0.758 -0.27707189333976   0.759 -0.27575350158651
0.760 -0.27443684570176   0.761 -0.27312192112045   0.762 -0.27180872329549   0.763 -0.27049724769768
0.764 -0.26918748981562   0.765 -0.2678794451556   0.766 -0.26657310924155   0.767 -0.26526847761488
0.768 -0.26396554583446   0.769 -0.26266430947649   0.770 -0.26136476413441   0.771 -0.26006690541881
0.772 -0.25877072895736   0.773 -0.25747623039471   0.774 -0.25618340539241   0.775 -0.25489224962879
0.776 -0.25360275879892   0.777 -0.25231492861449   0.778 -0.25102875480374   0.779 -0.24974423311139
0.780 -0.2484613592985   0.781 -0.24718012914245   0.782 -0.24590053843683   0.783 -0.24462258299133
0.784 -0.24334625863173   0.785 -0.24207156119973   0.786 -0.24079848655293   0.787 -0.23952703056473
0.788 -0.23825718912426   0.789 -0.23698895813626   0.790 -0.23572233352107   0.791 -0.23445731121448
0.792 -0.23319388716771   0.793 -0.23193205734729   0.794 -0.230671817735   0.795 -0.2294131643278
0.796 -0.22815609313775   0.797 -0.22690060019192   0.798 -0.22564668153233   0.799 -0.22439433321586
0.800 -0.22314355131421   0.801 -0.22189433191378   0.802 -0.22064667111562   0.803 -0.21940056503537
0.804 -0.21815600980317   0.805 -0.21691300156357   0.806 -0.21567153647551   0.807 -0.21443161071219
0.808 -0.21319322046104   0.809 -0.21195636192364   0.810 -0.21072103131565   0.811 -0.20948722486672
0.812 -0.20825493882046   0.813 -0.20702416943433   0.814 -0.2057949129796   0.815 -0.20456716574127
0.816 -0.20334092401803   0.817 -0.20211618412213   0.818 -0.20089294237939   0.819 -0.19967119512907
0.820 -0.19845093872384   0.821 -0.19723216952971   0.822 -0.19601488392596   0.823 -0.19479907830507
0.824 -0.19358474907266   0.825 -0.19237189264746   0.826 -0.19116050546116   0.827 -0.18995058395845
0.828 -0.18874212459688   0.829 -0.18753512384684   0.830 -0.18632957819149   0.831 -0.18512548412669
0.832 -0.18392283816093   0.833 -0.18272163681529   0.834 -0.18152187662339   0.835 -0.18032355413128
0.836 -0.17912666589743   0.837 -0.17793120849266   0.838 -0.17673717850005   0.839 -0.17554457251493

1.44 0.36464311358791   1.45 0.37156355643248   1.46 0.37843643572025   1.47 0.38526240079065
1.48 0.39204208777602   1.49 0.39877611995737   1.50 0.40546510810817   1.51 0.41210965082683
1.52 0.41871033485819   1.53 0.42526773540434   1.54 0.43178241642554   1.55 0.43825493093116
1.56 0.44468582126145   1.57 0.45107561936022   1.58 0.45742484703888   1.59 0.46373401623214
1.60 0.47000362924574   1.61 0.47623417899637   1.62 0.48242614924429   1.63 0.48858001481867
1.64 0.49469624183611   1.65 0.50077528791249   1.66 0.50681760236845   1.67 0.51282362642866
1.68 0.51879379341517   1.69 0.52472852893498   1.70 0.53062825106217   1.71 0.53649337051457
1.72 0.54232429082536   1.73 0.54812140850969   1.74 0.55388511322644   1.75 0.55961578793542
1.76 0.56531380905006   1.77 0.57097954658574   1.78 0.57661336430399   1.79 0.58221561985266
1.80 0.58778666490212   1.81 0.59332684527774   1.82 0.5988365010887   1.83 0.60431596685333
1.84 0.6097655716209   1.85 0.61518563909023   1.86 0.62057648772511   1.87 0.6259384308665
1.88 0.63127177684186   1.89 0.63657682907155   1.90 0.6418538861724   1.91 0.64710324205854
1.92 0.65232518603969   1.93 0.65752000291679   1.94 0.66268797307524   1.95 0.66782937257566
1.96 0.67294447324243   1.97 0.6780335427499   1.98 0.68309684470644   1.99 0.6881346387364
2.00 0.69314718055995

0.0 (undefined, infinite)   0.1 -2.302585092994   0.2 -1.6094379124341   0.3 -1.2039728043259
0.4 -0.91629073187416 0.5 -0.69314718055995 0.6 -0.51082562376599   0.7 -0.35667494393873
0.8 -0.22314355131421 0.9 -0.10536051565783 1.0 0.0   1.1 0.095310179804325
1.2 0.18232155679395   1.3 0.26236426446749   1.4 0.33647223662121   1.5 0.40546510810816
1.6 0.47000362924574   1.7 0.53062825106217   1.8 0.58778666490212   1.9 0.6418538861724
2.0 0.69314718055995   2.1 0.74193734472938   2.2 0.78845736036427   2.3 0.8329091229351
2.4 0.8754687373539   2.5 0.91629073187416   2.6 0.95551144502744   2.7 0.99325177301028
2.8 1.0296194171812   2.9 1.0647107369924   3.0 1.0986122886681   3.1 1.1314021114911
3.2 1.1631508098057   3.3 1.1939224684724   3.4 1.2237754316221   3.5 1.2527629684954
3.6 1.2809338454621   3.7 1.3083328196502   3.8 1.3350010667323   3.9 1.3609765531356
4.0 1.3862943611199   4.1 1.4109869737103   4.2 1.4350845252893   4.3 1.4586150226995
4.4 1.4816045409242   4.5 1.5040773967763   4.6 1.526056303495   4.7 1.547562508716
4.8 1.5686159179138   4.9 1.5892352051166   5.0 1.6094379124341   5.1 1.6292405397303
5.2 1.6486586255874   5.3 1.6677068205581   5.4 1.6863989535702   5.5 1.7047480922384
5.6 1.7227665977411   5.7 1.7404661748405   5.8 1.7578579175524   5.9 1.7749523509117
6.0 1.7917594692281   6.1 1.8082887711793   6.2 1.824549292051   6.3 1.8405496333975
6.4 1.8562979903656   6.5 1.8718021769016   6.6 1.8870696490324   6.7 1.9021075263969
6.8 1.9169226121821   6.9 1.9315214116032   7.0 1.9459101490553   7.1 1.9600947840473
7.2 1.974081026022   7.3 1.9878743481543   7.4 2.0014800002101   7.5 2.0149030205423
7.6 2.0281482472923   7.7 2.0412203288596   7.8 2.0541237336955   7.9 2.066862759473
8.0 2.0794415416798   8.1 2.0918640616784   8.2 2.1041341542702   8.3 2.1162555148026
8.4 2.1282317058493   8.5 2.1400661634963   8.6 2.1517622032595   8.7 2.1633230256605
8.8 2.1747517214842   8.9 2.1860512767381   9.0 2.1972245773362   9.1 2.2082744135228
9.2 2.219203484055   9.3 2.2300144001592   9.4 2.240709689276   9.5 2.2512917986065
9.6 2.2617630984738   9.7 2.2721258855093   9.8 2.2823823856765   9.9 2.2925347571405
10.0 2.302585092994

To calculate a value that is not listed in the table above, we can use a method that is similar to that previously described above for calculating a value that is not listed in the SIN x table, or e^x table.

Since the slope of the natural logarithm of x curve is equal to 1/x we will use:

        (y2 - y1) = m (x2 - x1)

        (y2 - y1) = 1/x (x2 - x1)

A SQUARE ROOT TABLE

0.000	0	0.001	0.031622776601684	0.002	0.044721359549996	0.003	0.054772255750517
0.004	0.063245553203368	0.005	0.070710678118655	0.006	0.077459666924148	0.007	0.083666002653408
0.008	0.089442719099992	0.009	0.094868329805051	0.010	0.1	0.011	0.10488088481702
0.012	0.10954451150103	0.013	0.11401754250991	0.014	0.11832159566199	0.015	0.12247448713916
0.016	0.12649110640674	0.017	0.13038404810405	0.018	0.13416407864999	0.019	0.1378404875209
0.020	0.14142135623731	0.021	0.14491376746189	0.022	0.14832396974191	0.023	0.15165750888103
0.024	0.1549193338483	0.025	0.15811388300842	0.026	0.16124515496597	0.027	0.16431676725155
0.028	0.16733200530682	0.029	0.17029386365926	0.030	0.17320508075689	0.031	0.17606816861659
0.032	0.17888543819998	0.033	0.18165902124585	0.034	0.18439088914586	0.035	0.1870828693387
0.036	0.1897366596101	0.037	0.19235384061671	0.038	0.19493588689618	0.039	0.19748417658132
0.040	0.2	0.041	0.20248456731317	0.042	0.20493901531919	0.043	0.20736441353328
0.044	0.20976176963403	0.045	0.21213203435596	0.046	0.21447610589527	0.047	0.21679483388679
0.048	0.21908902300207	0.049	0.22135943621179	0.050	0.22360679774998	0.051	0.22583179581272
0.052	0.22803508501983	0.053	0.23021728866443	0.054	0.23237900077245	0.055	0.23452078799117
0.056	0.23664319132398	0.057	0.23874672772627	0.058	0.24083189157585	0.059	0.24289915602982
0.060	0.24494897427832	0.061	0.24698178070457	0.062	0.24899799195977	0.063	0.25099800796022
0.064	0.25298221281347	0.065	0.25495097567964	0.066	0.2569046515733	0.067	0.2588435821109
0.068	0.26076809620811	0.069	0.26267851073127	0.070	0.26457513110646	0.071	0.26645825188948
0.072	0.26832815729997	0.073	0.27018512172213	0.074	0.27202941017471	0.075	0.27386127875258
0.076	0.2756809750418	0.077	0.27748873851023	0.078	0.27928480087538	0.079	0.2810693864511
0.080	0.28284271247462	0.081	0.28460498941515	0.082	0.28635642126553	0.083	0.28809720581776
0.084	0.28982753492379	0.085	0.29154759474227	0.086	0.2932575659723	0.087	0.29495762407505
0.088	0.29664793948383	0.089	0.29832867780353	0.090	0.3	0.091	0.30166206257997
0.092	0.30331501776206	0.093	0.30495901363954	0.094	0.30659419433512	0.095	0.30822070014845
0.096	0.30983866769659	0.097	0.31144823004795	0.098	0.31304951684997	0.099	0.31464265445105
0.100	0.31622776601684	0.101	0.31780497164141	0.102	0.31937438845343	0.103	0.32093613071762
0.104	0.32249030993194	0.105	0.32403703492039	0.106	0.32557641192199	0.107	0.32710854467592
0.108	0.3286335345031	0.109	0.33015148038438	0.110	0.33166247903554	0.111	0.33316662497915
0.112	0.33466401061363	0.113	0.33615472627943	0.114	0.33763886032268	0.115	0.33911649915626
0.116	0.34058772731853	0.117	0.34205262752974	0.118	0.34351128074635	0.119	0.34496376621321
0.120	0.34641016151378	0.121	0.34785054261852	0.122	0.34928498393146	0.123	0.350713558335
0.124	0.35213633723318	0.125	0.35355339059327	0.126	0.35496478698598	0.127	0.35637059362411
0.128	0.35777087639997	0.129	0.35916569992136	0.130	0.3605551275464	0.131	0.36193922141708
0.132	0.3633180424917	0.133	0.36469165057621	0.134	0.36606010435446	0.135	0.36742346141748
0.136	0.36878177829172	0.137	0.37013511046644	0.138	0.37148351242013	0.139	0.37282703764615
0.140	0.37416573867739	0.141	0.37549966711037	0.142	0.37682887362834	0.143	0.37815340802378
0.144	0.37947331922021	0.145	0.3807886552932	0.146	0.38209946349086	0.147	0.38340579025362
0.148	0.38470768123343	0.149	0.38600518131238	0.150	0.38729833462074	0.151	0.38858718455451
0.152	0.38987177379236	0.153	0.39115214431216	0.154	0.39242833740697	0.155	0.39370039370059
0.156	0.39496835316263	0.157	0.39623225512318	0.158	0.39749213828704	0.159	0.39874804074754
0.160	0.4	0.161	0.40124805295478	0.162	0.40249223594996	0.163	0.40373258476373
0.164	0.40496913462633	0.165	0.4062019202318	0.166	0.40743097574927	0.167	0.40865633483405
0.168	0.40987803063838	0.169	0.41109609582189	0.170	0.41231056256177	0.171	0.41352146256271
0.172	0.41472882706655	0.173	0.41593268686171	0.174	0.41713307229228	0.175	0.41833001326704
0.176	0.41952353926806	0.177	0.42071367935925	0.178	0.42190046219458	0.179	0.42308391602612
0.180	0.42426406871193	0.181	0.42544094772365	0.182	0.42661458015403	0.183	0.42778499272415
0.184	0.42895221179054	0.185	0.43011626335213	0.186	0.43127717305696	0.187	0.43243496620879
0.188	0.43358966777358	0.189	0.43474130238568	0.190	0.43588989435407	0.191	0.43703546766824
0.192	0.43817804600413	0.193	0.43931765272978	0.194	0.4404543109109	0.195	0.44158804331639
0.196	0.44271887242357	0.197	0.44384682042344	0.198	0.44497190922574	0.199	0.44609416046391
0.200	0.44721359549996	0.201	0.4483302354292	0.202	0.44944410108488	0.203	0.45055521304275

0.416 0.64498061986388   0.417 0.64575537163852   0.418 0.64652919500978   0.419 0.64730209330729
0.420 0.64807406984079   0.421 0.64884512790033   0.422 0.64961527075647   0.423 0.65038450166036
0.424 0.65115282384399   0.425 0.65192024052027   0.426 0.65268675488323   0.427 0.65345237010818
0.428 0.65421708935185   0.429 0.65498091575251   0.430 0.6557438524302   0.431 0.65650590248679
0.432 0.6572670690062   0.433 0.65802735505448   0.434 0.65878676368002   0.435 0.65954529791365
0.436 0.66030296076877   0.437 0.66105975524154   0.438 0.66181568431097   0.439 0.6625707509391
0.440 0.66332495807108   0.441 0.66407830863536   0.442 0.66483080554379   0.443 0.66558245169175
0.444 0.66633324995831   0.445 0.66708320320632   0.446 0.66783231428256   0.447 0.66858058601787
0.448 0.66932802122726   0.449 0.67007462271004   0.450 0.67082039324994   0.451 0.67156533561523
0.452 0.67230945255886   0.453 0.67305274681855   0.454 0.67379522111692   0.455 0.6745368781616
0.456 0.67527772064537   0.457 0.67601775124622   0.458 0.67675697262755   0.459 0.67749538743817
0.460 0.67823299831253   0.461 0.67896980787072   0.462 0.67970581871866   0.463 0.68044103344816
0.464 0.68117545463706   0.465 0.68190908484929   0.466 0.68264192663504   0.467 0.6833739825308
0.468 0.68410525505948   0.469 0.68483574673056   0.470 0.6855654600401   0.471 0.68629439747094
0.472 0.68702256149271   0.473 0.68774995456198   0.474 0.68847657912234   0.475 0.68920243760451
0.476 0.68992753242641   0.477 0.69065186599328   0.478 0.69137544069774   0.479 0.69209825891993
0.480 0.69282032302755   0.481 0.69354163537599   0.482 0.69426219830839   0.483 0.69498201415576
0.484 0.69570108523704   0.485 0.69641941385921   0.486 0.69713700231734   0.487 0.69785385289472
0.488 0.69856996786292   0.489 0.69928534948188   0.490 0.7   0.491 0.70071392165419
0.492 0.70142711667001   0.493 0.70213958726168   0.494 0.70285133563222   0.495 0.70356236397351
0.496 0.70427267446636   0.497 0.70498226928058   0.498 0.70569115057509   0.499 0.70639932049797
0.500 0.70710678118655   0.501 0.70781353476746   0.502 0.70851958335673   0.503 0.70922492905989
0.504 0.70992957397195   0.505 0.71063352017759   0.506 0.71133676975115   0.507 0.71203932475672
0.508 0.71274118724822   0.509 0.71344235926948   0.510 0.71414284285429   0.511 0.71484264002646
0.512 0.71554175279993   0.513 0.7162401831788   0.514 0.7169379331574   0.515 0.71763500472037
0.516 0.71833139984272   0.517 0.7190271204899   0.518 0.71972216861786   0.519 0.72041654617312
0.520 0.7211102550928   0.52 0.72180329730474   0.522 0.72249567472754   0.523 0.72318738927058
0.524 0.72387844283415   0.525 0.72456883730947   0.526 0.72525857457875   0.527 0.72594765651526
0.528 0.7266360849834   0.529 0.72732386183873   0.530 0.72801098892805   0.531 0.72869746808947
0.532 0.72938330115242   0.533 0.73006848993776   0.534 0.7307530362578   0.535 0.73143694191639
0.536 0.73212020870893   0.537 0.73280283842245   0.538 0.73348483283569   0.539 0.73416619371911
0.540 0.73484692283495   0.541 0.73552702193733   0.542 0.73620649277224   0.543 0.73688533707762
0.544 0.73756355658343   0.545 0.73824115301167   0.546 0.73891812807645   0.547 0.73959448348402
0.548 0.74027022093287   0.549 0.74094534211371   0.550 0.74161984870957   0.551 0.74229374239583
0.552 0.74296702484027   0.553 0.74363969770313   0.554 0.74431176263714   0.555 0.74498322128757
0.556 0.74565407529229   0.557 0.74632432628181   0.558 0.74699397587932   0.559 0.74766302570075
0.560 0.74833147735479   0.561 0.74899933244296   0.562 0.74966659255965   0.563 0.75033325929216
0.564 0.75099933422074   0.565 0.75166481891865   0.566 0.75232971495216   0.567 0.75299402388067
0.568 0.75365774725667   0.569 0.75432088662584   0.570 0.75498344352708   0.571 0.7556454194925
0.572 0.75630681604756   0.573 0.75696763471102   0.574 0.75762787699503   0.575 0.75828754440516
0.576 0.75894663844041   0.577 0.75960516059332   0.578 0.76026311234993   0.579 0.76092049518987
0.580 0.76157731058639   0.581 0.76223356000638   0.582 0.76288924491043   0.583 0.76354436675284
0.584 0.76419892698171   0.585 0.76485292703892   0.586 0.76550636836019   0.587 0.76615925237512
0.588 0.76681158050723   0.589 0.767463354174   0.590 0.76811457478686   0.591 0.7687652437513
0.592 0.76941536246685   0.593 0.77006493232714   0.594 0.77071395471991   0.595 0.77136243102708
0.596 0.77201036262475   0.597 0.77265775088327   0.598 0.77330459716725   0.599 0.77395090283557
0.600 0.77459666924148   0.601 0.77524189773257   0.602 0.77588658965083   0.603 0.77653074633269
0.604 0.77717436910902   0.605 0.77778174593052   0.606 0.77846001824114   0.607 0.7791020472313
0.608 0.77974354758472   0.609 0.78038452060507   0.610 0.78102496759067   0.611 0.78166488983451
0.612 0.78230428862432   0.613 0.78294316524254   0.614 0.78358152096639   0.615 0.78421935706791
0.616 0.78485667481394   0.617 0.78549347546622   0.618 0.78612976028134   0.619 0.78676553051084
0.620 0.78740078740118   0.621 0.78803553219382   0.622 0.78866976612521   0.623 0.78930349042684
0.624 0.78993670632526   0.625 0.7905694150421   0.626 0.7912016177941   0.627 0.79183331579317

0.840 0.91651513899117 0.841 0.91706052144883 0.842 0.91760557975636 0.843 0.91815031449104
0.844 0.91869472622847 0.845 0.91923881554251 0.846 0.91978258300535 0.847 0.92032602918748
0.848 0.92086915465771 0.849 0.92141195998316 0.850 0.92195444572929 0.851 0.92249661245991
0.852 0.92303846073715 0.853 0.92357999112151 0.854 0.92412120417183 0.855 0.92466210044535
0.856 0.92520268049763 0.857 0.92574294488265 0.858 0.92628289415275 0.859 0.92682252885868
0.860 0.92736184954957 0.861 0.92790085677296 0.862 0.92843955107481 0.863 0.928977793299949
0.864 0.92951600308978 0.865 0.93005376188691 0.866 0.93059120993055 0.867 0.93112834775878
0.868 0.93166517590817 0.869 0.93220169491371 0.870 0.93273790530888 0.871 0.93327380762561
0.872 0.9338094023943   0.873 0.93434469014385 0.874 0.93487967140162 0.875 0.93541434669349
0.876 0.93594871654381 0.877 0.93648278147545 0.878 0.9370165420098   0.879 0.93754999866674
0.880 0.93808315196469 0.881 0.93861600242059 0.882 0.93914855054991 0.883 0.93968079686668
0.884 0.94021274188345 0.885 0.94074438611134 0.886 0.94127573006001 0.887 0.94180677423769
0.888 0.94233751915118 0.889 0.94286796530585 0.890 0.94339811320566 0.891 0.94392796335314
0.892 0.94445751624941 0.893 0.9449867723942   0.894 0.94551573228583 0.895 0.94604439642123
0.896 0.94657276529594 0.897 0.94710083940413 0.898 0.94762861923857 0.899 0.94815610529069
0.900 0.94868329805051 0.901 0.94921019800674 0.902 0.9497368056467   0.903 0.95026312145637
0.904 0.95078914592038 0.905 0.95131487952202 0.906 0.95184032274326 0.907 0.95236547606473
0.908 0.95289033996573 0.909 0.95341491492424 0.910 0.95393920141695 0.911 0.9544631999192
0.912 0.95498691090507 0.913 0.9555103348473   0.914 0.95603347221737 0.915 0.95655632348545
0.916 0.95707888912043 0.917 0.95760116958993 0.918 0.95812316536028 0.919 0.95864487689655
0.920 0.95916630466254 0.921 0.95968744912081 0.922 0.96020831073262 0.923 0.96072888995804
0.924 0.96124918725583 0.925 0.96176920308357 0.926 0.96228893789755 0.927 0.96280839215287
0.928 0.96332756630338 0.929 0.96384646080172 0.930 0.9643650760993   0.931 0.96488341264632
0.932 0.96540147089177 0.933 0.96591925128346 0.934 0.96643675426797 0.935 0.96695398029069
0.936 0.96747092979583 0.937 0.96798760322641 0.938 0.96850400102426 0.939 0.96902012363005
0.940 0.96953597148327 0.941 0.97005154502222 0.942 0.97056684468407 0.943 0.97108187090482
0.944 0.97159662411929 0.945 0.97211110476118 0.946 0.97262531326303 0.947 0.97313925005623
0.948 0.97365291557105 0.949 0.9741663102366   0.950 0.9746794344809   0.951 0.97519228873079
0.952 0.97570487341204 0.953 0.97621718894926 0.954 0.97672923576598 0.955 0.9772410142846
0.956 0.97775252492643 0.957 0.97826376811165 0.958 0.97877474425937 0.959 0.97928545378761
0.960 0.97979589711327 0.961 0.9803060746522   0.962 0.98081598681914 0.963 0.98132563402777
0.964 0.98183501669069 0.965 0.98234413521943 0.966 0.98285299002445 0.967 0.98336158151516
0.968 0.98386991009991 0.969 0.98437797618598 0.970 0.98488578017961 0.971 0.985393322486
0.972 0.9859006035093   0.973 0.98640762365262 0.974 0.98691438331803 0.975 0.98742088290658
0.976 0.98792712281828 0.977 0.98843310345213 0.978 0.98893882520609 0.979 0.98944428847712
0.980 0.98994949366117 0.981 0.99045444115315 0.982 0.990959131347   0.983 0.99146356463564
0.984 0.99196774141098 0.985 0.99247166206396 0.986 0.99297532698451 0.987 0.99347873656158
0.988 0.99398189118313 0.989 0.99448479123615 0.990 0.99498743710662 0.991 0.99548982917959
0.992 0.9959919678391   0.993 0.99649385346825 0.994 0.99699548644916 0.995 0.997496867163
0.996 0.99799799598997 0.997 0.99849887330933 0.998 0.99899949949937 0.999 0.99949987493746
1.000   1

$\sqrt{10}$ = $\sqrt{10^1}$ = 3.1622776601684

$\sqrt{100}$ = $\sqrt{10^2}$ = $\sqrt{(10)(10)}$ = $\sqrt{10}$ $\sqrt{10}$ = 10

$\sqrt{1000}$ = $\sqrt{10^3}$ = $\sqrt{(100)(10)}$ = $\sqrt{100}$ $\sqrt{10}$ = 31.622776601684

$\sqrt{10,000}$ = $\sqrt{10^4}$ = $\sqrt{(1,000)(10)}$ = $\sqrt{1000}$ $\sqrt{10}$ = 100

$\sqrt{100,000}$ = $\sqrt{10^5}$ = $\sqrt{(10,000)(10)}$ = $\sqrt{10,000}$ $\sqrt{10}$ = 316.22776601684

$\sqrt{1,000,000}$ = $\sqrt{10^6}$ = $\sqrt{(100,000)(10)}$ = $\sqrt{100,000}$ $\sqrt{10}$ = 1000

A CUBE ROOT TABLE

The following table is for three decimal point places. A fourth place would require a table having ten times more values.

0.000	0	0.001	0.1	0.002	0.12599210498949	0.003	0.14422495703074
0.004	0.15874010519682	0.005	0.17099759466767	0.006	0.18171205928321	0.007	0.19129311827724
0.008	0.2	0.009	0.20800838230519	0.010	0.21544346900319	0.011	0.22239800905693
0.012	0.22894284851067	0.013	0.23513346877208	0.014	0.24101422641752	0.015	0.24662120743305
0.016	0.25198420997897	0.017	0.25712815906582	0.018	0.26207413942089	0.019	0.26684016487219
0.020	0.27144176165949	0.021	0.27589241763811	0.022	0.28020393306554	0.023	0.28438669798516
0.024	0.28844991406148	0.025	0.29240177382129	0.026	0.29624960684074	0.027	0.3
0.028	0.30365889718757	0.029	0.30723168256858	0.030	0.31072325059539	0.031	0.31413806523914
0.032	0.31748021039364	0.033	0.32075343299958	0.034	0.32396118012775	0.035	0.32710663101886
0.036	0.33019272488946	0.037	0.3332221851646	0.038	0.3361975406799	0.039	0.33912114430142
0.040	0.34199518933534	0.041	0.34482172403827	0.042	0.34760266448865	0.043	0.35033980603867
0.044	0.35303483353261	0.045	0.35568933044901	0.046	0.35830478710159	0.047	0.36088260801387
0.048	0.36342411856643	0.049	0.3659305710023	0.050	0.36840314986404	0.051	0.37084297692662
0.052	0.37325111568172	0.053	0.37562857542211	0.054	0.37797631496846	0.055	0.38029524607614
0.056	0.38258623655448	0.057	0.38485011312768	0.058	0.38708766406278	0.059	0.38929964158733
0.060	0.39148676411689	0.061	0.39364971831022	0.062	0.39578916096804	0.063	0.39790572078964
0.064	0.4	0.065	0.40207257585891	0.066	0.40412400206222	0.067	0.40615481004457
0.068	0.40816551019173	0.069	0.41015659297023	0.070	0.41212852998086	0.071	0.41408177494229
0.072	0.41601676461038	0.073	0.41793391963812	0.074	0.41983364538084	0.075	0.42171633265087
0.076	0.42358235842549	0.077	0.4254320865115	0.078	0.42726586816979	0.079	0.42908404270262
0.080	0.43088693800638	0.081	0.43267487109222	0.082	0.43444814857686	0.083	0.43620706714548
0.084	0.43795191398879	0.085	0.43968296721582	0.086	0.44140049624421	0.087	0.44310476216936
0.088	0.44479601811386	0.089	0.44647450955845	0.090	0.44814047465572	0.091	0.44979414452754
0.092	0.4514357435474	0.093	0.45306548960835	0.094	0.45468359437763	0.095	0.4562902635387
0.096	0.45788569702133	0.097	0.4594700892207	0.098	0.46104362920584	0.099	0.46260650091827
0.100	0.46415888336128	0.101	0.46570095078038	0.102	0.46723287283553	0.103	0.46875481476536
0.104	0.47026693754415	0.105	0.47176939803165	0.106	0.47326234911634	0.107	0.47474593985234
0.108	0.47622031559046	0.109	0.4776856181035	0.110	0.47914198570628	0.111	0.48058955337053
0.112	0.48202845283505	0.113	0.48345881271116	0.114	0.48488075858399	0.115	0.48629441310943
0.116	0.48769989610733	0.117	0.48909732465087	0.118	0.4904868131524	0.119	0.49186847344587
0.120	0.49324241486609	0.121	0.49460874432487	0.122	0.49596756638423	0.123	0.49731898332686
0.124	0.49866309522386	0.125	0.5	0.126	0.50132979349646	0.127	0.50265256953135
0.128	0.50396841995795	0.129	0.50527743472086	0.130	0.50657970191009	0.131	0.50787530781327
0.132	0.50916433696595	0.133	0.51044687220015	0.134	0.5117229946912	0.135	0.51299278400301
0.136	0.51425631813165	0.137	0.51551367354758	0.138	0.51676492523636	0.139	0.51801014673803
0.140	0.51924941018511	0.141	0.52048278633942	0.142	0.52171034462762	0.143	0.5229321531756
0.144	0.52414827884178	0.145	0.52535878724929	0.146	0.52656374281714	0.147	0.52776320879041
0.148	0.52895724726942	0.149	0.53014591923809	0.150	0.53132928459131	0.151	0.5325074021615
0.152	0.53368032974439	0.153	0.53484812412394	0.154	0.53601084109654	0.155	0.53716853549448
0.156	0.53832126120873	0.157	0.53946907121096	0.158	0.54061201757502	0.159	0.54175015149772
0.160	0.54288352331898	0.161	0.54401218254148	0.162	0.54513617784964	0.163	0.54625555712814
0.164	0.54737036747984	0.165	0.54848065524326	0.166	0.5495864660095	0.167	0.55068784463874
0.168	0.55178483527622	0.169	0.55287748136789	0.170	0.55396582567545	0.171	0.55504991029115
0.172	0.55612977665212	0.173	0.55720546555426	0.174	0.55827701716584	0.175	0.5593444710407
0.176	0.56040786613108	0.177	0.56146724080015	0.178	0.56252263283419	0.179	0.56357407945442
0.180	0.56462161732862	0.181	0.56566528258229	0.182	0.56670511080971	0.183	0.56774113708454
0.184	0.56877339597031	0.185	0.56980192153051	0.186	0.57082674733849	0.187	0.57184790648713
0.188	0.57286543159824	0.189	0.57387935483172	0.190	0.57488970789448	0.191	0.57589652204924
0.192	0.57689982812296	0.193	0.57789965651521	0.194	0.57889603720624	0.195	0.5798889997649

0.404 0.73925417917137 0.405 0.73986362229914 0.406 0.74047206305221 0.407 0.74107950554206
0.408 0.74168595385324 0.409 0.74229141204362 0.410 0.74289588414466 0.411 0.74349937416159
0.412 0.74410188607369 0.413 0.74470342383451 0.414 0.74530399137208 0.415 0.74590359258916
0.416 0.74650223136345 0.417 0.7470999115478   0.418 0.74769663697046 0.419 0.74829241143526
0.420 0.74888723872185 0.421 0.74948112258591 0.422 0.75007406675932 0.423 0.75066607495043
0.424 0.75125715084421 0.425 0.75184729810249 0.426 0.75243652036411 0.427 0.75302482124518
0.428 0.75361220433924 0.429 0.75419867321743 0.430 0.75478423142876 0.431 0.75536888250019
0.432 0.75595262993692 0.433 0.75653547722251 0.434 0.75711742781909 0.435 0.75769848516751
0.436 0.75827865268758 0.437 0.75885793377818 0.438 0.75943633181749 0.439 0.76001385016311
0.440 0.76059049215228 0.441 0.76116626110202 0.442 0.76174116030933 0.443 0.76231519305129
0.444 0.76288836258531 0.445 0.76346067214923 0.446 0.76403212496151 0.447 0.76460272422136
0.448 0.76517247310896 0.449 0.76574137478553 0.450 0.76630943239355 0.451 0.76687664905691
0.452 0.767443027881   0.453 0.76800857195294 0.454 0.76857328434166 0.455 0.76913716809812
0.456 0.76970022625536 0.457 0.77026246182874 0.458 0.770823877816   0.459 0.77138447719747
0.460 0.77194426293616 0.461 0.772250323797793 0.462 0.77306140525159 0.463 0.773361876766908
0.464 0.77417532812556 0.465 0.77473108949958 0.466 0.77528605465318 0.467 0.77584022643205
0.468 0.77639360766563 0.469 0.77694620116725 0.470 0.77749800973426 0.471 0.77804903614814
0.472 0.77859928317465 0.473 0.77914875356392 0.474 0.7796974500506   0.475 0.78024537535394
0.476 0.78079253217797 0.477 0.78133892321156 0.478 0.78188455112855 0.479 0.78242941858791
0.480 0.78297352823377 0.481 0.78351688269562 0.482 0.78405948458836 0.483 0.78460133651244
0.484 0.78514244105397 0.485 0.7856828007848   0.486 0.78622241826267 0.487 0.78676129603128
0.488 0.78729943662043 0.489 0.78783684254609 0.490 0.78837351631052 0.491 0.78890946040238
0.492 0.78944467729681 0.493 0.78997916945556 0.494 0.79051293932705 0.495 0.79104598934652
0.496 0.79157832193608 0.497 0.79210993950484 0.498 0.79264084444897 0.499 0.79317103915185
0.500 0.7937005259841   0.501 0.79422930730373 0.502 0.79475738545619 0.503 0.79528476277449
0.504 0.79581144157928 0.505 0.79633742417892 0.506 0.79686271286962 0.507 0.79738730993548
0.508 0.79791121764858 0.509 0.79843443826911 0.510 0.7989569740454   0.511 0.79947882721406
0.512   0.8   0.513 0.80052049461658 0.514 0.80104031326566 0.515 0.80155945813766
0.516 0.80207793141168 0.517 0.80259573525558 0.518 0.80311287182602 0.519 0.80362934326857
0.520 0.80414515171781 0.521 0.80466029929735 0.522 0.80517478811994 0.523 0.80568862028756
0.524 0.80620179789146 0.525 0.80671432301227 0.526 0.80722619772006 0.527 0.80773742407439
0.528 0.80824800412444 0.529 0.80875793990901 0.530 0.80926723345665 0.531 0.8097758867857
0.532 0.81028390190439 0.533 0.81079128081087 0.534 0.8112980254933   0.535 0.81180413792992
0.536 0.81230962008914 0.537 0.81281447392954 0.538 0.8133187014   0.539 0.81382230443977
0.540 0.81432528497847 0.541 0.81482764493623 0.542 0.81532938622369 0.543 0.81583051074213
0.544 0.81633102038347 0.545 0.81683091703038 0.546 0.81733020255631 0.547 0.81782887882558
0.548 0.81832694769342 0.549 0.81882441100602 0.550 0.81932127060065 0.551 0.81981752830563
0.552 0.82031318594047 0.553 0.82080824531588 0.554 0.82130270823386 0.555 0.82179657648771
0.556 0.82228985186215 0.557 0.82278253613333 0.558 0.82327463106891 0.559 0.82376613842809
0.560 0.82425705996171 0.561 0.82474739741226 0.562 0.82523715251394 0.563 0.82572632699276
0.564 0.82621492256654 0.565 0.82670294094496 0.566 0.82719038382969 0.567 0.82767725291434
0.568 0.82816354988457 0.569 0.82864927641815 0.570 0.82913443418497 0.571 0.82961902484712
0.572 0.83010305005894 0.573 0.83058651146705 0.574 0.83106941071041 0.575 0.83155174942038
0.576 0.83203352922076 0.577 0.83251475172783 0.578 0.8329954185504   0.579 0.83347553128986
0.580 0.833395509154026 0.581 0.83443410088829 0.582 0.83491256091336 0.583 0.83539047318768
0.584 0.83586783927625 0.585 0.83634466073692 0.586 0.83682093912046 0.587 0.83729667597059
0.588 0.83777187282401 0.589 0.83824653121044 0.590 0.83872065265271 0.591 0.83919423866676
0.592 0.83966729076168 0.593 0.84013981043978 0.594 0.84061179919662 0.595 0.84108325852103
0.596 0.84155418989519 0.597 0.84202459479466 0.598 0.84249447468838 0.599 0.84296383103878
0.600 0.84343266530175 0.601 0.84390097892674 0.602 0.84436877335676 0.603 0.84483605002843
0.604 0.84530281037202 0.605 0.84576905581149 0.606 0.84623478776454 0.607 0.84670000764262
0.608 0.84716471685098 0.609 0.84762891678872 0.610 0.84809260884881 0.611 0.84855579441814
0.612 0.84901847487755 0.613 0.84948065160187 0.614 0.84994232595993 0.615 0.85040349931464

0.828 0.93902418730036   0.829 0.93940206427768   0.830 0.9397796374953   0.831 0.94015690756292
0.832 0.9405338750883   0.833 0.94091054067726   0.834 0.94128690493367   0.835 0.94166296845947
0.836 0.94203873185468   0.837 0.94241419571742   0.838 0.94278936064388   0.839 0.94316422722837
0.840 0.94353879606331   0.841 0.94391306773924   0.842 0.94428704284482   0.843 0.94466072196687
0.844 0.94503410569032   0.845 0.94540719459828   0.846 0.94577998927201   0.847 0.94615249029094
0.848 0.94652469823267   0.849 0.946896613673   0.850 0.94726823718591   0.851 0.94763956934358
0.852 0.9480106107164   0.853 0.94838136187297   0.854 0.94875182338013   0.855 0.94912199580293
0.856 0.94949187970468   0.857 0.94986147564691   0.858 0.95023078418943   0.859 0.95059980589029
0.860 0.95096854130582   0.861 0.95133699099061   0.862 0.95170515549756   0.863 0.95207303537784
0.864 0.95244063118092   0.865 0.95280794345458   0.866 0.95317497274491   0.867 0.953354171959631
0.868 0.95390818455153   0.869 0.95427436815162   0.870 0.954640270936   0.871 0.95500589344243
0.872 0.955371236207   0.873 0.9557362997642   0.874 0.95610108464686   0.875 0.95646559138619
0.876 0.9568298205118   0.877 0.95719377255166   0.878 0.95755744803215   0.879 0.95792084747805
0.880 0.95828397141256   0.881 0.95864682035727   0.882 0.95900939483222   0.883 0.959371169535585
0.884 0.95973372244507   0.885 0.96009547661519   0.886 0.96045695838   0.887 0.96081816825173
0.888 0.96117910674107   0.889 0.96153977435718   0.890 0.9619001716077   0.891 0.96226029899875
0.892 0.96262015703491   0.893 0.96297974621928   0.894 0.96333906705346   0.895 0.96369812003753
0.896 0.96405690567009   0.897 0.96441542444828   0.898 0.96477367686772   0.899 0.96513166342261
0.900 0.96548938460563   0.901 0.96584684090804   0.902 0.96620403281964   0.903 0.96656096082877
0.904 0.96691762542233   0.905 0.96727402708579   0.906 0.96763016630321   0.907 0.96798604355718
0.908 0.9683416593289   0.909 0.96869701409817   0.910 0.96905210834335   0.911 0.96940694254143
0.912 0.96976151716798   0.913 0.97011583269719   0.914 0.97046988960187   0.915 0.97082368835345
0.916 0.97117722942197   0.917 0.97153051327613   0.918 0.97188354038325   0.919 0.97223631120928
0.920 0.97258882621886   0.921 0.97294108587523   0.922 0.97329309064034   0.923 0.97364484097478
0.924 0.97399633733779   0.925 0.97434758018733   0.926 0.97469856998   0.927 0.97504930717112
0.928 0.97539979221466   0.929 0.97575002556333   0.930 0.97610000766851   0.931 0.97644973898029
0.932 0.97679921994749   0.933 0.97714845101763   0.934 0.97749743263695   0.935 0.97784616525042
0.936 0.97819464930175   0.937 0.97854288523337   0.938 0.97889087348648   0.939 0.97923861450098
0.940 0.97958610871556   0.941 0.97993335656766   0.942 0.98028035849346   0.943 0.98062711492793
0.944 0.9809736263048   0.945 0.98131989305658   0.946 0.98166591561454   0.947 0.98201169440876
0.948 0.9823572298681   0.949 0.98270252242022   0.950 0.98304757249156   0.951 0.98339238050738
0.952 0.98373694689175   0.953 0.98408127206753   0.954 0.98442535645644   0.955 0.98476920047897
0.956 0.98511280455447   0.957 0.98545616910112   0.958 0.98579929453591   0.959 0.98614218127471
0.960 0.98648482973219   0.961 0.9868272403219   0.962 0.98716941345623   0.963 0.98751134954643
0.964 0.9878530490026   0.965 0.98819451223374   0.966 0.98853573964768   0.967 0.98887673165115
0.968 0.98921748864974   0.969 0.98955801104794   0.970 0.98989829924913   0.971 0.99023835365556
0.972 0.99057817466839   0.973 0.99091776268768   0.974 0.99125711811239   0.975 0.99159624134039
0.976 0.99193513276846   0.977 0.9922737927923   0.978 0.99261222180654   0.979 0.9929504202047
0.980 0.99328838837927   0.981 0.993362612672164   0.982 0.99396363562217   0.983 0.99430091547012
0.984 0.99463796665372   0.985 0.99497478956015   0.986 0.99531138457553   0.987 0.99564775208494
0.988 0.99598389247242   0.989 0.99631980612098   0.990 0.9966554934126   0.991 0.99699095472821
0.992 0.99732619044773   0.993 0.99766120095007   0.994 0.9979959866131   0.995 0.99833054781369
0.996 0.99866488492771   0.997 0.99899899832999   0.998 0.9993328883944   0.999 0.99966655549379
1.000   1

Table Of Fractional Powers Of (e)

For using this table, see the article on: Calculating Powers Of (e) Using A Small Table Of Constants.

x	e^x	e^0.x	e^0.0x	e^0.00x	e^0.000x
0	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1	**2.718281828459045**	1.105170918075648	1.010050167084168	1.001000500166708	1.000100005000167
2	7.38905609893065	1.22140275816017	1.020201340026756	1.002002001334	1.000200020001334
3	20.08553692318767	1.349858807576003	1.030454533953517	1.003004504503377	1.0003000450045
4	54.59815003314424	1.49182469764127	1.040810774192388	1.004008010677342	1.000400080010668
5	148.4131591025766	1.648721270700128	1.051271096376024	1.005012520859401	1.000500125020836
6	403.4287934927351	1.822118800390509	1.06183654654536	1.006018036054065	1.000600180036005
7	1096.633158428459	2.013752707470477	1.072508181254217	1.007024557266849	1.000700245057177
8	2980.957987041728	2.225540928492468	1.083287067674959	1.008032085504274	1.00080032008535
9	8103.083927575384	2.45960311115695	1.09417428370521	1.009040621773868	1.000900405121527

x	e^0.0000x	e^0.00000x	e^0.000000x	e^0.0000000x	e^0.00000000x
0	1.000000000000000	1.0000000000000	1.000000000000000	1.000000000000000	1.000000000
1	1.00001000005	1.0000010000005	1.000000100000005	1.00000001	1.000000001
2	1.000020000200001	1.000002000002	1.00000020000002	1.00000002	1.000000002
3	1.000030000450005	1.0000030000045	1.000000300000045	1.000000030000001	1.000000003
4	1.000040000800011	1.000004000008	1.00000040000008	1.000000040000001	1.000000004
5	1.000050001250021	1.0000050000125	1.000000500000125	1.000000050000001	1.000000005
6	1.00006001800036	1.000006000018	1.00000060000018	1.000000060000002	1.000000006
7	1.000070002450057	1.0000070000245	1.000000700000245	1.000000070000003	1.000000007
8	1.000080003200085	1.000008000032	1.00000080000032	1.000000080000003	1.000000008
9	1.000090004050122	1.0000090000405	1.000000900000405	1.000000090000004	1.000000009

Note that $\sqrt{e}$ = e^(1/2) = e^0.5

EXTRAS AND LATE ENTRIES

These topics and-or articles are either not specifically math oriented and-or late entries in this book, but they may be interesting and useful for many people.   The word "maths" now seems to includes all useful things of science.   Distribution and access of knowledge can sometimes be a problem due to a variety of reasons, and it is therefore practical to just include what I could in this one book, and that is why it has a generous amount of pages.

SOME TIPS TO TYPE EASIER

This discussion is about typing better so as to record your thoughts, data, and communicate better via typed messages. Many have never learned to formally type and rather "chicken pluck" each key with one finger rather than use both hands and most fingers. The notes below will help you use and-or remember the keyboard, key layout or positioning.

Standard key layout or positions:

```
1  2   3  4  5  6  7  8  9  0
 Q  W  E  R  T  Y  U  I  O  P
  A  S  D  F  G  H  J  K  L  ;
   Z  X  C  V  B  N  M  ,  .  /
            (space key)
```

People who do much typing should place their hands on what is called the "home row" of keys. These keys are:

    Left Hand   Right Hand   : excluding the thumb finger

     A S D F   J K L ;   ,   :   **;** = semicolon   and   **:** = colon, press shift and semicolon for colon

    Once you place your fingers on these keys, you can feel a slight bump on the F and J keys on most modern
    keyboards. This will help ensure your fingers are position correctly on the "home row" of keys. The home row
    is also called the middle set or row of keys. The row above it is often called the upper row. The row of keys
    below the home row is often called the lower set of keys, or simply the bottom row.

    All the other keys are pressed by moving a "home row" finger up, down, left or right such as for the G and H keys.
    The finger used is usually the one of closest to the key and-or the one most comfortable.

    Key or finger   The other keys typically used or pressed by that same finger.

        A   Q , Z
        S   W , X
        D   E
        F   R , T , C , V , G , B

        J   H , Y , U , N , M
        K   I , **,** (comma)
        L   O , **.** (period)
        ;   P , / (forward slash) , **"** (quotation marks , **?** (question mark)

    By using the above method, all 26 letters of the English alphabet can be typed. The shift keys which are just left
    land right of the bottom row are usually used to type in a capital or uppercase letter by first pressing a shift key with
    a "pinky" or small finger that is most opposite from your thumb finger, and then also pressing the desired key. If the

COMMON MUSIC NOTATION OF THE TIMING AND DURATION OF NOTES

[Fig 253]

A music symbol can represent a duration or beat, a note, or a rest where no note is played. The green line is used to indicate the timing of the start of one of the 4 beats in a measure.

1   2   3   4   : beat number

Whole Note, has the duration of the entire or whole measure.
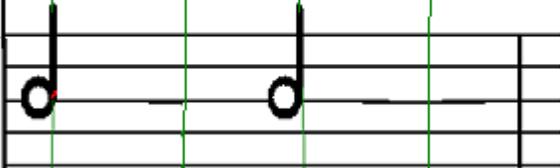
Half Notes
Half of a 4 beat measure.
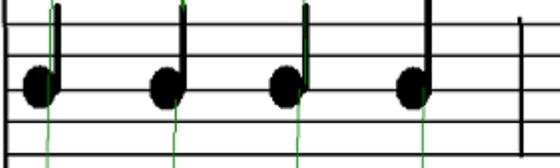
Quarter Notes
There are 4 beats in a common measure of music.

Each quarter note has a duration of 1/4 = 0.25 = 25% of the measure.
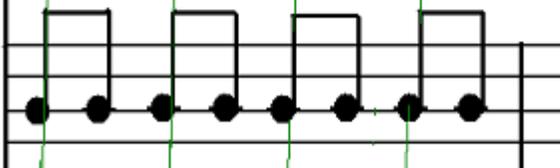
Eighth Notes
An eighth note has half the duration of 1 beat. There are two eight notes in one beat.
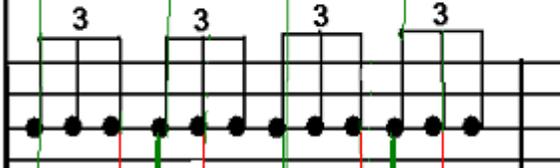
Quarter Note Triplets

A triplet puts three notes or durations in the count or duration of two notes.

For two notes, each will have 1/2 = 0.5 = 50% of the beat.
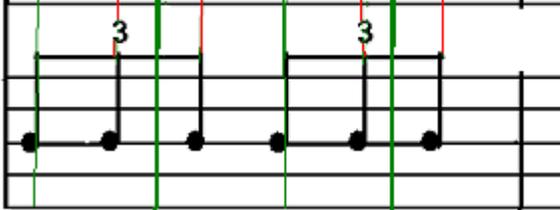
For three notes in a beat, each will have just 1/3 = 0.333 = 33.3% of the beat. The timing of each note of a triplet is at 0%, 33.3% and 66.7%

Half Note Triplets

Though there are usually many notes indicated on "music staffs" which are a notation system for representing music, I have only used the "B" note here so as to keep things simple

In your mind, you can count the eighth notes as: 1, 2, 3, 4, 5 , 6, 7, 8, however, they are more commonly counted as two sets of 4 as: 1, 2, 3, 4 , and 1, 2, 3, 4.

generally first learn the higher frequency, thinner strings on a guitar.   Being familiar with music notation helps people quickly learn songs, a guide, and-or how they were intended to be played in a formal manner.   Guitarists and or other musical instrument players can improvise by just finding or knowing the chords of a song, and then playing some pleasing sounding notes in the scale associated with the known or indicated chord.   A guitarist may even play, or need to play, each chord and-or note of the song using a certain number of half-steps higher or lower on the fretboard so as to change the entire pitch of the entire song, perhaps so as the singer can sing it easier.

EXAMPLE GUITAR CHORDS

A chord is two or more sounds or notes played at the same time.   The notes chosen to be played are usually those that have a pleasing sound.

A C major chord includes these notes in the C major scale: **1**, **3** and **5**.

```
     Note #:   1   2   3   4   5   6   7   8
  Note name:   C   D   E   F   G   A   B   C
```

Ex. C major chord = C, E, G   : For this chord and or scale, C is called the "root note". here, root = initial or base
                                         note. C is the basis of that chord, and the starting or initial note of that scale.

The notes of any chord can be played and-or noted with musical notation as any combination or order of the notes associated with that chord. For example, and with the first note being lowest in frequency and on the music staff:

Ex. C major chords = Cmaj = (C, E, G) and-or (C, G, E)   and-or
                                 (E, C, G) and-or (E, G, C) and-or
                                 (G, C, E) and-or (G, E, C)

A C minor chord includes these notes in the C minor scale: 1, 3b and 5.

In relationship to the major scale, a minor scale has a flatted third note, and this means that it is one fret lower on the guitar fretboard.   It also means that the third note is a half-step (or "1 chromatic step") less than the corresponding note in the major scale.

Ex. A C minor chord = Cm = (C, Eb, G)

A major chord can be thought of as having a direct or powerful sound and feeling, whereas a minor chord can be thought of as having a soft, moody or serious sound and feeling.

[FIG 255]

Notice for example that of the F chord shown starts on a blank space between the lines of musical staff, and that the next two notes of that chord are also on the next consecutive spaces. It could be said that from space to space is a "third note higher" or three steps in the scale, and that from a line to line is a "third note higher" or three steps in the scale.

There is another system of music notation for guitar playing and it is called Guitar Tablature or Guitar Tab. This notation is similar to a combination of both the standard musical notation system as shown in the above image on the right with a five line music staff, and also similar to the 6 strings of the guitar as shown on the left. The combined result is a 6 line music staff where each line is the string, and the note to play on that string is indicated by the fret number and-or fret position on that string. A bass guitar is similar to a guitar, and it has the 4 thicker strings as that of a guitar: E, A , D, G, of which for the bass guitar also thicker and usually longer so as to produce a lower frequency sound.

The "(musical) key of C" is often used for teaching what notes are on the musical notation staff. The key of C does not have any sharped or flatted notes. Other keys will have at least one sharped note or one flatted note. For example the scale or key of G has these following notes: G, A , B , C , D , E , F#, G.   We see that the scale contains an F-sharp note. Still there is a pattern as with the C scale or key:

root note (1), whole-step (2), whole-step (3), half-step (4), whole-step (5), whole-step (6), whole-step (7), half-step (8).

A whole-step is two semi-tones higher in frequency, and since each fret is a half-step or semi-tone in a scale, a whole-step is two semi-tomes and-or frets higher on the guitar fretboard. The key or scale of F has one flatted note, and that is Bb ("B-flat"): F, G, A Bb, C, D, E, F.

Here are the approximate frequencies of the standard guitar tuning based on a standardized A note being 440 hz.:

String	Frequency in hz.	
Bass E	82	
A	110	:The A string on a guitar may sometimes be noted as 220 or 440 hz. such as on some "reed tuners".
D	147	
G	196	
B	247	:this is practically 250hz, and which is two octaves lower than 1000hz...500hz..250hz..125hz...62.5hz
High E	330	

These frequencies shown above can be verified by the discussion about string length and frequency in the book.

Here are some of the common major and minor chords and their notes.

	Major Chord Notes	Minor Chord Notes	8 Note Of The Common "Western" Major Scale	
	(1 , 3, 5)	(1 , b3, 5)	**1** 2 **3** 4 **5** 6 7 8	
C	(C, E, G)	(C, Eb, G)	**C**, D, **E** , F, **G**, A , B , C	
D	(D, F#, A)	(D, F, A)	D, E , F#, G, A , B , C#, D	
E	(E, G#, B)	(E, G, B)	E , F#, G#, A , B , C#, D#, E	
F	(F, A, C)	(F, Ab, C)	F, G, A , Bb , C , D , E , F	: Bb = A# , "B flat equals A sharp"
G	(G, B, D)	(G, Bb, D)	G, A , B , C , D , E , F#, G	
A	(A, C#, E)	(A, C, E)	A , B , C# , D , E , F# ,G# , A	
B	(B, D#, F#)	(B, D, F#)	B , C# , D# , E , F# ,G# , A#, B	
C	(C, E, G)	(C, Eb, G)		

As shown above, the major chord has the following notes of its major scale: (1,   3,   5). As shown above, the minor chord also has the following notes of its minor scale: (1,   3,   5). In reference to the major scale, this 3 note of the minor scale is

Observe the patterns of the position of notes in a particular scale on the fretboard so as to help play the notes in scale. Improvising and-or "playing along" with recordings is a way to get involved with the music. Often a player may need to play both chords and single notes of a scale, and of which are could located near or at the chord being played.

On a scientific level, it is possible to create reference sound frequencies. If you had a wheel with say 60 bumps and-or gear teeth equally spaced around its circumference, rim or edge, and then spun that wheel at 1 revolution or rotation per second while holding a rod against that wheels edge or circumference, you would hear a 60hz sound tone. This is due to the 60 bumps and-or sound pulses per second created when the rod collided with the edge of the wheel and some of that energy from the collision was converted to sound energy. More bumps and-or a faster rotation can be used to create higher frequencies. This method discussed here was actually the method used to create the various notes and their corresponding frequencies with the first electronic organs, but instead of a stick against the circumference of the organs various diameter sized, spinning tone-wheels, a electro-magnetic transducer (ie., energy converter) or "(energy) pickup" was used near the circumference edge of each wheel, and its construction was essentially a coil of wire wrapped about a magnet. The varying or AC signal output from this pickup usually had a (smooth, undistorted) sine waveform, much like that from a bell. These organs were not completely electric like modern organs which use only electronic oscillators, but they were both partly mechanical, and partly electrical, hence being electromechanical. The initial invention of the tone-wheel is credited to Thaddeus Cahil in 1896, and the organ was called the Telharmonium. Because of its large size and cost, only a few Telharmonium organs were built. When electronic signal amplification was made practical after the invention of the vacuum tube amplifier, electronic organs would become smaller and more practical, such as the electromechanical organs produced by the Hammond Co. in 1935. Hammond organs were relatively affordable and were often installed in church buildings that had limited space, and since a full sized pipe organ was very large. The sound of an electromechanical Hammond organ can heard within many popular musical recordings, particularly from the 1950's through the 1970's.

Ex. To make a 1000hz tone, a large diameter wheel with 100 bumps about it will need to spin at 10 revolutions per second = 10rps:

Each physical bump or vibration on the tone-wheel will produce 1 electrical pulse or vibration.

Frequency = (revolutions per second)(number of bumps on the wheels edge for one revolution) =

Frequency = (10 revolutions / second) (100 bumps, pulses or vibrations / revolution) =

Frequency = $\frac{10 \text{ revolutions}}{1s}$ x $\frac{100 \text{ bumps}}{\text{revolution}}$ = $\frac{1000 \text{ bumps}}{1 \text{ second}}$ =

Frequency = vibrations per second = bumps per second = waves per second = cycles per second = 1000 hz

Note that this wheel has a rotational frequency or "angular velocity" of 10 revolutions per second, while the outputs sound of this system has a frequency of 1000 cycles per second. Each note on the organ's keyboard had a corresponding tone-wheel.

The "pickup" in an electric guitar is essentially a small electricity and-or signal (ie.,having a frequency part) generator. It is usually made by winding a coil of wire around a permanent (no electricity and-or current needed to produce the magnetism) magnet. When the string made of metal moves and-or vibrates near the pickup, the magnetic field will move slightly and produce an electric signal in the coil, and which is then sent to an electronic amplifier so as to make it a stronger signal with more power, particularly current, so as to move and-or vibrate an audio speaker.

The first assigned note, the A-note and its corresponding frequency is most likely due to the standardized 60 seconds in a minute. The multiples of the 60 hz frequency, musical note and-or its octaves (double the frequency) are then: 60, 120, 240, 480 and 960hz. Through the years, the frequencies of the standardized notes have slightly varied due to various reasons. Today, 440hz is the modern, standardized frequency of the A-note and-or tone, and 55hz is a lower frequency octave of the A-note. Musicians are generally not concerned with the actual frequency of a note(s), but are rather

COMMON GUITAR SCALE LENGTHS AND FRET SPACINGS

These tables were made by using a computer program made by the author of the Mathization ebook.

In general, the longer the scale length, the more the tension upon the strings is needed for standard tuning (EADGBE). This is due to that the length of the open or unfetted string is then longer which lowers its natural or fundamental frequency and when at the same standard tuning tension, and then more tension will be needed to raise it up to the desired frequency and-or note. An alternative to raising the tension on the strings so as to obtain a higher frequency and-or note is to use a mechanical device called a capo, and which is like movable clamp that can be set by the player and it is used to simulate the nut of a shorter scale length stringed instrument. New players and-or people with small hands may wish to use a capo and-or obtain a short scale guitar which usually has a thinner neck. A problem now with a very short scale length guitar is if the strings will have enough tension to feel right when tuned to standard tuning, but it is not usually an issue for many. With a capo, another less used option is to tune each string of the guitar a fret (ie., half-tone) or more lower than the standard tuning and place the capo up from the nut at the same number of steps of lower tuning, and so as to obtain a standard tuning. Using this method will also reduce the tension of the strings. The author of this ebook has a computer program to calculate the fret location and-or spacings given an input scale length. Shown below are the fret spacings for a 24.75" (regular) and 23.0" (medium) scale length guitar, and a 30.0 " "short-scale" bass guitar. A standard bass scale length is 34", a medium bass scale length is 32", and a short-scale base length is 30". Generally, the shorter the scale length, the neck thickness and width is also thinner.

SCALE LENGTH = **24.75** inches = 628.65 millimeters = 62.865 centimeters = 62 cm + 8_7_ mm   =~ 63 cm = 0.63 m

1 inch = 1" = 2.54 cm = 25.4 mm

```
FRET SIZE" In.+ 32nds   mm    FROM NUT" In.+ 32nds   mm

 1  1.389  1 + 12.45  35.283      1.389   1 + 12.45  35.283
 2  1.311  1 + 9.96  33.303       2.700   2 + 22.41  68.587
 3  1.238  1 + 7.60  31.434       3.938   3 + 30.01  100.020
 4  1.168  1 + 5.38  29.670       5.106   5 +  3.39  129.690
 5  1.103  1 + 3.28  28.004       6.208   6 +  6.67  157.695
 6  1.041  1 + 1.30  26.433       7.249   7 +  7.97  184.127
 7  0.982  0 + 31.43  24.949      8.231   8 +  7.40  209.076
 8  0.927  0 + 29.67  23.549      9.158   9 +  5.07  232.625
 9  0.875  0 + 28.00  22.227      10.034  10 +  1.07  254.852
10  0.826  0 + 26.43  20.980      10.860  10 + 27.51  275.832
11  0.780  0 + 24.95  19.802      11.639  11 + 20.45  295.634
12  0.736  0 + 23.55  18.691      12.375  12 + 12.00  314.325
-----------------------------------------------------------------------
13  0.695  0 + 22.23  17.642      13.070  13 +  2.23  331.967
14  0.656  0 + 20.98  16.652      13.725  13 + 23.20  348.618
15  0.619  0 + 19.80  15.717      14.344  14 + 11.01  364.335
16  0.584  0 + 18.69  14.835      14.928  14 + 29.69  379.170
17  0.551  0 + 17.64  14.002      15.479  15 + 15.34  393.172
18  0.520  0 + 16.65  13.216      16.000  15 + 31.99  406.389
19  0.491  0 + 15.72  12.475      16.491  16 + 15.70  418.863
20  0.464  0 + 14.83  11.774      16.954  16 + 30.54  430.638
21  0.438  0 + 14.00  11.114      17.392  17 + 12.54  441.751
22  0.413  0 + 13.22  10.490      17.805  17 + 25.75  452.241
23  0.390  0 + 12.47  9.901       18.195  18 +  6.23  462.142
24  0.368  0 + 11.77  9.345       18.562  18 + 18.00  471.487
```

Press y to continue, q for quit:

```
 1   1.684  1 + 21.88  42.768     1.684   1 + 21.88  42.768
 2   1.589  1 + 18.86  40.367     3.273   3 +  8.74  83.135
 3   1.500  1 + 16.00  38.102     4.773   4 + 24.74  121.237
 4   1.416  1 + 13.31  35.963     6.189   6 +  6.05  157.200
 5   1.336  1 + 10.77  33.945     7.525   7 + 16.81  191.145
 6   1.261  1 + 8.36   32.040     8.787   8 + 25.18  223.185
 7   1.191  1 + 6.10   30.241     9.977   9 + 31.28  253.426
 8   1.124  1 + 3.96   28.544    11.101  11 +  3.24  281.970
 9   1.061  1 + 1.94   26.942    12.162  12 +  5.18  308.912
10   1.001  1 + 0.04   25.430    13.163  13 +  5.22  334.342
11   0.945  0 + 30.24  24.003    14.108  14 +  3.46  358.345
12   0.892  0 + 28.54  22.655    15.000  14 + 32.00  381.000
---------------------------------------------------------------
13   0.842  0 + 26.94  21.384    15.842  15 + 26.94  402.384
14   0.795  0 + 25.43  20.184    16.637  16 + 20.37  422.568
15   0.750  0 + 24.00  19.051    17.387  17 + 12.37  441.618
16   0.708  0 + 22.65  17.982    18.094  18 +  3.02  459.600
17   0.668  0 + 21.38  16.972    18.763  18 + 24.41  476.573
18   0.631  0 + 20.18  16.020    19.393  19 + 12.59  492.592
19   0.595  0 + 19.05  15.121    19.989  19 + 31.64  507.713
20   0.562  0 + 17.98  14.272    20.551  20 + 17.62  521.985
21   0.530  0 + 16.97  13.471    21.081  21 +  2.59  535.456
22   0.501  0 + 16.02  12.715    21.582  21 + 18.61  548.171
23   0.472  0 + 15.12  12.001    22.054  22 +  1.73  560.172
24   0.446  0 + 14.27  11.328    22.500  22 + 16.00  571.500
```

Press y to continue, q for quit:

If we have (2/3) of a string length considered as G, and then take (2/3) of it, we then have a string length of just (2/3)(2/3) = 4/9 of the original string length. 4/9 of a string is 0.444... , and if we double it to get an octave lower, we will have 0.8888.... of that string length, and this is where the D or second note of the C scale would be. This will be less than the full (100% = 1) string length by the relative (ie., percentage, fractional) value of : (1 - 0.888888888) = 0.111111111 = =~ 11.11% The equal ratio chromatic music scale has a value for the second tone of being at 0.890898718 L

Where are the octaves or twice the frequency located on a string?

This is probably where Pythagoras started his musical scale.

Octave   Location Along A Length (L) Of Vibrating String Producing Sound Vibrations

0              $1/1 = 1L = 100\%L$

1              $1/2 = 0.5L = 50\%L$: after taking half of the above length

2              $1/4 = 1/(2^2) = 0.25L = 25\%L$: after taking half of the above length

3              $1/8 = 1/(2^3) = 0.125 = 0.125\%L$: after taking half of the above length

We see that these are fractions composed of a simple integer in both the numerator and denominator. For any string length (L) ,the octave is always located at half its length. From the above pattern, we see that the denominator is equal to octave (n):

$L_n = \dfrac{L}{2^n}$: the higher the octave, tones and-or frequencies, the shorter the string length and string length corresponds to the wavelength of its fundamental or natural frequency.

Pythagoras surely then reasoned that given any string or "scale length", then the octaves of any similar, but different "pitched" note, for example note A , will always be located at the same relative distance along its length from the start of the entire octave (ie., the 8 accepted, common or main tones of a musical scale, and before the concepts of the 12 note chromatic scale was used so as to create a scale of tones when any note is considered as the starting note). An example of the 8 common notes of a complete octave scale are: C=1, D=2, E=3, F=4 , G=5, A=6, B=7 , C=8

Pythagoras reasoned that if the octave note, a pleasing sounding note, is located at 1/2 of the string length or 1/2 up from the first note, then other pleasing notes should be located at 2/3L or 1/3 up from the first note , 3/4L or 1/4 up from the first note, and 4/5L or 1/5 of up from the first note. These are series of whole number fractions: 1/2 , 1/3 , 1/4 , 1/5

Ex. A musical string is 25.5 in long. The first octave higher in tone from the end of that open string is at
    25.5 / 2 = 12.75 in = 12 in + (12/16)in . This considers notes 1 and 8 of a complete octave scale.
    Halfway on this length segment of the string, there is the 8th note / 2 = 4th note, and this will be located
    at: 12.75 / 2 = 6.375 = 6in + (6/16)in   If the string was tuned to a C note, the 4th note in that scale
    would be F.   Considering this position as the start of a new length of a string, it would be a string length
    of: (25.5 - 6.375) = 19.125, and at half of this would be the 4th note in the F scale, hence note B.
    Pythagoras is said to have used the 5th note of every scale for his notes and positions analysis.

    [FIG 255A]

2F if both twice the fundamental string frequency and is the length of the next octave of it.   4F is four times the fundamental string frequency and the length of that once cycle would be the length of the next (higher in pitch) octave or four time the fundamental frequency of the entire string.

**Basic Northern Hemisphere
Star Map or Chart**
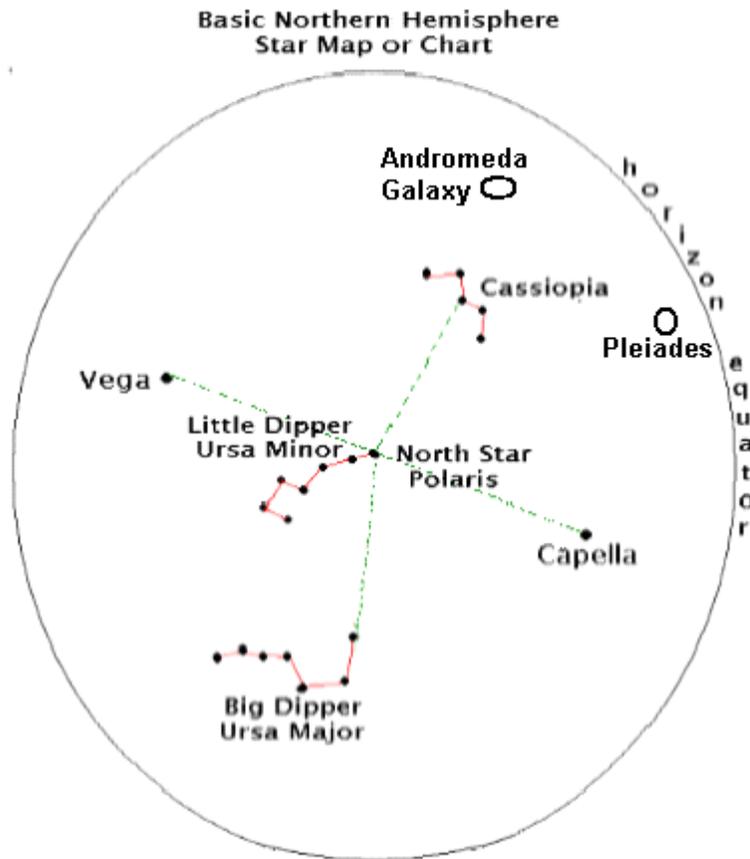
Vega and Capella are indicated due to their positions and easy to find brightness. These two stars are nearly on a line with each other which includes the **North Star** called **Polaris**. In brief, there are several celestial or astronomical coordinate or location systems in use for one reason or another, and all use a reference location and or plane.   The distance to each star or planet is generally not used or needed in these basic location systems where direction is more important - such as where to aim a telescope to view a particular star, planet or galaxy. Considered in these systems is the solar or ecliptic plane of which the planets and their orbit travel are relatively near to within a few degrees. The North star to the South star (ie., Earth's polar axis) plane is used in some of the systems. One common term often used in these concepts is "azimuth", and this is simply an angle value such as to a star.

Since astronomical observers may be at different latitudes on the Earth, the apparent angle to a particular star will not be the same as that of another observer, and hence why a common (0°) reference plane(s) must be used so that all may properly locate an object.   For example, the Sun may appear as directly overhead to a person near the Equator, but it will appear as say 45° above the horizon (at local noon, 12pm time) to an observer located halfway to the north pole between the equator and the north pole.

A simple astronomical location system for people in the same geographical area, within a few hundred miles of each other, would be the angle rotated clockwise from the north-star's horizontal direction or plane, and the angle to the object above the local horizon. Even so, the planets and stars appear to change celestial position by 15 degrees per hour, and therefore, the same observation time should also be considered for these coordinates or position.

A more general or complete discussion of the topic of celestial coordinates is beyond the scope of this book, and for more information, it can be researched elsewhere.

and is actually a three star system. The star C-Centauri is commonly called **Proxima Centauri** which is a relatively small red-dwarf star and was found using a telescope in 1915 and is the closest star known to our solar system, and is about 4.25 light-years away. As of the year 2022, three exoplanets have been found to orbit Proxima Centauri, hence they compose the closest exo-planetary solar system. In the above image. Proxima Centauri is located just south-east of Alpha-Centauri, and at about the 5 o'clock position relative to it. By using the apparent visual effect called parallax. and taking two observations of Proxima Centauri at 6 months apart as Earth orbits about the Sun, its distance can be found using right-triangle trigonometry to a relatively high accuracy since it is relatively close to us as compared to all other stars, and the parallax effect is more apparent. Throughout the year, at most latitude locations in the northern hemisphere of Earth, Proxima Centauri cannot be seen and is effectively below the horizon.

How To Calculate Your Local, True Or Solar Noon Time If you Know Your Longitude

At solar noon, the Sun is directly in line with your latitude, and it will be at the highest or greatest angle in the sky for that particular day. Every location that has the same longitude on Earth will have the same, true or actual local time. First, find out what is your (east, west)longitude. You may also wish to know your (north, south) latitude.

We usually set out clocks not to our local solar noon, but by the time zone. All the different longitude locations in a time zone will have the same clock time, and this helps people stay on schedule and or in reference with each other and events. Only where the hourly time zone starts will the Sun be directly overhead at 12:00pm. If you live to the west in the time zone your true local (at your latitude) solar noon could be several more minutes later. The Earth rotates at 15° per hour, and from this we have that the Earth rotates 0.25° per minute, or 0.004667° per second.
.
Here is an example.

If your longitude is 75.842°W, what is your true, local solar noon time on your ("time zone" or standard) clock?

Since each timezone is 15° apart, your time zone is: 75.842° / 15° = 5.0561333... , and the whole value of 5 indicates:

you are in timezone 5 west of the Greenwich time zone1, and your clock time is set to be 5 hours earlier than the Greenwich clock ("timezone") time. To be more specific, you are 5.0561333... hours earlier than Greenwich. Since this amount of hours has a fractional value, you do not live on the longitude line at the start of that time zone, but are actually 0.0561333...hours offset from this, and your local, true solar noon will be 0.0561333 hours later than the standardized clock ("timezone") noon. Let us now convert this amount of hours to its equivalent amount of minutes. We can use a proportion or equivalent fraction type of equation to do this:

$$\frac{1 \ hr}{60min} = \frac{0.0561333 \ hr}{x \ min} \quad , \quad \text{after solving for x min, we have:}$$

$$x \ min =\sim 3.368 \ min \quad : \text{your local, true solar noon will be 3.368min past or later than clock or "timezone" noon.}$$

The above is a theorized value, but since the orbit of Earth about the Sun is elliptical, true solar noon may not be this calculated value, but needs to be adjusted. The topic of the **Equation of Time** considers this adjustment, and it and its value it may be thought of as: The Adjustment, Difference or Offset Of True or Actual Solar Noon Time From The 24 Hour, Standard Clock Or Axis Rotation Time. Note that for any day of the year, and regardless of Earths (slightly) elliptical orbit, the Earth always rotates at the same angular velocity on its axis, and that is 1 rotation per day = 360°/24hrs. When Earth is closer to the Sun in the winter time, the Sun will rise later than when the Earth is more distant in its orbit during the summer. Clock time that we commonly use throughout the year is based on the constant 24 hours in day, and does not consider the "equation of time" or changes in our orbit and the current (angle) position of the Sun with respect to Earth during its elliptical (non-circular) orbit during a year. Since the "equation of time (adjustment)" is not considered on our ("time zone") clocks, it is therefore the equivalent time as if the Earth had a pure circular, non-elliptical, orbit about the Sun. There are some special mechanical clocks made which do consider the equation of time, and those clocks are called "Equation Clocks". For these clocks, the adjustment or time offset is displayed on a separate dial, and it can be combined to the displayed time. If the clock was set to true local or solar noon, the adjustment is simply added, otherwise if the clock was set to "time zone noon", then you need to first calculate the true solar local noon from your known longitude as shown in this article above, and then to also combine it with the equation of time adjustment.

The zero time offset days are separated by 90 days and they occur at roughly the time of the start of each season: March 20, June 20, Sept 20, and Dec 20. The equation of time is sinusoidal-like, but having two waves, each with its own amplitude (peak value), and one wave then following the other. Each of the two time offset waves have a 180 days period or time length. There is a spring and summer wave, and a fall and winter wave. Between the zero dates, at about every 45 days, are the maximum offset or equation of time adjustment values, and these peaks are therefore 90 days apart.

Due to Earth's elliptical orbit about the Sun, as the Earth gets closer to the Sun due to its increased gravitational force and resulting acceleration caused by the Sun, the speed of the orbit of the Earth about the Sun increases in the winter months,

The topic of the equation of time has been studied for years by astronomers and much data about it has been measured and recorded. The equation of time can get somewhat complicated and has not much practical use for the average person, but it is sometimes important to some astronomers, precision sundials (ie., sun clocks) makers, and navigators. If your region uses the concept of Daylight Savings Time, you may also need to adjust your calculated and expected, true solar noon time value by an entire one hour more or less.

The offset times or difference times between the ("time zone", 24h) clock and the solar or Sun time can be measured using a sundial or a vertical pole in the ground. Observe a clock when it indicates noon or 12pm, and also observe the shadow on the sundial when it is indicated as your local (longitudinal) solar noon. For the fundamental equation of time and-or the difference than the expected value on the standard (24 hour) clock), calculate the time difference between solar noon and clock noon. Clock noon is also called "(24 hour) standard time" and sometimes as the local "(24 hour clock) averaged solar time". As for a vertical pole in the ground, its shadow will be the shortest length at solar noon, and where the Sun is at its highest angle in the sky for that day.

Key Data Points For Earth's Equation Of Time :(Offset value or time between solar or true noon and clock time noon)

Day   Time Offset In Minutes

February 11   -14   = - (14 min.)
April 15   0
 May 14   +3.5 = (3 min. + 30 seconds)
June 13   0
 July 26   -6.5 = (6 min. + 30s)
September 1   0
November 3   +16.5 = (16 min and 30s)
December 25   0

[FIG 259] Equation of time, approximation graph.
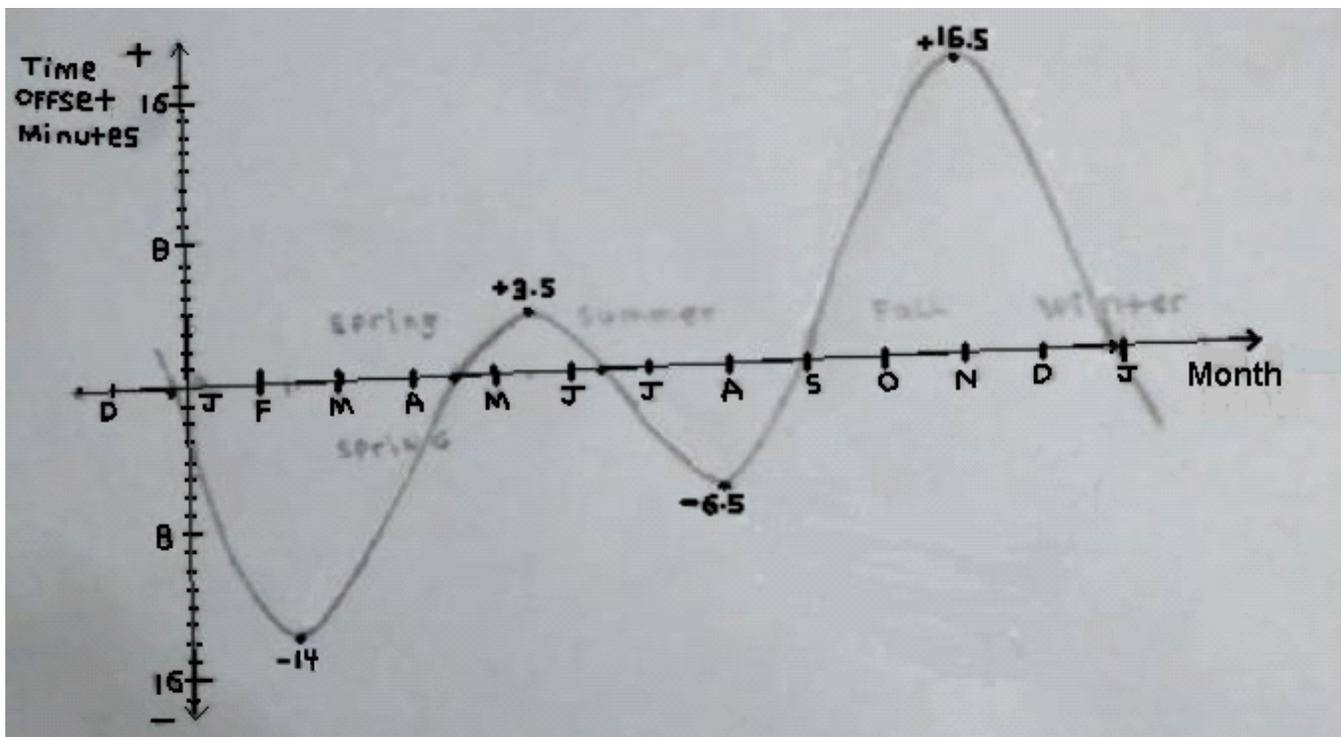
COMMON WHOLE NUMBER DIVISIONS AND-OR FRACTIONS

When the numerator is 1, the value of the faction is also called as, and equal to, the reciprocal of the denominator.
Note these helpful example where reciprocals are fundamentally involved:

$\dfrac{2}{3} = \dfrac{1}{3} + \dfrac{1}{3} = 2\left(\dfrac{1}{3}\right)$: hence twice or 2 times the reciprocal of 3 and which is (1/3).

$\dfrac{1}{20} = \dfrac{1}{2\,(10)} = \dfrac{1}{2}\dfrac{(1)}{(10)} = 0.5\,(0.1) = 0.5\,(10^\wedge\!-1) = 0.05$

$\dfrac{1}{0.5} = \dfrac{1\,(10^\wedge 0)}{5\,(10^\wedge\!-1)} = \dfrac{1}{5}\,(10^\wedge\!+1) = (0.2)\,(10) = 2$

1/1 = 1
1/2 = 0.5
1/3 = 0.333333333333333
1/4 = 0.25
1/5 = 0.2
1/6 = 0.166666666666667
1/7 = 0.142857142857143
1/8 = 0.125
1/9 = 0.111111111111111
1/10 = 0.1
1/11 = 0.0909090909090909
1/12 = 0.0833333333333333
1/13 = 0.0769230769230769
1/14 = 0.0714285714285714
1/15 = 0.0666666666666667
1/16 = 0.0625
1/17 = 0.0588235294117647
1/18 = 0.0555555555555556
1/19 = 0.0526315789473684
1/20 = 0.05
1/21 = 0.0476190476190476
1/22 = 0.0454545454545455
1/23 = 0.0434782608695652
1/24 = 0.0416666666666667
1/25 = 0.04
1/26 = 0.0384615384615385
1/27 = 0.037037037037037
1/28 = 0.0357142857142857
1/29 = 0.0344827586206897
1/30 = 0.0333333333333333
1/31 = 0.032258064516129
1/32 = 0.03125
1/33 = 0.0303030303030303
1/34 = 0.0294117647058824
1/35 = 0.0285714285714286
1/36 = 0.0277777777777778
1/37 = 0.027027027027027
1/38 = 0.0263157894736842
1/39 = 0.0256410256410256

To convert a numeric (decimal) fractional part of an inch to sixteenths
of an inch, divide by (1"/16) = 0.0625".   1inch = 16, sixteenths of an inch
Ex. 0.5 in = 0.5 / 0.0625 = 8 sixteenths of an inch = (8/16)in = 8(1/0.0625)
Ex. (3/16) in = 0.1875 in , 0.1875in / 0.0625 = 3 sixteenths of 1 inch

Given an amount of kilograms (kg), 3 digits of the (decimal) fractional part
is equal the number of grams (g) extra to add to the total number of
kilograms.   1kg = 1000g ,   kgs = whole amount of kg + fraction of 1kg
5.345 kg = 5kg + 0.345kg = 5kg + 345g
The 3 digits past the grams part is the number of milligrams (mg).
1g = 1000mg.   1mg = 0.001g = 1mg = 0.000001kg = 1 millionth of a kg
Ex. 5.345678kg = 5kg + 345.678g = 5kg + 345g + 678mg
Ex. 345.6g = 345g + 0.6g   = 345g + 600mg

1/94 = 0.0106382978723404
1/95 = 0.0105263157894737
1/96 = 0.0104166666666667
1/97 = 0.0103092783505155
1/98 = 0.0102040816326531
1/99 = 0.0101010101010101
1/100 = 0.01

2/1 = 2
2/2 = 1
2/3 = 0.666666666666667
2/4 = 0.5
2/5 = 0.4
2/6 = 0.333333333333333
2/7 = 0.285714285714286
2/8 = 0.25
2/9 = 0.222222222222222
2/10 = 0.2
2/11 = 0.181818181818182
2/12 = 0.166666666666667

Given an amount of meters (m), 2 digits of the (decimal) fractional part is equal to the number of centimeters. 1m = 100cm, 1cm = 0.01m
Ex. 3.45m = 3m + 0.45m = 3m + 45cm
 1cm = (1/10)m = 0.01m = 10mm , 1mm = 0.1cm = 0.001m
Ex. 3.4567m = 3m + 45.67cm = 3m + 45cm + 6.7mm
       3.4567m = 3m + 45cm + 6mm + 700um
 1mm = 0.001m ,   1um = (1/1000)mm = 0.001mm = 0.000001m
 1um = (1/1000000)m = 0.000001m

3/1 = 3
3/2 = 1.5
3/3 = 1
3/4 = 0.75
3/5 = 0.6
3/6 = 0.5
3/7 = 0.428571428571429
3/8 = 0.375
3/9 = 0.333333333333333
3/10 = 0.3
3/11 = 0.272727272727273
3/12 = 0.25

4/1 = 4
4/2 = 2
4/3 = 1.33333333333333
4/4 = 1
4/5 = 0.8
4/6 = 0.666666666666667
4/7 = 0.571428571428571
4/8 = 0.5
4/9 = 0.444444444444444
4/10 = 0.4
4/11 = 0.363636363636364
4/12 = 0.333333333333333

9/1 = 9
9/2 = 4.5
9/3 = 3
9/4 = 2.25
9/5 = 1.8
9/6 = 1.5
9/7 = 1.28571428571429
9/8 = 1.125
9/9 = 1
9/10 = 0.9
9/11 = 0.818181818181818
9/12 = 0.75

10/1 = 10
10/2 = 5
10/3 = 3.33333333333333
10/4 = 2.5
10/5 = 2
10/6 = 1.66666666666667
10/7 = 1.42857142857143
10/8 = 1.25
10/9 = 1.11111111111111
10/10 = 1
10/11 = 0.909090909090909
10/12 = 0.833333333333333

11/1 = 11
11/2 = 5.5
11/3 = 3.66666666666667
11/4 = 2.75
11/5 = 2.2
11/6 = 1.83333333333333
11/7 = 1.57142857142857
11/8 = 1.375
11/9 = 1.22222222222222
11/10 = 1.1
11/11 = 1
11/12 = 0.916666666666667

12/1 = 12
12/2 = 6
12/3 = 4
12/4 = 3
12/5 = 2.4
12/6 = 2
12/7 = 1.71428571428571
12/8 = 1.5
12/9 = 1.33333333333333
12/10 = 1.2
12/11 = 1.09090909090909
12/12 = 1

COMMON CUBE ROOTS

1 1	2 1.25992104989487	3 1.44224957030741
4 1.5874010519682	5 1.7099759466767	6 1.81712059283214
7 1.91293118277239	**8 2**	9 2.0800838230519
10 2.15443469003188	11 2.22398009056932	12 2.28942848510666
13 2.35133468772076	14 2.41014226417523	15 2.46621207433047
16 2.51984209978975	17 2.57128159065824	18 2.6207413942089
19 2.66840164872194	20 2.71441761659491	21 2.75892417638112
22 2.80203933065539	23 2.84386697985157	24 2.88449914061482
25 2.92401773821287	26 2.96249606840737	**27 3**
28 3.03658897187566	29 3.07231682568585	30 3.10723250595386
31 3.14138065239139	32 3.1748021039364	33 3.20753432999583
34 3.23961180127748	35 3.27106631018859	36 3.30192724889463
37 3.33222185164595	38 3.36197540679896	39 3.39121144301417
40 3.41995189335339	41 3.44821724038273	42 3.47602664488645
43 3.50339806038672	44 3.53034833532606	45 3.55689330449006
46 3.58304787101595	47 3.60882608013869	48 3.63424118566428
49 3.65930571002297	50 3.68403149864039	51 3.70842976926619
52 3.73251115681725	53 3.75628575422107	54 3.77976314968462
55 3.80295246076139	56 3.82586236554478	57 3.8485011312768
58 3.8708766406278	59 3.89299641587326	60 3.91486764116886
61 3.93649718310217	62 3.95789160968041	63 3.97905720789639
64 4	65 4.02072575858906	66 4.04124002062219
67 4.06154810044568	68 4.08165510191735	69 4.10156592970235
70 4.12128529980856	71 4.14081774942285	72 4.16016764610381
73 4.17933919638123	74 4.19833645380841	75 4.21716332650875
76 4.23582358425489	77 4.25432086511501	78 4.27265868169792
79 4.29084042702621	80 4.30886938006377	81 4.32674871092222
82 4.34448148576861	83 4.36207067145484	84 4.37951913988789
85 4.39682967215818	86 4.4140049624421	87 4.43104762169363
88 4.44796018113863	89 4.46474509558454	90 4.48140474655716
91 4.49794144527541	92 4.514357435474	93 4.53065489608349
94 4.54683594377634	95 4.56290263538697	96 4.57885697021333
97 4.59470089220704	98 4.61043629205845	99 4.62606500918274
100 4.64158883361278		

Press A Key To Exit Program

```c
for(;;){
        printf("%3d %-20.15g ", (int)n , pow(n,(1.0/3.0)) );
        /* basic format specifier format used above:
           %[-left justify in the total width][total character width]%[.total number of decimal digits]
        */

        t=t+1;   /* the number of enteries on each line */
        if(t>=4){  t=1;  printf("\n");  };

        n=n+1.0;
        if(n>=101.0){ break; };
    };

printf("\n\nPress A Key To Exit Program");  fflush(stdin);  getch();

return;
};
/*------------------------------------------------------------------------------*/
```

COMMON GEOMETRIC STRUCTURES MADE FROM A LINE SEGMENT

Part of all things math or science is learning at least some of the vast knowledge already available, and then applying it to make some knew discoveries and-or ideas that can be shared with others in a cooperative effort.

[FIG 261]

Common Geometric Structures Made From A Line Segment

1   line , L = length = distance

All these structures have a perimeter of P = L
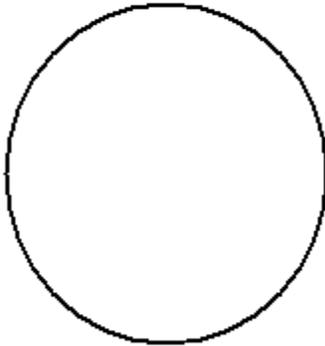
2   circle , where s = C = L
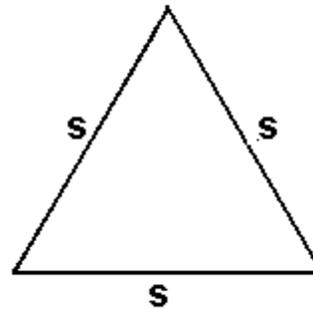
3   folded line , L / 2 = C / 2
    or squashed circle

    Note that C = (pi) D , therefore

    D = C / (pi) = C / 3.14..

    D = L / 3.14...

4   triangle , s = L / 3 = C / 3
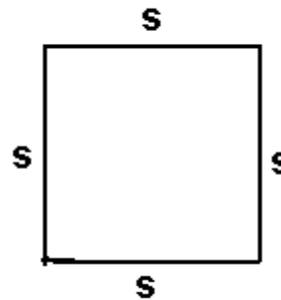
5   square , s = L / 4 = C / 4

Calculating the volume and-or equation of a sphere is not easy as other geometrical objects, and this is due to that the sides of the sphere are not linear or straight, but are curved.   The most common derivation for the volume of a sphere is found by a large sum of volumes of a (infinitely) thin cross sectional disks of the sphere, and where each disk has an area of (pi)r^2 and a very small thickness which is the same value for each disk.   Volume of each disk = (the area of each disk) (thickness of each disk).   These disk volumes are all effectively summed using integration, and the reader can explore this elsewhere and further if need be.   The most common derivation of the formula for the surface area of a sphere is somewhat similar.

With an experiment, a sphere can be placed within a cylinder or cube of the same height (here, = 2r), and then a fluid such as water can be placed in the region surrounding the sphere. The volume of the sphere will be equal to the total empty volume of the cylinder or cube less (ie., subtract) the amount of water in the surrounding region when the sphere is in it.

Vcontainer = Vobject + Vsurrounding_region   ,   mathematically:
Vcontainer = Vsphere + Vsurrounding_region   ,   mathematically:
   Vsphere = Vcontainer - Vsurrounding_region

If the sphere radius r=1, a "unit sphere", and if the container is a cylinder as shown in the above figure:

Vsurrounding_region = Vsr = Vcylinder - Vsphere   , Let us now find and express these values and
                                                                                              mathematical relationships expressed as ratios:

Vsr = 2(pi)r^3 - $\frac{4(pi)(r^3)}{3}$ If r=1:

Vsr = 2(pi) - $\frac{4(pi)}{3}$ = $\frac{6(pi) - 4(pi)}{3}$ = $\frac{2(pi)}{3}$ = 0.6667 (pi) =~ 2.0943951

$\frac{Vsr}{Vcylinder}$ = $\frac{(2(pi)r^3) / 3}{2(pi)r^3}$ = $\frac{1}{3}$ = 0.333333...   , mathematically: Vsr = (1/3) Vcylinder

$\frac{Vsr}{Vsphere}$ = $\frac{(2(pi)r^3) / 3}{(4(pi)r^3) / 3}$ = 0.5   , and the reciprocal of this is:

$\frac{Vsphere}{Vsr}$ = $\frac{(2/3) Vcylinder}{(1/3) Vcylinder}$ = (2/3) (3/1) = 6/3 = 2

$\frac{Vsphere}{Vcylinder}$ = $\frac{(4(pi)r^3) / 3}{2(pi)r^3}$ = $\frac{2}{3}$ = 0.66667 =~ 66.7%   , mathematically: Vs = (2/3) Vcylinder
                                                                                          , the reciprocal is:

$\frac{Vcylinder}{Vsphere}$ = $\frac{3}{2}$ = 1.5

Values for the particular example where r=1:

Vsr = Vcylinder - Vsphere

Vsr = 6.2818 units^3 - 4.18879 units^3 =~ 2.09301   : as shown above

Below is a similar analysis using the volume of a cube:

Note that the volume of a cube does not have (pi) variable in its equation.

$$\frac{Vpyramid}{Vcylinder} = \frac{1}{3} = 0.3333... \approx 33.33\%$$

If (1/3) of a cylinder is occupied by a cone, then: 1 - (1/3) = (3/3) - (1/3) = (2/3) of the cylinder is not occupied by the cone.

[FIG 264]

pyramid in a cube
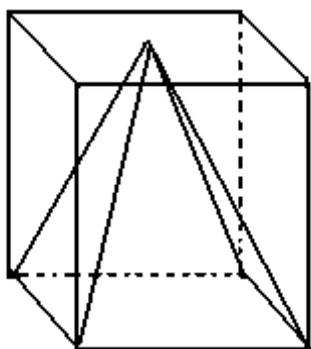
side = s

$$\frac{Vpyramid}{Vcube} = \frac{(s^2)(s)/3}{s^3} = \frac{s^3/3}{s^3} = \frac{1}{3} = 0.333333.... \approx 33.33\%$$, and the reciprocal of this is:

$$\frac{Vcube}{Vpyramid} = 3$$

Given many various triangles, if the base and height are the same value, their areas will all be the same. Likewise, given a cone or pyramid in a cylinder, prism or cube (a special instance of a prism, rectangular solid shape), as long as the height is the same, the volume of the pyramid or cone is the same. For example, the tip of the cone or pyramid may be at one of the upper corners of the cylinder or prism.

Extra: Since **Vs** = 4/3 (pi) r^3 = 1.33333 (pi) r^3 = **4.18879 r^3**   : volume of a sphere given its radius

r = cube-root (Vs / 4.18879) =
r = cube-root Vs / cube root 4.18879 = (cube-root Vs) / 1.611991928 =
r = 0.620350501 (cube-root Vs)   : radius of a sphere given its volume

Ex. A volume of 1 liter = 1000 mL = 1000 cc = 1000 cm^3 = (10 cm)^3 = (10 cm)(10 cm)(10 cm) = 1000 cm^3
Now considering a sphere having this same volume as this cube shape:
r = 0.620350501 (cube-root 1000 cm^3) = 0.620350501 (cube root 1000)(cube root 1cm^3)
r = 6.2035 cm   , d = 2r = 12.407 cm , note that this is greater than 10cm which was the base of the equivalent cube shaped volume and is 12.407 cm / 10 cm =

Ds = diameter of a sphere = 1.2407 times wider than a cube of the same volume
To imagine how this can be, imagine a cube shape of clay, and if you push the 8 corners and edges inward so as to make a sphere shape, that material will be added to the existing central volume of material, hence making it slightly wider than the cube was initially.

360° = 1 rotation or revolution = an angle of 2 (pi) =~ 6.28 radians

$$\frac{1 \text{ rev}}{s} = \frac{360°}{s} = \frac{2(pi) \text{ rads}}{s} =~ \frac{6.28 \text{ rads}}{s}$$

rotatonal or angular velocity = w =

$$w = \frac{\text{angular distance}}{\text{time}} = \frac{\text{rads}}{s} = \frac{\text{rads}}{s} \frac{\text{(revs)}}{\text{(revs)}} = \frac{\text{rads}}{\text{rev}} \frac{\text{revs}}{s} \quad , \text{ and:}$$

Given a number (N) of rotations or revolutions per second = rps:

$$w = \frac{2(pi) \text{ rads}}{\text{(rev)}} \frac{\text{(N revs)}}{(s)} =~ \frac{6.28 N \text{ radians}}{s} = 6.28 \text{ (rps) radians} = \frac{6.28 \text{ (rot) radians}}{s}$$

Given a rotating circle where r=1 and-or 100% for general, relative or proportional analysis:

angular velocity for 1 rotation = w = (angular distance for 1 rotation = 360°) / t =
   w = 2(pi) rads / s   = 2(pi) / s = 6.28 rads / s = 1 rotation / s = 1rps

linear v for 1 rotation = v = linear d / t = Circumference / s = 2(pi) r / s , If r=1m: c / s = 2(pi)(1) m / s = 6.28 meters / s

If r=1, v = linear distance / s , and if there is 1 rot / s or= 1 rev / s = 360° / s, this can interpreted as: 2(pi)(r) / s =
   1Cm / s . w = 6.28 rads / s, also, at r=1,   1 Cm / s = 6.28m / s , and
     Work = (Force)(distance) = Fd = F (6.28m) joules , and if this was done in 1s,   Pw = W/t = E/t = 6.28F J/s =
       6.28F watts

If there are N rotations or N revolutions per second, we simply multiply (d) or (C), and (w) by (N) to find:
   v = d / t = NCm / s   and   w = (6.28N rads) / s

Pw = Ej / s = Work / s = (force)(distance) / s = F (velocity) = Fv , and in terms of torque , rotating or twisting forces:
    Pw = Fv = Tw = T (6.28 N) rads / s = T (6.28)(rot / s) rads / s

   , if given a rpm value , 1 rps = 60 rpm , multiplying both sides by N:
               N rps = N 60 rpm ,   dividing both sides by 60 and switching sides:
               N rpm = (N / 60) rps   , Ex.: If given 60 rpm , 60 rpm = (60/60) rps = 1 rps

     Pw = T w = T 6.28 (N rpm) rads = T (6.28)(N rpm / 60) rads / s = $\frac{T (6.28)(N \text{ rpm})}{60}$ rads / s

(N rot) / s = (N rps) = w can be thought of as in terms of a linear equivalence of: (nc) / s = d / t = v , and
rpm = (rot / min) can be thought of as: (nc) / min = nc / 60s   = (N rot) / 60s

AN IMPROVED ALGORITHM TO SOLVE DIFFICULT EQUATIONS

Previously in this book, a method and-or algorithm called EQSOLVE2.C was shown how to evaluate difficult expressions and-or equations. Below is an alternate and-or improved version. It uses only one main loop, and has been placed in a more usable example program.

```
/* ------------------------------------------------------------------------------------
FESA.c ,   FAST NUMERICAL ALGORITHM (FANA)   or
             FAST EQUATION SOLVING ALGORITHM (FESA)

   An non algebraic, mechanical-like numerical method or algorithm to zero into
   or approach the result and having a high precision.

   Compiler used:  TinyC  ,  standard or fundamental C only.

   When using just 1 term, the result essentially finds the log of that number, and
   which is the exponent.
   Ex.  If result=number=1000 ,  and a=10,  then x=logarithm=exponent=3
   Note that the number of a logarithm cannot be a negative value.
   A logarithm can be negative in value, such as when the number is less than 1.
   The logarithm of 1 as the number, using any base, is 0.  If the number is less
   than 1, then the logarithm will be negative in value.

   (c). JPA Nov. 2022  ,  Pa , USA
------------------------------------------------------------------------------------*/


#include "stdio.h"
#include "stdlib.h"
#include "float.h"
#include "math.h"


/*------------------------------------------------------------------------------------*/
double equation(double x, int terms);  /* (x) is a temporary value to use in the equation, and it will find the true value of it */
double FANA(double RESULT, int terms);   /* This algorithm does not directly use the weights for
                                             the precision, and does not include loop conditions,
                                             and the original version shown in this book did.  Terms
                                             is the desired number of terms of the expression to utilize.
                                             RESULT is the value of the expression containing (x).
                                             The return value is a temporary result using the temporary
                                             value of (x) in the equation. */
/*------------------------------------------------------------------------------------*/
/* Variables used in the equation:  */
/* Equation variable initializations can be hard coded using a value, or with user input in main(), etc */

double x=0.0;        /* x is to be found, it will be set to an initial very low, improbably and-or negative value,
                         below. This can be set higher if the solution speed is very critical. */
double a = 0.0;
double b=0.0;
double c=0.0;
double d=0.0;
double e=0.0;
double f=0.0;
double g=0.0;
double h=0.0;
```

```c
        if(TERMS>4){   printf("\nEnter e: ");  fflush(stdin); scanf("%lf",&e);
                       if(e==0.0){ printf("\a"); exit(0); };
                 };

        if(TERMS>5){   printf("\nEnter f: ");  fflush(stdin); scanf("%lf",&f);
                       if(f==0.0){ printf("\a"); exit(0); };
                 };

        if(TERMS>6){   printf("\nEnter g: ");  fflush(stdin); scanf("%lf",&g);
                        if(g==0.0){ printf("\a"); exit(0); };
                 };

        if(TERMS>7){   printf("\nEnter h: ");  fflush(stdin); scanf("%lf",&h);
                        if(h==0.0){ printf("\a"); exit(0); };
                  };

        if(TERMS>8){   printf("\nEnter i: ");  fflush(stdin); scanf("%lf",&i);
                        if(i==0.0){ printf("\a"); exit(0); };
                 };

        if(TERMS>9){   printf("\nEnter j: ");  fflush(stdin); scanf("%lf",&j);
                        if(j==0.0){ printf("\a"); exit(0); };
                 };

     x = result = FANA(result,TERMS);    /* call the function and-or algorithm to solve for (x) with the given data used */

/*---------------- Display The Results -------------------------------------*/

     printf("\n\nThe value of x = %.15g",result);  /* use g or lf*/
     printf("\n\n");

     /* x=result; */

     /* below, g or lf can be used,   lf might be better */

  if(TERMS>=1){  r = pow(a,x);  printf("\na^x = %.15g^%.15g = %.15g",a,x,r);  };
  if(TERMS>1){  r = pow(b,x);   printf("\n\nb^x =  %.15g^%.15g = %.15g",b,x,r);  };
  if(TERMS>2){  r = pow(c,x);   printf("\n\nc^x =  %.15g^%.15g = %.15g",c,x,r);  };
  if(TERMS>3){  r = pow(d,x);   printf("\n\nd^x =  %.15g^%.15g = %.15g",d,x,r);  };
  if(TERMS>4){  r = pow(e,x);   printf("\n\ne^x =  %.15g^%.15g = %.15g",e,x,r);  };
  if(TERMS>5){  r = pow(f,x);   printf("\n\nf^x =  %.15g^%.15g = %.15g",f,x,r);  };
  if(TERMS>6){  r = pow(g,x);  printf("\n\ng^x =  %.15g^%.15g = %.15g",g,x,r);  };
  if(TERMS>7){  r = pow(h,x);  printf("\n\ng^x =  %.15g^%.15g = %.15g",h,x,r);  };
  if(TERMS>8){  r = pow(i,x);   printf("\n\ni^x =  %.15g^%.15g = %.15g",i,x,r);  };
  if(TERMS>9){  r = pow(j,x);   printf("\n\nj^x =  %.15g^%.15g = %.15g",j,x,r);  };

  printf("\n\nPress A Key ");  fflush(stdin); getch();
 };

return;
};


/*----------------------------------------------------------------------------------------------------------------------*/
```

```
double equation(double x, int terms)
{
double r=0.0;    /*  :to hold the result of the equation used, and it is: */

/* A probable modification would be to allow a term to be either
   negative or positive, possibly after asking the user.   */

    /* This coding below allows successive "fall through" processing, like a switch statement could */

   if(terms>=1){  r = pow(a,x);    };    /* The equation is hard coded here, but could be composed of  variables that can
                                             be called TermN that are double floats that a parser can set after simplifying a
                                             user input expression via a text string of ASCII characters or letters. */
   if(terms>1){  r = r + pow(b,x);  };
   if(terms>2){  r = r + pow(c,x);  };
   if(terms>3){  r = r + pow(d,x);  };
   if(terms>4){  r = r + pow(e,x);  };
   if(terms>5){  r = r + pow(f,x);   };
   if(terms>6){  r = r + pow(g,x);  };
   if(terms>7){  r = r + pow(h,x);  };
   if(terms>8){  r = r + pow(i,x);   };
   if(terms>9){  r = r + pow(j,x);   };

return r;
};
/*-------------------------------------------------------------------------------------------------------------------------*/
```

Here is an example and output of the above program, and showing the intermediate or temporary results :

This program and algorithm in it is: (c) J.P. Albertson , 2022, author of
the Mathization ebook. Donations will help me and this effort to help
others, and the verified link is in the ebook and-or in my channel
description. The .pdf format book can be viewed on a phone or computer,
and is filled with basic math, algebra, trigonometry, advanced topics, and
science. It is a helpful supplement to any course, home learning and-or work.

This Fast Numerical Algorithm fill find the value of (x) in this expression:

a^x + b^x + c^x + d^x + e^x + f^x + g^x + h^x + i^x + j^x = result

Enter the number of initial terms to use: (1 to 10): 3

Enter the positive valued result of the expression or 0 to quit.

result = 110

Enter a: 5.0

Enter b: 6.0

Enter c: 7.0

MORSE CODE

Since antiquity, signals other than verbal sound signals have been a way to communicate, especially over long distance by using things such as sound, light, smoke, hand, and flag signals. Once an electric signaling system was developed not long after the discovery of the battery (ie. voltaic pile) in about 1800, an electric buzzer (audio sounder, much like a vibrating speaker having one tone), made it practical to communicate nearly immediately (ie., with nearly no time delays) over long distances, day or night, from city to city, from valley to valley.

A signal will requires a pulse or brief instance of the method used such as light, and the duration of it needs to be experimented with so as it can be received and-or recorded well enough to be deciphered later and to have meaning. With a signaling pattern or code system, more information can be communicated. In order to count a number of pulses, there also needs to be a lack of a pulse in between each pulse transmitted. We could code or express each pulse as a value of 1, and a lack of a pulse as 0.   Samuel Mores recorded his two types of pules as a dot or dash, either written or on a paper tape.

Pulses of electricity could be sent and the specific number of pulses sent could represent a certain predefined or coded (another form, non-verbal, non-text) message. Even today as of the year 2022, in most hospitals, an electric buzzer or signaling system can be used by a patient in a certain indicated room to request some attention from the medical staff located down the hall, or on a different floor, or in a different building.   Many homes, rooms, gates and buildings still have a door buzzer or tone system for signaling (ie., electronically requesting) attention is needed, and these are commonly called "door bells" or "door buzzers".

Samuel Morse invented the **Morse Code** communication system so that each text letter of the alphabet, and number, could have an associated code or coding sequence to it, and this made the telegraph much more useful and practical . Later, his Morse Code was adapted to the standardized International Morse Code system. Morse's system used pulses, but two different types of short and long pulses. A short or quick pulse called a "dot [.]" could be considered as analogous to a 0, and a longer pulse called a "dash" [-] could be considered as analogous to a 1.   Due to that there may be several of the same types of the pulse in a row, such as three dashes or three dots, there is still a necessary spacing and-or timing delay between pulse each so as to being able to recognize or distinguish each, and it is standardized as the time of 1 dot or short pulse between each pulse. A dash or long pulse is standardized as being 3 short pulses or dots long in time duration. Between each letter, the timing space or duration of silence (ie., no signal) is equivalent to three pules, and between each word, the timing space is 7 pulses or dots.

A . _	X _ . . _	Each letter has up to 4 pulses. Each pulse is one of two pulse types:
B _ . . .	Y _ . _ _	A dot signal may be said as a "dit" or "di" or "de"
C _ . _ .	Z _ _ . .	A dash signal may be said as a "dah" or "da"
D _ . .		**SOS = . . . _ _ _ . . .** : use to quickly request help, keep transmitting.
E .		If your location is unknown, radio source direction can be used to find you.
F . . _ .		
G _ _ .	1 . _ _ _ _	Each number always has 5 pulses.
H	2 . . _ _ _	
I . .	3 . . . _ _	
J . _ _ _	4 . . . . _	
K _ . _	5	: Each dot can be considered as a count of 1
L . _ . .	6 _	: Each preceding dash of a number can be thought of as a count of 2
M _ _	7 _ _ . . .	
N _ .	8 _ _ _ . .	Note the opposite pulse types of: **(S , O)** , (A , N) , (B , J) , **(E , T)**
O _ _ _	9 _ _ _ _ .	(F , Q) , (G , U) , (K , R) , (L , Y), **(M , I)** , (D , W) , (P , X) , (D , G) ,
P . _ _ .	0 _ _ _ _ _	(G , W)
Q _ _ . _		Note the similarities of: **(E , I , S , H)** , (S , V) , (F , U) , (G , Q) ,
R . _ .	S . . .	T _	(A , R) , (P , W) , (I , U) , (M , G , Z , 7) , **(T , M , O)** , (J , O) , (J , P) ,
U . . _	V . . . _	W . _ _	**(E , A , R)** , **(T , N , D , B)** , (T , N , K , C) , (O , J) , (G , U , W) ,
			(A , W , J) , (A , U , V) , **(E , A , W , J)**

Letter = (Row pulses, Column Pulses)   : Rows are horizontal = left to right , Columns are vertical = up and down.
J = (2, 4) = ...   : = J is located at: row 2 , column 4

A way to signal or indicate you are ready to send a message is to perhaps shine a lamp continuously until the receiver shines theirs which signals that they have received and acknowledged your communication request signal, and then communication can begin. A set time(s) of communication can be also be used to help communication on a regular or synchronous (ie. synchronized, happening at the same time, "synced", coincidental) basis, instead of being asynchronous (ie., unsynchronous, non-synchronous, without specific timing and coordination, hence random-like or randomly).

A computer can be used to code, send, receive and decode pulses to letters like this and each pulse is analogous to receiving a binary number 1 or logical true which is usually represented as 1. To distinguish each row and column value, and each letter, a silence or 0 needs to be sensed, and the specific duration of it will determine if the next pulse is the start of another pulse or letter. It is possible to use a certain frequency or tone to signal that a new pulse will be sent, and tone(s) and-or pulsed-tones can be used to signal the specific row or column value is being sent.   With this latter method using pulsed-tones, the timing and-or communication synchronization becomes less of an issue.   In a modern computerized communication system, it may be rated as being able to transmit up to 10 bits or pulses per second. The timing of each bit is then: 10 bits/second   , and after dividing the numerator and denominator by 10, we find that; 10bits/s = 1 bit / 0.1s. After each tenth of a second or 0.1s, the computer will be instructed to send or sense (ie., "read") the next bit value of either 0 or 1. A longer delay could be used to signal the next byte of data will be communicated. After receiving 8 bits or a byte of data, it can be stored in a file. A special byte value may even signal a (communication) control code, such as start of transmission, end of transmission, error, resend, new line of text, etc.   The ASCII communication code for text messages has several control codes having a byte value in the range of 0 to 31.

Again, for effective communication, a language or code as a standard must me used.

```
echo Copying to drive d:\
copy %1 d:\    /y

rem echo.
rem echo Copying to drive e:\
rem copy %1 e:\ /y        /* Currently on my system, E is the CD read only drive   */

echo.
echo Copying to drive f:\
copy %1 f:\    /y

echo.
echo Finished
echo.

rem normal batch file termination here:
rem When the pause command is encountered during processing, the user is prompted to press a key to resume.
pause
exit

:HERE1
echo off
echo To use this batch file, please drag and drop the file you
echo want to copy to your drives onto the batch file icon.
echo.
pause
exit
```

This batchfile below allows the user to quickly go to the current directory's or folder's command line of which the batchfile is placed in.

```
@echo off

rem  ToThisDOS.bat
rem  This batchfile allows the user to quickly go to the current directory's or folder's command line.
rem  (c) JPA 2023  , Author of the Mathization , Math and Science ebook

echo.----------------------------------------------------------------------------

echo Type HELP to see many of the available DOS or computer system commands.
echo Type HELP command_name to see about a specific commands usage.
echo.
echo CTRL C  or  CTRL BREAK  can sometimes be used to terminate a command.
echo.
echo Type EXIT to return to system.
echo.----------------------------------------------------------------------------
echo.

cmd.exe
```

```
                fflush(stdin);
                getch(); exit(0);
              };

/* If the input file does exist: */
rewind(fp1);
fseek(fp1,0, SEEK_END);     /* : Get filesize.  Set file access position at the end of the file marker. */
filesize = ftell(fp1);
L=filesize;
printf("\nFilesize = %lu bytes.",filesize);
if(filesize==0L){  printf("\n\aThe input file is empty.  Press a key.");   getch();  fclose(fp1); exit(0);  };
rewind(fp1);    /* : ensure the file pointer and-or access (read or write) position is at the first byte in the file */

fp2=fopen(outputfile,"rb");   /* See if the output file already exists. */

if(fp2 != NULL){   printf("\nThis output file already exists.  Press 1 for overwrite, or 0 to exit.");
                   fflush(stdin);  c=getch();  c=c-48;  if(c!=1){ fclose(fp2); exit(0);  };
                   fclose(fp2);   /* :since we want to now open the file for writing access */
              };
fp2=fopen(outputfile,"wb");   /* If file already exists, it will be erased to 0 bytes, and so as to be written over.  */
                              /* "wb" will allow writing access, beginning at the first byte of the file. */

printf("\n\nPlease wait until finished processing.");

/*----------------------------- Copy the data in the input file to the output file.  ---------------*/

/* A loop to copy (read and write) the file byte by byte:  */

while(L>0){
                e=fread(&c,1,1,fp1);     /* : Read a byte from the input file.  */
                if(e==0){  printf("\n\nFILE READ ERROR"); fflush(stdin);
                        printf("\nPress A Key");  getch(); exit(0);  };

                e=fwrite(&c,1,1,fp2);    /* : Write a byte to the output file.    */
                if(e==0){  printf("\n\nFILE WRITE ERROR"); fflush(stdin);
                        printf("\nPress A Key");  getch(); exit(0);  };

                /* printf("\n%lu %d ",L,c)   : optional , to display each file address and the byte stored there     */
                /*                            The user can press pause on the keyborad to temporarily halt the program,
                                              or CTRL and Break , or CTRL and C to stop the program.
*/

              L = L - 1;
            };

/* ------------------------------------------------------------------------------------------------------ */

fclose(fp1);
fclose(fp2);

printf("\n\nFinished, press a key: ");  fflush(stdin); getch();

};
/*----------------------------------------------------------------------------------------------------*/
```

```
            printf("\nWavelength = %.15g meters = %.15g feet",(c/f) , (c/f) * 3.28084 );
        };

    if(choice==2){ printf("\n\nEnter the wavelength in meters:  ");
            fflush(stdin); scanf("%lf",&w);
            if( (w<0.0) || (w > c) ){  printf("\a"); continue; };    /* let 300 million meters  be max.. */
            printf("\n");
            printf("\nFrequency = %.15g hz",(c/w));
        };

    printf("\n\nPress A Key");  fflush(stdin); getch();

    };  /* end of main loop */

return;
};
/*---------------------------------------------------------------------------------*/
```

Here is an example of the above programs input and output:

Frequency Or Wavelength Calculator Program
(c) JPA 2022 , Author of the Mathization,
Math And Science book

The Speed Of Light Is 299,792,458 m/s = about 300,000,000 m/s

1 Input Frequency To Find Its Wavelength.

2 Input Wavelength In Meters To Find Its Frequency.

    1 m = 100 cm = 3.28084 ft = 39.37 in
    1 cm = 0.01 m = 0.3937 in = 6.3 sixteenths of an inch
    1 ft = 12 in = 0.3048 m = 30.38 cm
    1 in = 0.08333...ft = 0.0254 m = 2.54 cm exact

0 Exit

Enter Choice: 1

Enter the frequency in hz: 1000000

Wavelength = 299.792458 meters = 983.57108790472 feet

Press A Key

```
/*----------------------------------------------------------------------------*/
```

Program output:

0 1 2 3 4 5 6 7 8 9

Press a key:

A COMPUTER PROGRAM TO CONVERT A NUMBER OF SECONDS TO Y:D:H:M:S FORMAT

Many other conversions can be thought of and put into this program example, and-or other programs.

```
/*-------------------------------------------------------------------------------
TimeConverter.c

This example program converts an input number of seconds to Y:D:H:M:S format
or:   Years:Days:Hours:Minutes:Seconds  format

For reference:

1 minute = 60 seconds
1 hour = 60 minutes = 3600 seconds
1 day = 24 hours = 24 (3600s) = 86400s
1 year = 365 days  =  365.0 (86400s) = 31,536,000s     When using the basic amount 365.0 days.
                                                       See below.

As a test of this program , the user can input 31626061 for the seconds value,
and the result should be: Y:D:H:M:S = 1:1:1:1:1    As another test, 86460 seconds
should give 0:1:0:1:0    Another test of 90061 should give:  0:1:1:1:1

NOTE and for floating point values in general, scientific notation and-or powers
of 10 can be input using letter e or E which means the exponent of 10. 10^e or 10^E
Ex.  To input 3600 , you could use:  3.6E3 or 3.6e3  which can be interpreted as
3.6 (10e3)  or=  3.6 (10^3)

(c) Feb. 2023, JPA
-------------------------------------------------------------------------------*/

#include "stdio.h"
#include "conio.h"

/*----------------------------------------------------------------------------*/

void main(void)
{
double input=0.0;
double inputseconds=0.0;
double Y=0.0;  /* years */ /* Here using non-integer float values so as to
                            have input and output  fractions of a value. */
double D=0.0;  /* days */
double H=0.0;  /* hours */
double M=0.0; /* minutes */
double S=0.0;  /* seconds */
```

```c
/* Get the number of years. */          /* For equivalent HOURS */
    Y = input / 31536000.0;        /*  A 365.0 day year = 31536000 s */
    printf("\n%ld seconds = %.15g Years = Y",(long int) inputseconds,Y);

    /* the long int typecast effectively removes the fractional part of the floating point number. */
    /* printf("    The whole number of years = %ld",(long int)Y); */

   /* get the remaining seconds ,after the year amount of seconds is subtracted */
     temp = input -  (double) ( ((long int) Y) * 31536000.0);
     input = temp;    /* as like a new number of seconds input */
    /*  printf("\remaining seconds = %.15g",temp); */

/*---------------------------------------------------------------------------------------------------------*/

    /* For equivalent DAYS leftover or remaining.  */

    D=input / 86400.0;
    temp = input - (double)( ((long int) D) * 86400.0);  /* subtract the number of seconds in those days */
    input = temp;    /* as like a new number of seconds input to work with*/

    printf("\nRemaining Days = D = %.15g",D);
    /*  printf("    The whole number of Days  = %ld",(long int) D); */

/*---------------------------------------------------------------------------------------------------------*/

  /* For the equivalent HOURS left. */

    H=input/3600.0;
    temp = input - (double)( ((long int) H) * 3600.0);  /* subtract the number of seconds in those hours */
    input = temp;    /* as like a new number of seconds input */

    printf("\nRemaining Hours = H = %.15g",H);
    /*  printf("    The whole number of  Hours  = %ld",(long int) H);  */

/*---------------------------------------------------------------------------------------------------------*/

  /* For the equivalent MINUTES  left. */

    M=input/60.0;

    temp = input - (double)( ((long int) M) * 60.0);  /* subtract the number of seconds in those minutes */
    input = temp;    /* as like a new number of seconds input */

    printf("\nRemaining Minutes = M = %.15g",M);
    /*  printf("    The whole number of  Minutes  = %ld",(long int) M);  */

/*---------------------------------------------------------------------------------------------------------*/

  /* For the equivalent number of  SECONDS  left or remaining.  */

    S=input;

    temp = input - (double)( ((long int) M) );  /* subtract the number of seconds in those minutes */
```

A Computer Program To Convert Between Fahrenheit, Celsius, and Kelvin Temperatures

```c
/*-----------------------------------------------------------------------------
temperatures.c

A Computer Program To Convert Between Fahrenheit, Celsius, and Kelvin Temperatures

Note, the Celsius and Kelvin temperature scales are linear.  Each 1 degree rise in
Celsius is 1 degree rise in Kelvin and vice-versa.  This is not the case with the
Fahrenheit scale in relation to the Celsius and Kelvin scales.  When converting from
Fahrenheit to Kelvin or vice-versa, it is best to first convert its value to Celsius.

(c) JPA 2003

Programming Language:   ANSI C
Compiler:  TinyC


-------------------------------------------------------------------------------*/

#include "stdio.h"
#include "conio.h"

/*-----------------------------------------------------------------------------*/
void main(void)
{
double K = 0.0;          /* Kelvin      */
double F = 0.0;           /* Fahrenheit */
double C = 0.0;            /* Celsius      */
double input = 0.0;
double temp=0.0;
char c;

printf("\n\nTemperature program to convert between Fahrenheit, Celsius, and Kelvin");
printf("\n(c) JPA 2023, for the Mathization eBook");

for(;;){     /* loop1  */

    F = 0.0;  C = 0.0; C = 0.0;  input = 0.0;  temp=0.0;   c=-1;   /* : reset for reuse */

    printf("\n\n---------------------------------------------------------");
    printf("\n\n1 Enter a Fahrenheit Temperature. ");
    printf("\n2 Enter a Celsius Temperature. ");
    printf("\n3 Enter a Kelvin Temperature. ");
    printf("\n\n0 Exit. ");

    printf("\n\nEnter Selection:  ");
    fflush(stdin);  scanf("%c",&c);
    c=c-48;  /* convert ASCII to numeric selection */
    if(c==0){  printf("\n\n");  break; };
    if( (c<0) || (c>3) ){  printf("\a"); continue;  };

   if(c==1){   printf("\nEnter Fahrenheit Value:  ");
              fflush(stdin);
              scanf("\n%lf",&F);
```

Enter Fahrenheit Value: 212

F = 0 = C = -17.778 = K = 255.37

SOME COMPUTER PROGRAM IDEAS TO CONSIDER

First, it is perhaps better for a programmer to write their own functions to do something, even if a function already exists, and they can create a new, modified and-or enhanced version. After all, every program consists of individual functions to help it perform what the program needs to do, and to keep it readable and orderly. Programmers should keep a collection of all the functions they created, and so as to have them available for later use, editing and-or study.

1. A program to **split a file** into smaller file size segments, and is also capable of rebuilding a file from those file segments being joined together into one file. This is helpful for very large files, such as video files, that are larger than the 700Mb space on a compact disk (CD) storage. It can also be a way to send a file that is greater in size than a standard file size attachment for email, such as say 20Mb.   Each file segment must be sent as a separate attachment.   Years ago, it was very common to receive and install a large program such as the Windows operating system from about 10 floppy disks. After installing the first disk, the computer program would display a message prompt such as "Please insert disk #2, and press a key", and so on.

2. Create a file editor such as a **byte editor**. Make a function that reads and-or writes a bytes at a certain address in a file and the value   For example, the function arguments might be:

          unsigned char readbyte(fFILE * fp , unsigned long address);
          void writebyte(FILE * fp , unsigned long address);

   The basic concepts from the computer program in this book that can copy a file can be considered. As the user to input the address (either the offset or logical address of the byte, or physical address value), and-or they byte value if it is to be edited.

3. **Create a database**. The data for each record can be placed in a single data file, or a single, readable/editable (such as by non-computer programmers, etc) text file. For example, each record or group of related data might include a persons name on one line and some other numerical data on the following lines. This data can be searched for given a persons name, record or account number or some other number or keyword(s).   The data can be read and processed to display things such as a certain record or some average of a specific piece of data in all the record.   It is even possible to load all the data into computer memory for faster access, and then save the data to a file when completed. You may wish to allow random comments or notes to be included in this data, such as for future reference, and you will need to delimit or set it apart from the other data by using with some special character(s) or words. For a basic and helpful start to this, you may first consider the computer program in this book called: A Computer Program To Calculate Some Statistics Of Data. A C language programmer may wish to look up the **struct** keyword in C, and it is essentially for creating a user defined data type that is composed of one or more of C's data types, and is analogous to creating a record of related information. struct essentially means a (data) structure. An array of the standard data types can also be used as a record of data, but it is usually for data elements having the same data type such as char (1 byte sized data) or int (2 byte sized data).

4. This book has programs, such as the calculator to solve one expression at a time which contains only one or two numeric operands such as 5 * 3 and the result is 15. This book also shows some ways so as to solve equations within a program, but it would also very helpful to allow the user to input a text string consisting of longer and more complex expressions such as 5 * (3 + 1). What is needed to solve general user input expressions is to **create an equation parser** code within a program and-or entire program which separates the (text, string form) equation into its more manageable composing terms, and then calculate the value of each term, and then combines these term values so as

to produce the result of the entire equation. This would make a better calculator, and possibly one which includes optional variables and loops.   Considerations for a parser should include expression evaluation from left to right as for the standard order of operations, and grouping symbols such a parenthesis surrounding an inner expression which have a higher precedence and must be calculated first.   When you write a computer program, the language compiler already has an equation parser and solver built into it, but its for use by the programmer and generally not the user of the created programs.

5. **Create our own simple computer language**.   The easiest way to do this would be to create a Basic-like language where each line of the input text-source file program is processed from top down or one after the other. The language interpreter program created should at least include the allowing of user input such as text strings, variables, loops of which can be made using the goto command and program location labels, screen display output, basic math calculations, and advanced programmers might wish to include a general equation parser for solving for the result of user input expressions. This interpreter program can use the compilers built in string, math and other functions, etc., and-or write your own as shown in this book. An example of a user-made string function would be one that converts a text string to a numerical value so that it can be used elsewhere in the program, such as in a numerical expression.

6. **Write simple program for a company**. This company can be either imaginary or real. First find or choose a topic to make a program about and what the result and-or output of the program should be. Consider what equations to use, and the values(s) to be input by the user. Allow the program to have a "help screen" or "about" menu option so as to display what the program is about, how to use it, possible helpful data, recommendations and-or options. The date, author, copyright, and-or owner of the program and contact information can also be displayed . Then consider some other programs to create for the same company and-or other companies. If a company pays you for a program, you will need to negotiate with them if they are also buying the copyright also, and so as to have all rights to the use of that program. In a way, you will then have to become like a business person. If possible, advertise your programming work so that you and others may have a chance to gain from it.

CONCLUSION

Many new math concepts and ideas take some time for the mind to "adjust" or comprehend to where those ideas seem simpler, practical and available for more immediate or better use. Nobody is born knowing much about any subject, and everyone can learn it on their own time and abilities, and no one born should feel that they are behind or lacking. Some math-minded people may have a general or basic knowledge of much of math and may only specialize in one certain subject of it, say statistics for example. There is also then a great need for what can be called as "math programmers" so as to make computer programs for say the statistics, industry or anyone in need of a computer program.

Hopefully. this book has or will improve your math understanding and interest, and so as to prevent some misunderstandings towards the usefulness of knowing some mathematics. Learning new things should be fun, but it can also be stressful, often due to an overload of information, so try to learn in a comfortable manner, and put the books aside and let your mind digest, processes and clarify new knowledge. There is sometimes a fear and apathy to math, and it may be because it is perceived as being so vast and then can only be unattainable. Take into consideration that this book includes many of the common and simple basics of math that even advanced mathematicians always rely upon.   Every book will eventually be set on the shelf, either before, during or after they are read, still, just having the availability, quick access and some time to put useful knowledge within a mind is very important to you and others so that it can be quickly applied in some useful way when needed.

The field of science and or math is so vast and still being developed, and that it has now become impractical to be very knowledgeable and proficient in all of it, and it is much better for a person to focus on just one certain field of science and-or math, and then to perhaps focus on just a select amount of equations and-or some aspect of a process and teamwork, and so as to apply the results and be useful for others.

Each book also seems to lack in some way and yet excels in other ways. Some math conscience, ability or minimal requirement, even if just adding and subtracting, seems to be almost everywhere because of its usefulness at providing solutions. Knowing some math will provide you with extra confidence, knowledge and professionalism in any work or field of study. After writing this book, I began to realize a word such as "mathization" could be a generic-like word for all science things and books of math(s), and which is sometimes collectively called as "polymath". I can not write about all of these topics, and out of respect for all others, maybe I should change the book title to something like "Jon's Math Book". With this understanding of some math and science within this book, it is hoped that you can now more easily understand, explore and-or apply it. Vast knowledge belongs in a library, but practical, useful and immediate knowledge belongs within you and a book like this.

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